# MINIMIZATION OF THE NUMBER OF PERIODIC POINTS FOR SMOOTH SELF-MAPS OF CLOSED SIMPLY-CONNECTED 4-MANIFOLDS 

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#### Abstract

Let $M$ be a smooth closed simply-connected $m$-dimensional manifold, $f$ be a smooth self-map of $M$ and $r$ be a given natural number. The invariant $D_{r}^{m}[f]$ defined by the authors in [Forum Math. 21 (2009)] is equal to the minimum of \#Fix $\left(g^{r}\right)$ over all maps $g$ smoothly homotopic to $f$. In this paper we calculate the invariant $D_{r}^{4}[f]$ for the class of smooth self-maps of 4-manifolds with fast grow of Lefschetz numbers and for $r$ being a product of different primes.


1. Introduction. One of the fundamental problems in periodic point theory is to find minimal number of periodic points in the homotopy class of a given map. Let $f$ be a self-map of a compact manifold M. B. Jiang introduced in 1983 the invariant $N F_{r}(f)$ which estimates from above \#Fix $\left(g^{r}\right)$ for all $g$ homotopic to $f$ [14]. J. Jezierski proved in 2006 that the invariant is the best estimation if the dimension of $M$ is at least 3 [12]. This means that $N F_{r}(f)$ is equal to the minimal number of elements in $\operatorname{Fix}\left(g^{r}\right)$ over all $g$ homotopic to $f$. In the last years the invariant was computed in many special cases, see for example: [10], [13], [16], [18].

In the recent papers [4], [6] the authors developed the theory for the smooth (i.e. $C^{1}$ ) category, searching for the minimum in smooth homotopy class. As a result, two counterparts of $N F_{r}(f)$ were found: $D_{r}^{m}[f]$ for simply-connected manifolds [4] and its generalization $N J D_{r}^{m}[f]$ for non simply-connected ones [6]. The crucial demanding for effective computation of the invariants is the knowledge of all sequences of local fixed point indices of iterations at a periodic $p$-orbit for smooth maps in the given dimension $m$, called $D D^{m}(p)$ sequences. This information was provided in dimension 3 in the paper [9], which made it possible to compute the value of $D_{r}^{3}[f]$ for $S^{2} \times I[4], S^{3}[5]$, a two-holed 3-dimensional closed ball [3] and $N J D_{r}^{3}[f]$ for $\mathbb{R} P^{3}$ [7]. Recently, in [8] we provided the list of all possible sequences of local indices of iterations in arbitrary dimension, which allows one to calculate the invariants for self-maps of higher dimensionional manifolds. In this paper we partially realize this

[^0]programme for simply-connected manifolds and $m=4$. We calculate $D_{r}^{4}[f]$ under the assumption that the so-called periodic expansion of $\left\{L\left(f^{n}\right)\right\}_{n=1}^{\infty}$, the sequence of the Lefschetz numbers of iterations, has only non-zero coefficients. This property holds for example for maps with fast grow of the sequence of Lefschetz numbers, such as self-maps of $S^{4}$ with degree $d$ satisfying $|d|>1$.

The paper is organized in the following way. First, in Section 2 we give the definition of $D_{r}^{m}[f]$ which is expressed in terms of $D D^{m}(p)$ sequences. Next, in Section 3 we provide the list of all $D D^{m}(1)$ sequences and prove that in order to calculate $D_{r}^{4}[f]$ it is enough to use only $D D^{4}(1)$ sequences. Finally, in Section 4 we calculate $D_{r}^{4}[f]$ for $r$ being a product of different primes (Theorem 4.8).
2. The invariant $D_{r}^{m}[f]$. The notion of Differential Dold sequences ( $D D$ sequences in short) introduced in [4] is used in the definition of the invariant $D_{r}^{m}[f]$. A $D D^{m}(p)$ sequence is a sequence of integers that can be locally realized as a sequence of indices on an isolated $p$-orbit for some smooth map.

Definition 2.1. A sequence of integers $\left\{c_{n}\right\}_{n=1}^{\infty}$ is called a $D D^{m}(p)$ sequence if there is a $C^{1} \operatorname{map} \phi: U \rightarrow \mathbb{R}^{m}\left(U \subset \mathbb{R}^{m}\right)$ and its isolated $p$-orbit $P$ such that $c_{n}=\operatorname{ind}\left(\phi^{n}, P\right)$. If this equality holds for $n \mid r$, where $r$ is fixed, then the finite sequence $\left\{c_{n}\right\}_{n \mid r}$ will be called a $D D^{m}(p \mid r)$ sequence.

Let $r$ be fixed. The minimal decomposition of the sequence of Lefchetz numbers of iterations into $D D^{m}(p \mid r)$ sequences gives the value of $D_{r}^{m}[f]$.

Definition 2.2. Let $\left\{L\left(f^{n}\right)\right\}_{n \mid r}$ be a finite sequence of Lefschetz numbers. We decompose $\left\{L\left(f^{n}\right)\right\}_{n \mid r}$ into the sum:

$$
\begin{equation*}
L\left(f^{n}\right)=c_{1}(n)+\ldots+c_{s}(n), \tag{1}
\end{equation*}
$$

where $c_{i}$ is a $D D^{m}\left(l_{i} \mid r\right)$ sequence for $i=1, \ldots, s$. Each such decomposition determines the number $l=l_{1}+\ldots+l_{s}$. We define the number $D_{r}^{m}[f]$ as the smallest $l$ which can be obtained in this way.

The invariant $D_{r}^{m}[f]$ is equal to the minimal number of $r$-periodic points in smooth homotopy class of $f$.

Theorem 2.3. ([4]) Let $M$ be a smooth closed connected and simply-connected manifold of dimension $m \geq 3$ and $r \in \mathbb{N}$ a fixed number. Then,

$$
D_{r}^{m}[f]=\min \left\{\# \operatorname{Fix}\left(g^{r}\right): g \text { is smoothly homotopic to } f\right\}
$$

Periodic expansion is a convenient method of storing the data connected with the sequence of indices of iterations. Each such sequence can be expanded as a combination of some basic periodic sequences $\left\{\operatorname{reg}_{k}\right\}_{n}$ taken with integral coefficients.

Definition 2.4. For a given $k$ we define the basic sequence:

$$
\operatorname{reg}_{k}(n)=\left\{\begin{array}{rlr}
k & \text { if } & k \mid n \\
0 & \text { if } & k \nmid n .
\end{array}\right.
$$

A sequence of indices of iterations (as well as a sequence of Lefchetz numbers of iterations) may be written down in the form of periodic expansion (cf. [15]), namely:

$$
\begin{equation*}
\operatorname{ind}\left(f^{n}, x_{0}\right)=\sum_{k=1}^{\infty} a_{k} \operatorname{reg}_{k}(n) \tag{2}
\end{equation*}
$$

where $a_{n}=\frac{1}{n} \sum_{k \mid n} \mu(k) \operatorname{ind}\left(f^{(n / k)}, x_{0}\right), a_{n}$ are integers, $\mu$ is the classical Möbius function, i.e. $\mu: \mathbb{N} \rightarrow \mathbb{Z}$ is defined by the following three properties: $\mu(1)=1$, $\mu(k)=(-1)^{s}$ if $k$ is a product of $s$ different primes, $\mu(k)=0$ otherwise.

The fact that the coefficients $a_{n}$ are integers follows from the result of Dold [2].
The invariant $D_{r}^{m}[f]$ is defined in terms of $D D^{m}(p)$ sequences. On the other hand, it is enough to know only the forms of $D D^{m}(1)$ sequences, because every $D D^{m}(p)$ sequence can be obtained from some $D D^{m}(1)$ one.
Definition 2.5. We will say that the $D D^{m}(p)$ sequence $\left\{\tilde{c}_{n}\right\}_{n}$ comes from the given $D D^{m}(1)$ sequence $\left\{c_{n}\right\}_{n}$ with the periodic expansion $c_{n}=\sum_{d=1}^{\infty} a_{d} \operatorname{reg}_{d}(n)$ if the periodic expansion of $\left\{\tilde{c}_{n}\right\}_{n}$ has the form:

$$
\tilde{c}_{n}=\sum_{d=1}^{\infty} a_{d} \mathrm{reg}_{p d}(n)
$$

Theorem $2.6([4])$. Every $D D^{m}(p)$ sequence comes from some $D D^{m}(1)$ sequence.
3. Local indices of iterations in dimension 4. In this section we give the complete list of all sequences of local indices of iterations of a smooth map in dimension 4 i.e. the list of all $D D^{4}(1)$ sequences. Let us mention here that the forms of indices of iterations for continuous maps are known since 1991 [1], and recently indices of iterations have been found also for other important classes of maps, such as holomorphic maps [22] and planar homeomorphisms [17], [21].

Definition 3.1. Let $H$ be a finite subset of natural numbers, we introduce the following notation.
$\operatorname{By} \operatorname{LCM}(H)$ we mean the least common multiple of all elements in $H$ with the convention that $\operatorname{LCM}(\emptyset)=1$. We define the set $\bar{H}$ by: $\bar{H}=\{\operatorname{LCM}(Q): Q \subset H\}$.

For natural $s$ we denote by $L(s)$ any set of natural numbers of the form $\bar{L}$ with $\# L=s$ and $1,2 \notin L$.

By $L_{2}(s)$ we denote any set of natural numbers of the form $\bar{L}$ with $\# L=s+1$ and $1 \notin L, 2 \in L$.

Theorem 3.2 (Main Theorem I in [8]). Let $f$ be a $C^{1}$ self-map of $\mathbb{R}^{m}$, where $m$ is even. Then the sequence of local indices of iterations $\left\{\operatorname{ind}\left(f^{n}, 0\right)\right\}_{n=1}^{\infty}$ has one of the following forms:

$$
\begin{gathered}
\left(A^{e}\right) \operatorname{ind}\left(f^{n}, 0\right)=\sum_{k \in L_{2}\left(\frac{m-4}{2}\right)} a_{k} \operatorname{reg}_{k}(n) . \\
\left(B^{e}\right) \operatorname{ind}\left(f^{n}, 0\right)=\sum_{k \in L\left(\frac{m-2}{2}\right)} a_{k} \operatorname{reg}_{k}(n) . \\
\left(C^{e}\right),\left(D^{e}\right),\left(E^{e}\right) \quad \operatorname{ind}\left(f^{n}, 0\right)=\sum_{k \in L_{2}\left(\frac{m-2}{2}\right)} a_{k} \operatorname{reg}_{k}(n),
\end{gathered}
$$

where

$$
a_{1}=\left\{\begin{array}{cl}
1 & \text { in the case }\left(C^{e}\right) \\
-1 & \text { in the case }\left(D^{e}\right) \\
0 & \text { in the case }\left(E^{e}\right)
\end{array}\right.
$$

$$
\left(F^{e}\right) \operatorname{ind}\left(f^{n}, 0\right)=\sum_{k \in L\left(\frac{m}{2}\right)} a_{k} \operatorname{reg}_{k}(n)
$$

where $a_{1}=1$.
By $[d, l]$ we denote the least common multiple of $d$ and $l$.
Theorem 3.3. The list of all $D D^{4}(1)$ sequences is the following:
(A) $\quad c_{A}(n)=a_{1} \operatorname{reg}_{1}(n)+a_{2} \operatorname{reg}_{2}(n) ;$
$(B) \quad c_{B}(n)=a_{1} \operatorname{reg}_{1}(n)+a_{d} \operatorname{reg}_{d}(n) ;$
$(C-E)_{o d d}$

$$
c_{X}(n)=\varepsilon_{X} \operatorname{reg}_{1}(n)+a_{2} \operatorname{reg}_{2}(n)+a_{d} \operatorname{reg}_{d}(n)+a_{2 d} \operatorname{reg}_{2 d}(n)
$$

where $\varepsilon_{X} \in\{-1,0,1\}, X \in\{C, D, E\}, d$ is odd.
$(C-E)_{\text {even }}$

$$
c_{X}(n)=\varepsilon_{X} \operatorname{reg}_{1}(n)+a_{2} \operatorname{reg}_{2}(n)+a_{d} \operatorname{reg}_{d}(n)
$$

where $\varepsilon_{X} \in\{-1,0,1\}, X \in\{C, D, E\}$, $d$ is even.
$(F) c_{F}(n)=\operatorname{reg}_{1}(n)+a_{d} \operatorname{reg}_{d}(n)+a_{l} \operatorname{reg}_{l}(n)+a_{[d, l]} \operatorname{reg}_{[d, l]}(n)$, In all cases $d, l \geq 3$ and $a_{i} \in \mathbb{Z}$.

Proof. We apply Theorem 3.2 for $m=4$, obtaining the corresponding parts of the thesis. For example, to obtain the case $(F)$, we use $\left(F^{e}\right)$ and get:

$$
L\left(\frac{m}{2}\right)=L(2)=\overline{\{d, l\}}=\operatorname{LCM}\{Q \subset\{d, l\}\}=\{1, d, l,[d, l]\}
$$

Corollary 1. Let us notice that any $D D^{4}(1)$ sequence has one of the following forms:

1. $a_{1} \operatorname{reg}_{1}(n)+a_{d} \operatorname{reg}_{d}(n)$; for $a_{1}, a_{d} \in \mathbb{Z}$.
2. $\varepsilon \operatorname{reg}_{1}(n)+a_{2} \operatorname{reg}_{2}(n)+a_{d} \operatorname{reg}_{d}(n)+\gamma_{d} a_{2 d} \operatorname{reg}_{2 d}(n)$;
for $a_{2}, a_{d} \in \mathbb{Z}, \varepsilon=0, \pm 1, \gamma_{d}=0$ if $d$ is even and $\gamma_{d}=1$ if $d$ is odd.
3. $\operatorname{reg}_{1}(n)+a_{d} \operatorname{reg}_{d}(n)+a_{l} \operatorname{reg}_{l}(n)+a_{[d, l]} \operatorname{reg}_{[d, l]}(n)$;
for $a_{d}, a_{l} \in \mathbb{Z}, d, l \geq 3$.
The next two lemmas show that during the calculation of $D_{r}^{4}[f]$ we may consider only $D D^{4}(1)$ sequences, which makes the computation much easier.
Lemma 3.4 (Remark 4.6 in [4]). For $m \geq 3$ in Definition 2.2 of $D_{r}^{m}[f]$ we can equivalently use only $D D^{m}(p \mid r)$ sequences such that $p<2^{\left[\frac{m+1}{2}\right]}$.
Lemma 3.5. To calculate $D_{r}^{4}[f]$ it is enough to consider only $D D^{4}(1)$ sequences.
Proof. By Lemma 3.4 it is enough to consider only such $D D^{4}(p \mid r)$ sequences for which $p \leq 3$.

We show that
(1) every $D D^{4}(2 \mid r)$ sequence is a sum of at most two $D D^{4}(1 \mid r)$ sequences.
(2) every $D D^{4}(3 \mid r)$ sequence is a sum of at most three $D D^{4}(1 \mid r)$ sequences.

Proof of (1). Using Theorem 2.6 we find the forms of all $D D^{4}(2 \mid r)$ sequences, each of which comes from some $D D^{4}(1 \mid r)$ sequences of one of the types (A)-(F). Next, we represent each $D D^{4}(2 \mid r)$ sequence as a sum of at most two $D D^{4}(1 \mid r)$ sequences.
(A) $a_{2} \operatorname{reg}_{2}(n)+a_{4} \mathrm{reg}_{4}(n)$ is in fact the $D D^{4}(1 \mid r)$ sequence of the type $(D)_{\text {even }}$. (B) $a_{2} \operatorname{reg}_{2}(n)+a_{2 d} \mathrm{reg}_{2 d}(n)$ the same argument as above is true.
(C-E) For $d$ odd we have that $[4,2 d]=4 d$, then $\varepsilon_{X} \operatorname{reg}_{2}(n)+a_{4} \operatorname{reg}_{4}(n)+a_{2 d} \operatorname{reg}_{2 d}(n)+a_{4 d} \mathrm{reg}_{4 d}(n)=$ $\operatorname{reg}_{1}(n)+a_{4} \operatorname{reg}_{4}(n)+a_{2 d} \operatorname{reg}_{2 d}(n)+a_{4 d} \operatorname{reg}_{4 d}(n)$ $-\operatorname{reg}_{1}(n)+\varepsilon_{X} \operatorname{reg}_{2}(n)$
where on the right-hand side of the above formula we indicated that the first sum is realized by a sequence of the type ( F ) and the second by (A).

In the same way we deal with the case of $d$ even (every sequence is a sum of a sequence of the type (F) and (A)).
(F) Notice that $[2 d, 2 l]=2[d, l]$, thus we get

$$
\operatorname{reg}_{2}(n)+a_{2 d} \operatorname{reg}_{2 d}(n)+a_{2 l} \operatorname{reg}_{2 l}(n)+a_{2[d, l]} \operatorname{reg}_{2[d, l]}(n)=
$$

$$
\begin{equation*}
-\operatorname{reg}_{1}(n)+\operatorname{reg}_{2}(n)+ \tag{A}
\end{equation*}
$$

$$
\begin{equation*}
+\operatorname{reg}_{1}(n)+a_{2 d} \operatorname{reg}_{2 d}(n)+a_{2 l} \operatorname{reg}_{2 l}(n)+a_{[2 d, 2 l]} \operatorname{reg}_{[2 d, 2 l]}(n) \tag{F}
\end{equation*}
$$

Proof of (2). Let us now consider a $D D^{4}(3 \mid r)$ sequence.
Notice that by Theorem 2.6 and Corollary 1 it has always the form with no more than four basic sequences $\mathrm{reg}_{i}$, i.e.

$$
a_{p} \operatorname{reg}_{p}(n)+a_{q} \operatorname{reg}_{q}(n)+a_{r} \operatorname{reg}_{r}(n)+a_{s} \operatorname{reg}_{s}(n)
$$

where $p, q, r, s \geq 3$. Then we may represent this sequence as a sum of three $D D^{4}(1 \mid r)$ sequences in the following way:

$$
\begin{align*}
& -\operatorname{reg}_{1}(n)+a_{p} \operatorname{reg}_{s}+  \tag{B}\\
& +a_{q} \operatorname{reg}_{q}(n)+  \tag{D}\\
& +\operatorname{reg}_{1}(n)+a_{r} \operatorname{reg}_{r}(n)+a_{s} \operatorname{reg}_{s}(n)
\end{align*}
$$

This completes the proof.
4. Calculation of the invariant. We work under the following standing assumptions

## Standing Assumptions

1. $f: M^{4} \rightarrow M^{4}$ is a smooth self-map of a smooth closed connected and simplyconnected 4-manifold,
2. $r=p_{1} \ldots p_{s}$ is a product of different prime numbers,
3. in the periodic expansion of Lefschetz numbers

$$
L\left(f^{k}\right)=\sum_{i=1}^{\infty} a_{i} \operatorname{reg}_{i}(k)
$$

$a_{i} \neq 0$ for all $i \neq 1$ dividing $r$.
Remark 1. The assumption (3) is satisfied for a self-map $f: S^{4} \rightarrow S^{4}$ with $|\operatorname{deg}(f)|>1$ [20]. In general, it often takes place if the growth of $\left\{L\left(f^{k}\right)\right\}_{k}$ is quick.

We will find the formula for $D_{r}^{4}[f]$, under the above assumptions.
It turns out that first it is convenient to find the minimal decomposition of the sum

$$
\sum_{i \mid r} a_{i} \mathrm{reg}_{i}
$$

into $D D^{4}(1 \mid r)$ sequences modulo reg $_{1}$ i.e. we require that the equality holds only for all divisors $i \mid r$ different than 1 . In other words, we will temporarily ignore the coefficient at reg ${ }_{1}$.

Lemma 4.1. The two following numbers are equal:

1. the minimal number of summands in the decomposition of the sum

$$
\sum_{i \mid r} a_{i} \mathrm{reg}_{i}
$$

modulo $\mathrm{reg}_{1}$ into $D D^{4}(1 \mid r)$ sequences,
2. the minimal number $h(s)$ determining the family of pairs of subsets of $I_{s}=$ $\{1, \ldots, s\}$ :

$$
\left\{A_{1}, B_{1}\right\},\left\{A_{2}, B_{2}\right\}, \ldots,\left\{A_{h(s)}, B_{h(s)}\right\}
$$

such that

$$
\bigcup_{i=1}^{h(s)}\left\{A_{i}, B_{i}, A_{i} \cup B_{i}\right\}=2^{I_{s}} \backslash\{\emptyset\}
$$

i.e. for each nonempty subset $X \subset I_{s}$ there is an $i$ such that either $X=A_{i}$ or $X=B_{i}$ or $X=A_{i} \cup B_{i}$.
Proof. Let us notice that to get the minimal decomposition of

$$
\sum_{i \mid r} a_{i} \mathrm{reg}_{i} \text { modulo reg }{ }_{1}
$$

we should use as much as possible the most "greedy" $D D^{4}(1 \mid r)$ sequences, with the greatest number of basic expressions reg ${ }_{i}$ i.e. of the type (2) or (3) of Corollary 1. In both of these cases we have the sequences of the form:

$$
\begin{equation*}
\varepsilon \mathrm{reg}_{1}+a_{d} \mathrm{reg}_{d}+a_{l} \mathrm{reg}_{l}+\gamma a_{[d, l]} \mathrm{reg}_{[d, l]} \tag{3}
\end{equation*}
$$

where $d, l$ are divisors of $r$ different than $1, \gamma \in\{0,1\}$.
Since $r=p_{1} \cdots p_{s}$ is a product of different primes, there is a bijection $G: 2^{I_{s}} \rightarrow$ $\operatorname{Div}(r)$ between $\operatorname{Div}(r)$, the set of all divisors of $r$, and the family of all subsets of $I_{s}=\{1, \ldots, s\}$ :

$$
\{1, \ldots, s\} \supset A \rightarrow \Pi_{i \in A} p_{i} \in \operatorname{Div}(r)
$$

with the convention that $\Pi_{i \in \emptyset} p_{i}=1$. Moreover

$$
G(A \cup B)=[G(A), G(B)]
$$

As a result, every triple of divisors $d, l,[d, l]$ determining the sequence (3) is associated with a triple of subsets of $I_{s}: A_{j}, B_{j}, A_{j} \cup B_{j}$.

Now, a decomposition of the sum $\sum_{1 \neq i \mid r} a_{i} \operatorname{reg}_{i}(k)$ into $h(s) D D^{4}(1 \mid r)$ sequences of the form (3) is equivalent to the existence of $h(s)$ families of subsets of $I_{s}$

$$
\left\{A_{1}, B_{1}\right\},\left\{A_{2}, B_{2}\right\}, \ldots,\left\{A_{h(s)}, B_{h(s)}\right\}
$$

such that

$$
\bigcup_{i=1}^{h(s)}\left\{A_{i}, B_{i}, A_{i} \cup B_{i}\right\}=2^{I_{s}} \backslash\{\emptyset\}
$$

i.e. for each nonempty subset $X \subset I_{s}$ there is an $i$ such $X=A_{i}, X=B_{i}$ or $X=A_{i} \cup B_{i}$.

Now our problem reduces to the following combinatorial question:
Problem 4.1. Let $s$ be a natural number. Find the minimal number $h(s)$ such that there exist $h(s)$ families of subsets $\mathcal{A}_{1}, \ldots, \mathcal{A}_{h(s)} \subset 2^{I_{s}}$ satisfying

1. $\# \mathcal{A}_{i} \leq 2$ i.e. each family consists of at most two subsets,
2. for each nonempty subset $X \subset\{1, \ldots, s\}$ there exists $i \in\{1, \ldots, s\}$ such that $X$ is one of the sets $A_{i}, B_{i}$ or $A_{i} \cup B_{i}$, where $\mathcal{A}_{i}=\left\{A_{i}, B_{i}\right\}$.
Theorem 4.2. The minimal number searched in Problem 4.1 is given by the formula

$$
\begin{equation*}
h(s)=\frac{2^{s}+(-1)^{s+1}}{3} \tag{4}
\end{equation*}
$$

The proof of Theorem 4.2 is a consequence of the following three lemmas.
Lemma 4.3. The formula (4) for $h(s)$ can be given inductively as follows:

$$
h(2)=1, \quad h(s+1)=2 \cdot h(s)+(-1)^{s}
$$

Proof.

$$
\begin{gathered}
2 \cdot h(s)+(-1)^{s}=2 \cdot \frac{2^{s}+(-1)^{s+1}}{3}+(-1)^{s} \\
=\frac{2^{s+1}+2 \cdot(-1)^{s+1}+3 \cdot(-1)^{s}}{3}=\frac{2^{s+1}+(-1)^{s}}{3}=h(s+1) .
\end{gathered}
$$

Lemma 4.4. $h(s)$ given by the formula (4) is less or equal to the minimal number satisfying the conditions in Problem 4.1.
Proof. We notice that each family containing two subsets $\{A, B\} \subset 2^{I_{s}}$ determines at most three nonempty subsets $A, B, A \cup B \subset I_{s}$. Thus, to realize all nonempty subsets in $I_{s}$ we need at least $\left(2^{s}-1\right) / 3$ pairs. The last means that the minimal number in Problem 4.1 is greater or equal to $\left(2^{s}-1\right) / 3$. On the other hand, the least natural number $\geq\left(2^{s}-1\right) / 3$ is equal to $\left(2^{s}-1\right) / 3$ when $s$ is even and $\left(2^{s}+1\right) / 3$ when $s$ is odd. It remains to notice that in both cases we get $h(s)$.
Lemma 4.5. (I) For each $s \geq 2$ there exist $h(s)=\frac{2^{s}+(-1)^{s+1}}{3}$ families satisfying the conditions in Problem 4.1.
(II) Moreover, if $s$ is even then each family must contain two different subsets, while if $s$ is odd then $h(s)-1$ families must contain two different subsets and the last family can contain only one subset consisting of a single, arbitrarily chosen, element.

Proof. We will show inductively that (for $s \geq 2$ ): there exists a family $\mathcal{A}_{s}=$ $\left\{\left\{A_{i}, B_{i}\right\}: i=1, \ldots, h(s)\right\}$ whose elements are nonempty subsets $A_{i}, B_{i} \subset I_{s}$ realizing all nonempty subsets in $I_{s}$ and moreover

1. $A_{i} \neq B_{i}$ if $i=1, \ldots, h(s)$ and $s$ is even,
2. $A_{i} \neq B_{i}$ if $i=1, \ldots, h(s)-1$ and $s$ is odd.
3. $A_{h(s)}=B_{h(s)}=\{s\}$ for $s$ odd.

For $s=2$ all nonempty subsets of $I_{2}=\{1,2\}$ can be obtained from the family $\{\{1\},\{2\}\}$ which implies $h(2)=1$.

Now we assume that for even $s$ a family $\mathcal{A}_{s}=\left\{\left\{A_{i}, B_{i}\right\}: i=1, \ldots, h(s)\right\}$ where $A_{i} \neq B_{i}$ realizes all nonempty subsets in $I_{s}=\{1, \ldots, s\}$. Then the family

$$
\left.\mathcal{A}_{s+1}=\left\{\left\{A_{i}, B_{i}\right\},\left\{A_{i} \cup\{s+1\}, B_{i} \cup\{s+1\}\right\},\{\{s+1\}\}\right\}: i=1, \ldots, h(s)\right\}
$$

realizes all nonempty subsets in $I_{s+1}=\{1, \ldots, s, s+1\}$. Moreover,

$$
\# \mathcal{A}_{s+1}=2 \cdot \# \mathcal{A}_{s}+1=2 \cdot h(s)+1=2 \cdot h(s)+(-1)^{s}=h(s+1)
$$

since $s$ is even.

Now, the family

$$
\mathcal{A}_{s+2}=\left\{\left\{A_{i}^{\prime}, B_{i}^{\prime}\right\},\left\{A_{i}^{\prime} \cup\{s+2\}, B_{i}^{\prime} \cup\{s+2\}\right\},\{\{s+1\},\{s+2\}\}\right\}
$$

where $\left\{A_{i}^{\prime}, B_{i}^{\prime}\right\} \in \mathcal{A}_{s+1} \backslash\{\{s+1\}\}$
realizes all subsets in $I_{s+2}$ and moreover
$\# \mathcal{A}_{s+2}=2 \cdot\left(\# \mathcal{A}_{s+1}-1\right)+1=2 \cdot h(s+1)-1=2 \cdot h(s+1)+(-1)^{s+1}=h(s+2)$ since $s+1$ is odd.

This ends the proof of part (I). Part (II) of Lemma 4.5 follows from Lemma 4.4 and the observation that for $s+1$ odd in the above inductive construction, the family $\{\{s+1\}\}$, i.e. the last element in $\mathcal{A}_{s+1}$, consists of one subset containing a single element. It is evident that after a permutation $\{\{s+1\}\}$ can be replaced with $\{\{i\}\}$ for an arbitrarily prescribed $i \in I_{s+1}$.

## Proof of Theorem 4.2

Lemma 4.4 gives

$$
h(s) \leq \text { minimal number in Problem } 4.1
$$

while Lemma 4.5 proves the opposite inequality.

By Theorem 4.2 we obtain
Corollary 2. The minimal decomposition of the sum

$$
\sum_{i \mid r} a_{i} \mathrm{reg}_{i}
$$

modulo reg ${ }_{1}$ into $D D^{4}(1 \mid r)$ sequences contains exactly

$$
h(s)=\frac{2^{s}+(-1)^{s+1}}{3}
$$

sequences.
Moreover, by Lemma 4.5 (II) we get:
(A) if $s$ is even then the minimal decomposition must contain $h(s)$ sequences of the type

$$
\begin{equation*}
\varepsilon \cdot \operatorname{reg}_{1}+a_{d} \operatorname{reg}_{d}+a_{l} \operatorname{reg}_{l}+\gamma a_{[d, l]} \operatorname{reg}_{[d, l]} \tag{5}
\end{equation*}
$$

i.e. of the form (2) or (3) of Corollary $1(\gamma \in\{0,1\})$;
$(B)$ if $s$ is odd then the minimal decomposition must contain $h(s)-1$ sequences of the type (5) while the remaining sequence may be $a_{1} \operatorname{reg}_{1}(n)+a_{d} \mathrm{reg}_{d}(n)$ (i.e. of the type (1) of Corollary 1), where $d \neq 1$ is an arbitrarily prescribed divisor of $r$.

Remark 2. Let us notice that in all sequences (5), appearing in the minimal decomposition modulo 1 described in Corollary 2, the divisors $d, l$ must be different as they correspond to different subsets in Lemma 4.5, so both $\operatorname{reg}_{d}(n)$ and $\operatorname{reg}_{l}(n)$ appear with nonzero coefficients.

Now we are in a position to find the formula for $D_{r}^{4}[f]$, i.e. we take into account also the coefficient at $\mathrm{reg}_{1}$.

Let us remark that $D_{r}^{4}[f] \geq h(s)$. In fact, in the minimal realization modulo reg ${ }_{1}$ we need $h(s)$ of $D D^{4}(1 \mid r)$ sequences. The following lemmas make it precise when
these sequences are sufficient to obtain the decomposition with $a_{1} \mathrm{reg}_{1}$ and when one additional sequence, to realize $a_{1} \mathrm{reg}_{1}$, is necessary.
Lemma 4.6. Assume our Standing Assumptions are satisfied and $s$ is even, then

$$
D_{r}^{4}[f]= \begin{cases}h(s) & \text { if }(r \text { is odd and } L(f)=h(s)) \\ & \text { or }(r \text { is even and } h(s)-2 \leq L(f) \leq h(s)) \\ h(s)+1 & \text { otherwise. }\end{cases}
$$

Proof. By Corollary 2 (A) to realize

$$
\sum_{1 \neq i \mid r} a_{i} \mathrm{reg}_{i}
$$

we need at least $h(s) D D^{4}(1 \mid r)$ sequences of the type (2) or (3) of Corollary 1.
If we assume that $r$ is odd then they all must be of the type (3). Then the contribution of each of them to the coefficient at reg ${ }_{1}$ is 1 . If moreover $L(f)=h(s)$ then $D_{r}^{4}[f]=h(s)$, since $a_{1}=L(f)$. Otherwise, we need one sequence of the type (1) more to realize the difference $\left(a_{1}-h(s)\right) \cdot \operatorname{reg}_{1}(n)$.

Now we consider the case of even $r$. Then exactly one sequence in the minimal decomposition must be of the type (2) and the remaining $h(s)-1$ sequences are of the type (3). Their contribution to the coefficient at $\operatorname{reg}_{1}$ is $(h(s)-1)+\varepsilon$ where $\varepsilon=0,+1,-1$. Now, if $h(s)-2 \leq L(f) \leq h(s)$, then $a_{1}$ can be realized by these sequences. Otherwise we need one more sequence of the type (1).
Lemma 4.7. Assume our Standing Assumptions are satisfied and $s$ is odd, then

$$
D_{r}^{4}[f]=h(s)
$$

Proof. It is enough to show that $\sum_{i \mid r} a_{i} \mathrm{reg}_{i}(n)$ is the sum of exactly $h(s) D D^{4}(1 \mid r)$ sequences.

Since $s$ is odd, by Corollary $2(\mathrm{~B}), h(s)-1$ sequences of the types (2) or (3) of Corollary 1 realize

$$
\sum_{i} a_{i} \mathrm{reg}_{i}
$$

where the summation runs through the set $\operatorname{Div}(r) \backslash\{1, d\}$, for some $d \mid r$. Again by Corollary 2 (B), it remains to add one expression of the type (1) realizing the sum $a_{1} \mathrm{reg}_{1}+a_{d} \mathrm{reg}_{d}$.

We sum up our considerations in the following
Theorem 4.8. Let $f: M^{4} \rightarrow M^{4}$ be a smooth self-map of a smooth closed connected and simply-connected 4-manifold, $r=p_{1} \ldots p_{s}$ be a product of different prime numbers. We assume that the coefficients $a_{i}$ in the periodic expansion of $L\left(f^{k}\right)=\sum_{i=1}^{\infty} a_{i} \mathrm{reg}_{i}(k)$, are nonzero for all $i \mid r, i \neq 1$. Then

$$
D_{r}^{4}[f]= \begin{cases}h(s) & \text { if }(s \text { is odd }) \text { or }(r \text { is odd and } L(f)=h(s)) \\ & \text { or }(r \text { is even and } h(s)-2 \leq L(f) \leq h(s)) \\ h(s)+1 & \text { otherwise. }\end{cases}
$$

where $h(s)=\left(2^{s}+(-1)^{s+1}\right) / 3$.
Remark 3. If in Theorem 4.8 we drop the part (3) of the Standing Assumption according which $a_{i} \neq 0$ for all $i \neq 1$ dividing $r$ then the equality becomes the inequality and we get the estimation for $D_{r}^{4}[f]$ from above.

## REFERENCES

[1] I. K. Babenko and S. A. Bogatyi, The behavior of the index of periodic points under iterations of a mapping, Math. USSR Izv. 38 (1992), 1-26.
[2] A. Dold, Fixed point indices of iterated maps, Invent. Math. 74 (1983), 419-435.
[3] G. Graff, Minimal number of periodic points for smooth self-maps of two-holed 3-dimensional closed ball, Topol. Methods Nonlinear Anal. 33 (2009), no. 1, 121-130.
[4] G. Graff and J. Jezierski, Minimal number of periodic points for $C^{1}$ self-maps of compact simply-connected manifolds, Forum Math. 21 (2009), no. 3, 491-509.
[5] G. Graff and J. Jezierski, Minimal number of periodic points for smooth self-maps of $S^{3}$, Fund. Math. 204 (2009), 127-144.
[6] G. Graff and J. Jezierski, Minimizing the number of periodic points for smooth maps. Nonsimply connected case, Topology Appl. 158 (2011), no. 3, 276-290.
[7] G. Graff, J. Jezierski and M. Nowak-Przygodzki, Minimal number of periodic points for smooth self-maps of $\mathbb{R} P^{3}$, Topology Appl. 157 (2010), no. 10-11, 1784-1803.
8. G. Graff, J. Jezierski and P. Nowak-Przygodzki, Fixed point indices of iterated smooth maps in arbitrary dimension, J. Differential Equations (2011), doi:10.1016/j.jde.2011.05.024
[9] G. Graff and P. Nowak-Przygodzki, Fixed point indices of iterations of $C^{1}$ maps in $\mathbb{R}^{3}$, Discrete Cont. Dyn. Systems 16 (2006), no. 4, 843-856.
[10] E. Hart, P. Heath and E. Keppelmann, Algorithms for Nielsen type periodic numbers of maps with remnant on surfaces with boundary and on bouquets of circles. I., Fund. Math. 200 (2008), no. 2, 101-132.
[11] E. Hart and E. Keppelmann, Nielsen periodic point theory for periodic maps on orientable surfaces, Topology Appl. 153 (2006), no. 9, 1399-1420.
[12] J. Jezierski, Wecken's theorem for periodic points in dimension at least 3, Topology Appl. 153 (2006), no. 11, 1825-1837.
[13] J. Jezierski, Homotopy periodic sets of selfmaps of real projective spaces, Bol. Soc. Mat. Mexicana (3) 11 (2005), no. 2, 294-302.
[14] B. J. Jiang, "Lectures on the Nielsen Fixed Point Theory," Contemp. Math. 14, Amer. Math. Soc., Providence 1983.
[15] J. Jezierski and W. Marzantowicz, "Homotopy Methods in Topological Fixed and Periodic Points Theory," Topological Fixed Point Theory and Its Applications, 3. Springer, Dordrecht, 2005.
[16] H. J. Kim, J. B. Lee and W. S. Yoo, Computation of the Nielsen type numbers for maps on the Klein bottle, J. Korean Math. Soc. 45 (2008), no. 5, 1483-1503.
[17] P. Le Calvez, Dynamique des homéomorphismes du plan au voisinage d'un point fixe, Ann. Sci. École Norm. Sup. (4) 36, (2003), no. 1, 139-171.
[18] J. B. Lee and X. Zhao, Nielsen type numbers and homotopy minimal periods for maps on the 3-nilmanifolds, Sci. China Ser. A 51 (2008), no. 3, 351-360.
[19] J. B. Lee and X. Zhao, Nielsen type numbers and homotopy minimal periods for maps on 3-solvmanifolds, Algebr. Geom. Topol. 8 (2008), no. 1, 563-580.
[20] J. Llibre, J. Paranõs and J. A. Rodriguez, Periods for transversal maps on compact manifolds with a given homology, Houston J. Math. 24 (1998), no. 3, 397-407.
[21] F. Ruiz del Portal and J.M. Salazar, A Poincaré formula for the fixed point indices of the iterates of arbitrary planar homeomorphisms, Fixed Point Theory Appl., Volume 2010 (2010), Article ID 323069, 31 pages.
[22] G.Y. Zhang, Fixed point indices and periodic points of holomorphic mappings, Math. Ann. 337 (2007), no. 2, 401-433.

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