# Mountain pass type periodic solutions for Euler-Lagrange equations in anisotropic Orlicz-Sobolev space 

M. Chmara ${ }^{\text {a, },}$, J. Maksymiuk ${ }^{\text {a }}$<br>${ }^{a}$ Department of Technical Physics and Applied Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland


#### Abstract

Using the Mountain Pass Theorem, we establish the existence of periodic solution for Euler-Lagrange equation. Lagrangian consists of kinetic part (an anisotropic G-function), potential part $K-W$ and a forcing term. We consider two situations: $G$ satisfying $\Delta_{2} \cap \nabla 2$ at infinity and globally. We give conditions on the growth of the potential near zero for both situations.


Keywords: anisotropic Orlicz-Sobolev space, Euler-Lagrange equations, variational functional, Mountain Pass Theorem, Palais-Smale condition 2010 MSC: 46E40, 34C25, 37J45

## 1. Introduction

We consider the second order system

$$
\left\{\begin{array}{l}
\frac{d}{d t} L_{v}(t, u(t), \dot{u}(t))=L_{x}(t, u(t), \dot{u}(t)) \quad \text { for a.e. } t \in(-T, T)  \tag{ELT}\\
u(-T)=u(T)
\end{array}\right.
$$

where $L:[-T, T] \times \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ is given by

$$
L(t, x, v)=G(v)+V(t, x)+\langle f(t), x\rangle .
$$

We assume that $G$ is a differentiable G-function (in the sense of Trudinger [1]) and $V$ satisfies suitable growth conditions. If $G(v)=\frac{1}{p}|v|^{p}$ then the equation (ELT) reduces to p-laplacian. More general case is $G(v)=\phi(|v|)$, where $\phi$ is convex and nonnegative. In the above cases, $G$ depends on norm $|v|$ and its

[^0]growth is the same in all directions (isotropic). In this paper we consider the situation when the growth of $G$ is different in different directions (anisotropic) e.g. $G(x, y)=|x|^{p}+|y|^{q}$.

Existence of periodic solutions for the problem (ELT) was investigated in many papers, e.g.: [2] (anisotropic case), [3] (isotropic case), [4] ( $(p, q)$-laplacian), [5, 6] (p-laplacian), [7] (laplacian) and many others.

This paper is motivated by $[8,9,10]$, where the existence of homoclinic solution of $\frac{d}{d t} L_{v}(t, u(t), \dot{u}(t))=L_{x}(t, u(t), \dot{u}(t))$ is investigated (see also [11, 12]). In all these papers an intermediate step is to show, using the Mountain Pass Theorem, that corresponding periodic problem has a solution.

We want to adapt methods from [8] to anisotropic Orlicz-Sobolev space setting. It turns out, that the mountain pass geometry of action functional is strongly depended on Simonenko indices $p_{G}$ and $q_{G}$ (see section 2). To show that the action functional satisfies the Palais-Smale condition we need index $q_{G}^{\infty}$. Similar observation can be found in $[13,14,15,16]$ where the existence of solutions to elliptic systems via the Mountain Pass Theorem is considered. In [14] authors deal with an anisotropic problem. The isotropic case is considered in $[13,15,16]$.

We assume that:
$\left(A_{1}\right) G: \mathbb{R}^{N} \rightarrow[0, \infty)$ is a continuously differentiable G-function (i.e. G is convex, even, $G(0)=0$ and $G(x) /|x| \rightarrow \infty$, as $|x| \rightarrow \infty)$ satisfying $\Delta_{2}$ and $\nabla_{2}$ condition,
$\left(A_{2}\right) V(t, x)=K(t, x)-W(t, x)$, where $K, W \in \mathbf{C}^{1}\left([-T, T] \times \mathbb{R}^{N}, \mathbb{R}\right)$,
$\left(A_{3}\right)$ there exist $a \in \mathbf{L}^{1}([-T, T], \mathbb{R}), b>1$ and $\rho_{0}>0$ such that

$$
V(t, x) \geq b G(x)-a(t) \quad \text { for }|x| \leq \rho_{0}, t \in[-T, T]
$$

$\left(A_{4}\right)$ there exist $b_{1}>0$ and $p>1$ satisfying $|\cdot|^{p} \prec G$, such that

$$
\liminf _{|x| \rightarrow \infty} \frac{K(t, x)}{|x|^{p}} \geq b_{1} \text { uniformly in } t \in[-T, T]
$$

and

$$
\liminf _{|x| \rightarrow \infty} \frac{W(t, x)}{\max \{K(t, x), G(x)\}}>3 \text { uniformly in } t \in[-T, T]
$$

$\left(A_{5}\right)$ there exist $\nu \in \mathbb{R}, \mu>q_{G}^{\infty}+\nu$ and $\kappa \in \mathbf{L}^{1}([-T, T],[0, \infty))$ such that

$$
\left\langle V_{x}(t, x), x\right\rangle \leq\left(q_{G}^{\infty}+\nu\right) K(t, x)-\mu W(t, x)+\kappa(t) \quad \text { for }(t, x) \in[-T, T] \times \mathbb{R}^{N}
$$

$\left(A_{6}\right) \int_{-T}^{T} V(t, 0) d t=0$,
$\left(A_{7}\right) f \in \mathbf{L}^{G^{*}}\left([-T, T], \mathbb{R}^{N}\right)$.
Assumptions $\left(A_{3}\right),\left(A_{4}\right)$ and $\left(A_{5}\right)$ are essential for the Mountain Pass Theorem. We need $\left(A_{3}\right)$ to show that there exists $\alpha>0$ such that functional

$$
\begin{equation*}
\mathcal{J}(u)=\int_{-T}^{T} G(\dot{u})+V(t, u)+\langle f, u\rangle d t \tag{J}
\end{equation*}
$$

is greater than $\alpha$ on the boundary of some ball (see lemma 3.4). To do this we need to control behavior of V near zero.

Condition $\left(A_{4}\right)$ allows us to control the growth of $V$ at infinity. The first condition, together with $\left(A_{5}\right)$, is used to show that the Palais-Smale condition is satisfied. The latter condition is used to show that functional $\mathcal{J}$ is negative far from zero. Assumption $\left(A_{5}\right)$ is a modification of the well known AmbrosettiRabinowitz condition.

Let us denote by $C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}$ an embedding constant for $\mathbf{W}^{1} \mathbf{L}^{G} \hookrightarrow \mathbf{L}^{\infty}$ and define

$$
\rho:=\frac{\rho_{0}}{C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}}
$$

Now we can formulate our main theorems.
Theorem 1.1. Let $L:[-T, T] \times \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ satisfies $\left(A_{1}\right)-\left(A_{7}\right)$. Assume that $G$ satisfies $\Delta_{2}$ and $\nabla_{2}$ globally, and

$$
\int_{-T}^{T} G^{*}(f(t))+a(t) d t<\min \{1, b-1\} \begin{cases}(\rho / 2)^{q_{G}}, & \rho \leq 2  \tag{1}\\ (\rho / 2)^{p_{G}} & \rho>2\end{cases}
$$

Then (ELT) possesses a periodic solution.
The assumption that $G$ satisfies $\Delta_{2}$ and $\nabla_{2}$ globally can be relaxed if we assume that $\rho \geq 2$. In this case we need a stronger assumption on $f$ and $a$.

Theorem 1.2. Let $L:[-T, T] \times \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ satisfies $\left(A_{1}\right)-\left(A_{7}\right)$. Assume that $\rho \geq 2$ and

$$
\begin{equation*}
\int_{-T}^{T} G^{*}(f(t))+a(t) d t<\min \{1, b-1\}(\rho / 2) \tag{2}
\end{equation*}
$$

Then (ELT) possesses a periodic solution.
Theorem 1.1 generalizes Lemma 3.1 from [8]. Actually, assumption (1) has the same form as $\left(H_{5}\right)$ in [8], since in p-laplacian case $p_{G}=q_{G}=q_{G}^{\infty}=p$. Note that p-laplacian satisfies $\Delta_{2}$ and $\nabla_{2}$ globally. To the best of the authors knowledge there is no analogue of Theorem 1.2 in the literature.

Now we give two examples of potentials suitable for our setting.

Example 1.3. Consider the functions

$$
\begin{gathered}
G(x, y)=x^{2}+(x-y)^{4}, \quad K(t, x, y)=(2+\sin t) G(x, y)+\left|x^{2}+y^{2}\right|^{2} \cos ^{2} t \\
W(t, x, y)=\frac{\left|x^{2}+y^{2}\right|^{5 / 2}\left(e^{t^{2}\left(x^{2}+y^{2}-1\right)}-1\right)}{t^{2}+1}+\sin t .
\end{gathered}
$$

$G$ is differentiable $G$-function satisfying $\Delta_{2}$ and $\nabla_{2}$ globally. Here $V=K-W$ satisfies $\left(A_{2}\right)-\left(A_{6}\right)$, where $p_{G}=2, q_{G}^{\infty}=q_{G}=4, \mu=5, a(t)=\sin t, b=2$, $\kappa(t) \geq 5 \sin t$. On the other hand $K$ does not satisfy assumption $\left(H_{1}\right)$ and $W$ does not satisfy assumption $\left(H_{2}\right)$ from [8].

The next example shows that our results generalize Lemma 7 from [9].
Example 1.4. Set

$$
V(t, x)=c(t) G(x)-\lambda d(t) F(x),
$$

where $F$ is convex function satisfying $\Delta_{2}$ globally, $G \prec \prec F$, the functions $c(t)$, $d(t)$ are continuously differentiable, even on $\mathbb{R}, 0<c \leq c(t) \leq C, 0<d \leq d(t) \leq$ $D$, $t c^{\prime}(t)>0$ for $t \neq 0$ and $t c^{\prime}(t)<0$ for $t \neq 0$. Then $V$ satisfies conditions $\left(A_{2}\right)-\left(A_{7}\right)$.

Theorems 1.1 and 1.2 assert the existence of periodic solutions for

$$
\begin{aligned}
& \frac{d}{d t} \nabla G(\dot{u})-c(t) \nabla G(u)+\lambda d(t) \nabla F(u)=f(t), \\
& u(-T)=u(T)=0
\end{aligned}
$$

which is a generalization of the problem (2) from [9].

## 2. Some facts about G-functions and Orlicz-Sobolev spaces

Assume that $G: \mathbb{R}^{N} \rightarrow[0, \infty)$ satisfies assumption $\left(A_{1}\right)$. We say that

- $G$ satisfies the $\Delta_{2}$ condition if

$$
\begin{equation*}
\exists_{K_{1}>2} \exists_{M_{1} \geq 0} \forall_{|x| \geq M_{1}} G(2 x) \leq K_{1} G(x), \tag{2}
\end{equation*}
$$

- $G$ satisfies the $\nabla_{2}$ condition if

$$
\begin{equation*}
\exists_{K_{2}>1} \exists_{M_{2} \geq 0} \forall_{|x| \geq M_{2}} G(x) \leq \frac{1}{2 K_{2}} G\left(K_{2} x\right) . \tag{2}
\end{equation*}
$$

- $G$ satisfies $\Delta_{2}\left(\right.$ resp. $\left.\nabla_{2}\right)$ globally if $M_{1}=0\left(\right.$ resp. $\left.M_{2}=0\right)$.

Functions $G_{1}(x)=|x|^{p}, G_{2}(x)=|x|^{p_{1}}+|x|^{p_{2}}$ satisfy $\Delta_{2}$ and $\nabla_{2}$ globally. If $G$ does not satisfy $\Delta_{2}$ globally, then it could decrease very fast near zero. For example,

$$
G(x)= \begin{cases}|x|^{2} e^{-1 /|x|} & x \neq 0 \\ 0 & x=0\end{cases}
$$

satisfies $\Delta_{2}$ but does not satisfy $\Delta_{2}$ globally. For more details about $\Delta_{2}$ condition in case of N -function we refer the reader to [17].

Since $G$ is differentiable and convex,

$$
\begin{equation*}
G(x)-G(x-y) \leq\langle\nabla G(x), y\rangle \leq G(x+y)-G(x) \quad \text { for all } x, y \in \mathbb{R}^{N} \tag{3}
\end{equation*}
$$

A function $G^{*}(y)=\sup _{x \in \mathbb{R}^{N}}\{\langle x, y\rangle-G(x)\}$ is called the Fenchel conjugate of $G$. As an immediate consequence of the definition we have the Fenchel inequality:

$$
\forall_{x, y \in \mathbb{R}^{N}}\langle x, y\rangle \leq G(x)+G^{*}(y)
$$

Now we briefly recall the notion of anisotropic Orlicz space. For more details we refer the reader to [18] and [19]. The Orlicz space associated with $G$ is defined to be

$$
\mathbf{L}^{G}=\left\{u:[-T, T] \rightarrow \mathbb{R}^{N}: \int_{-T}^{T} G(u) d t<\infty\right\}
$$

The space $\mathbf{L}^{G}$ equipped with the Luxemburg norm

$$
\|u\|_{\mathbf{L}^{G}}=\inf \left\{\lambda>0: \int_{-T}^{T} G\left(\frac{u}{\lambda}\right) d t \leq 1\right\}
$$

is a reflexive Banach space. We have the Hölder inequality

$$
\int_{I}\langle u, v\rangle d t \leq 2\|u\|_{\mathbf{L}^{G}}\|v\|_{\mathbf{L}^{G^{*}}} \quad \text { for every } u \in \mathbf{L}^{G} \text { and } v \in \mathbf{L}^{G^{*}}
$$

Let us denote by

$$
\mathbf{W}^{1} \mathbf{L}^{G}:=\left\{u \in \mathbf{L}^{G}: \dot{u} \in \mathbf{L}^{G}\right\}
$$

an anisotropic Orlicz-Sobolev space of vector valued functions with norm

$$
\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}=\|u\|_{\mathbf{L}^{G}}+\|\dot{u}\|_{\mathbf{L}^{G}} .
$$

We introduce the following subset of $\mathbf{W}^{1} \mathbf{L}^{G}$

$$
\mathbf{W}_{T}^{1} \mathbf{L}^{G}:=\left\{u \in \mathbf{W}^{1} \mathbf{L}^{G}: u(-T)=u(T)\right\}
$$

We will also consider an equivalent norm given by

$$
\|u\|_{1, \mathbf{W}^{1} \mathbf{L}^{G}}=\inf \left\{\lambda>0: \int_{-T}^{T} G\left(\frac{u}{\lambda}\right)+G\left(\frac{\dot{u}}{\lambda}\right) d t \leq 1\right\}
$$

The following proposition will be crucial to Lemma 3.4.

## Proposition 2.1.

$$
\frac{1}{2}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}} \leq\|u\|_{1, \mathbf{W}^{1} \mathbf{L}^{G}} \leq 2\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}
$$

The proof for isotropic case can be found in [20, Proposition 9, p.177]. It remains the same for anisotropic case.

Functional $R_{G}(u):=\int_{-T}^{T} G(u) d t$ is called modular.
Proposition 2.2. [21, Proposition 2.7] $R_{G}(u)$ is coercive on $\mathbf{L}^{G}$ in the following sense:

$$
\lim _{\|u\|_{\mathbf{L}^{G} \rightarrow \infty}} \frac{R_{G}(u)}{\|u\|_{\mathbf{L}^{G}}}=\infty .
$$

Define the Simonenko indices for G-function

$$
p_{G}=\inf _{|x|>0} \frac{\langle x, \nabla G(x)\rangle}{G(x)}, \quad q_{G}=\sup _{|x|>0} \frac{\langle x, \nabla G(x)\rangle}{G(x)}, \quad q_{G}^{\infty}=\limsup _{|x| \rightarrow \infty} \frac{\langle x, \nabla G(x)\rangle}{G(x)} .
$$

It is obvious that $p_{G} \leq q_{G}^{\infty} \leq q_{G}$. Moreover, if $G$ satisfies $\Delta_{2}$ and $\nabla_{2}$ globally, then $1<p_{G}$ and $q_{G}<\infty$. The following results is crucial to Lemma 3.4.

Proposition 2.3. Let $G$ satisfies $\Delta_{2}$ and $\nabla_{2}$ globally.

1. If $\|u\|_{\mathbf{L}^{G}} \leq 1$, then $\|u\|_{\mathbf{L}^{G}}^{q_{G}} \leq R_{G}(u)$.
2. If $\|u\|_{\mathbf{L}^{G}}>1$, then $\|u\|_{\mathbf{L}^{G}}^{p_{G}} \leq R_{G}(u)$.

The proof can be found in appendix. More information about indices for isotropic case can be found in [22], [23] and [13]. For relations between Luxemburg norm and modular for anisotropic spaces we refer the reader to [19, Examples 3.8 and 3.9].

For, respectively, continuous and compact embeddings we will use the symbols $\hookrightarrow$ and $\hookrightarrow \hookrightarrow$.

Let $G_{1}$ and $G_{2}$ be G-functions. Define

$$
\begin{equation*}
G_{1} \prec G_{2} \Longleftrightarrow \exists_{M \geq 0} \exists_{K>0} \forall_{|x| \geq M} G_{1}(x) \leq G_{2}(K x) \tag{4}
\end{equation*}
$$

The relation $\prec$ allows to compare growth rate of functions $G_{1}$ and $G_{2}$.
It is well known that if $G_{1} \prec G_{2}$, then $\mathbf{L}^{G_{2}} \hookrightarrow \mathbf{L}^{G_{1}}$. Let $u \in \mathbf{W}^{1} \mathbf{L}^{G}$, $A_{G}: \mathbb{R}^{N} \rightarrow[0, \infty)$ be the greatest convex radial minorant of $G$ (see [2]). Then

$$
\|u\|_{\mathbf{L}^{\infty}} \leq C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}
$$

where $C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}=A_{G}^{-1}\left(\frac{1}{2 T}\right) \max \{1,2 T\}$.
The following proposition will be used in the proof of Lemma 3.2.

Proposition 2.4. (cf. [8]) For any $1<p \leq q<\infty$, such that $|\cdot|^{p} \prec G(\cdot) \prec|\cdot|^{q}$,

$$
\int_{-T}^{T}|u|^{p} d t \geq C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}^{p-q} C_{G, q}^{-q}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}^{p-q}\|u\|_{\mathbf{L}^{G}}^{q}
$$

for $u \in \mathbf{W}^{1} \mathbf{L}^{G} \backslash\{0\}$, where $C_{G, q}$ is an embedding constant from $\mathbf{L}^{q} \hookrightarrow \mathbf{L}^{G}$.
Proof. Let $u \in \mathbf{W}^{1} \mathbf{L}^{G} \backslash\{0\}$. Since $G \prec|\cdot|^{q}$,

$$
\int_{-T}^{T}|u|^{q} d t=\|u\|_{\mathbf{L}^{q}}^{q} \geq C_{G, q}^{-q}\|u\|_{\mathbf{L}^{G}}^{q}
$$

From Hölder's inequality and embedding $\mathbf{W}^{1} \mathbf{L}^{G} \hookrightarrow \mathbf{L}^{\infty}$ we obtain

$$
\begin{aligned}
& \int_{-T}^{T}|u|^{q} d t=\int_{-T}^{T}|u|^{p}|u|^{q-p}, d t \leq \\
& \leq\|u\|_{\mathbf{L}^{\infty}}^{q-p} \int_{-T}^{T}|u|^{p} d t \leq\left(C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}\right)^{q-p} \int_{-T}^{T}|u|^{p} d t
\end{aligned}
$$

## 3. Proof of the main results

Let $\mathcal{J}: \mathbf{W}_{T}^{1} \mathbf{L}^{G} \rightarrow \mathbb{R}$ be given by

$$
\begin{equation*}
\mathcal{J}(u)=\int_{-T}^{T} G(\dot{u})+K(t, u)-W(t, u)+\langle f, u\rangle d t \tag{J}
\end{equation*}
$$

From $\left(A_{1}\right),\left(A_{2}\right)$ and $\left[19\right.$, Thm. 5.5] we have $\mathcal{J} \in C^{1}$ and

$$
\mathcal{J}^{\prime}(u) \varphi=\int_{-T}^{T}\langle\nabla G(\dot{u}), \dot{\varphi}\rangle d t+\int_{-T}^{T}\left\langle V_{x}(t, u)+f(t), \varphi\right\rangle d t
$$

It is standard to prove that critical points of $\mathcal{J}$ are solutions of (ELT).
Our proof is based on the well-known Mountain Pass Theorem (see [24]).
Theorem 3.1. Let $X$ be a real Banach space and $I \in C^{1}(X, \mathbb{R})$ satisfies the following conditions:

1. $I(0)=0$,
2. I satisfies Palais-Smale condition,
3. there exist $\rho>0, e \in X$ such that $\|e\|_{X}>\rho$ and $I(e)<0$,
4. there exists $\alpha>0$ such that $\left.I\right|_{\partial B_{\rho}(0)} \geq \alpha$.

Then I possesses a critical value $c \geq \alpha$ given by

$$
c=\inf _{g \in \Gamma} \max _{s \in[0,1]} I(g(s)),
$$

where $\Gamma=\{g \in C([0,1], X) ; \quad g(0)=0, g(1)=e\}$.
We divide the proof into sequence of lemmas.
Lemma 3.2. $\mathcal{J}$ satisfies the Palais-Smale condition, i.e. every sequence $\left\{u_{n}\right\} \subset$ $\mathbf{W}_{T}^{1} \mathbf{L}^{G}$ such that $\left\{\mathcal{J}\left(u_{n}\right)\right\}$ is bounded and $\mathcal{J}^{\prime}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$ contains a convergent subsequence.
Proof. From $\left(A_{5}\right)$ and $\left(\mathcal{J}^{\prime}\right)$ we get

$$
\begin{align*}
\int_{-T}^{T} \mu W(t, u)- & \left(q_{G}^{\infty}+\nu\right) K(t, u) d t \leq \\
& \leq-\mathcal{J}^{\prime}(u) u+\int_{-T}^{T}\langle\nabla G(\dot{u}), \dot{u}\rangle d t+\int_{-T}^{T}\langle f(t), u\rangle+\kappa(t) d t . \tag{5}
\end{align*}
$$

From the definition of the functional we obtain

$$
\begin{aligned}
& \mu \int_{-T}^{T} G(\dot{u}) d t+\left(\mu-q_{G}^{\infty}-\nu\right) \int_{-T}^{T} K(t, u) d t= \\
& \quad=\mu \mathcal{J}(u)+\int_{-T}^{T} \mu W(t, u)-\left(q_{G}^{\infty}+\nu\right) K(t, u) d t-\mu \int_{-T}^{T}\langle f(t), u\rangle d t
\end{aligned}
$$

Applying (5), the Hölder inequality and $\left(A_{7}\right)$ we have

$$
\begin{align*}
& \mu \int_{-T}^{T} G(\dot{u}) d t+\left(\mu-q_{G}^{\infty}-\nu\right) \int_{-T}^{T} K(t, u) d t \leq \\
& \quad \leq \mu \mathcal{J}(u)-\mathcal{J}^{\prime}(u) u+\int_{-T}^{T}\langle\nabla G(\dot{u}), \dot{u}\rangle d t+C_{\kappa}+(1-\mu) C_{f}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}} \tag{6}
\end{align*}
$$

where $C_{\kappa}=\int_{-T}^{T} \kappa(t) d t$ and $C_{f}=2\|f\|_{\mathbf{L}^{G^{*}}}$. By the definition of $q_{G}^{\infty}$, there exists $M>0$ such that

$$
\langle x, \nabla G(x)\rangle \leq\left(q_{G}^{\infty}+\nu\right) G(x), \text { for }|x|>M .
$$

Hence

$$
\begin{align*}
& \int_{-T}^{T}\langle\nabla G(\dot{u}), \dot{u}\rangle d t=\int_{\{|\dot{u}|>M\}}\langle\nabla G(\dot{u}), \dot{u}\rangle d t+\int_{\{|\dot{u}| \leq M\}}\langle\nabla G(\dot{u}), \dot{u}\rangle d t \leq \\
& \quad \leq\left(q_{G}^{\infty}+\nu\right) \int_{\{|\dot{u}|>M\}} G(\dot{u}) d t+C_{\nabla G} \leq\left(q_{G}^{\infty}+\nu\right) \int_{-T}^{T} G(\dot{u}) d t+C_{\nabla G} \tag{7}
\end{align*}
$$

where $C_{\nabla_{G}}=\max { }_{|x| \leq M} 2 T M \nabla G(x)$. Applying (7) we can rewrite (6) as

$$
\begin{array}{rl}
\left(\mu-q_{G}^{\infty}-\nu\right) \int_{-T}^{T} & G(\dot{u})+K(t, u) d t \leq \\
& \leq \mu \mathcal{J}(u)-\mathcal{J}^{\prime}(u) u+C_{\kappa}+(1-\mu) C_{f}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}+C_{\nabla G} . \tag{8}
\end{array}
$$

From $\left(A_{4}\right)$, given any $0<\varepsilon_{1}<b_{1}$, there exists $\delta_{1} \geq 0$ such that

$$
K(t, x) \geq\left(b_{1}-\varepsilon_{1}\right)|x|^{p}-\delta_{1}, \quad \text { for } x \in \mathbb{R}^{N} \text { and } t \in[-T, T] .
$$

By Proposition 2.4 we obtain

$$
\begin{align*}
& \int_{-T}^{T} K(t, u) d t \geq \int_{-T}^{T}\left(b_{1}-\varepsilon_{1}\right)|u|^{p} d t-2 T \delta_{1} \geq \\
& \geq\left(b_{1}-\varepsilon_{1}\right) C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}^{p-q} C_{G, q}^{-q} \frac{\|u\|_{\mathbf{L}^{G}}^{q}}{\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}^{q-p}}-2 T \delta_{1}, \tag{9}
\end{align*}
$$

for any $q$ such that $G \prec|\cdot|^{q}$. Finally, applying (9) to (8) we obtain

$$
\begin{align*}
\left(\mu-q_{G}^{\infty}-\nu\right) & \left(\int_{-T}^{T} G(u) d t+\left(b_{1}-\varepsilon_{1}\right) C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}^{p-q} C_{G, q}^{-q} \frac{\|u\|_{\mathbf{L}^{G}}^{q}}{\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}^{q-p}}\right)+ \\
& -(\mu-1) C_{f}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}+\mathcal{J}^{\prime}(u) u \leq \mu \mathcal{J}(u)+C_{\kappa}+C_{\nabla_{G}}+C_{\delta_{1}}, \tag{10}
\end{align*}
$$

where $C_{\delta_{1}}=2 T \delta_{1}\left(\mu-q_{G}^{\infty}-\nu\right)$.
Let $\left\{u_{n}\right\} \subset \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ will be a Palais-Smale sequence for $\mathcal{J}$. There exist $C_{J}$, $C_{J^{\prime}}>0$ such that

$$
\mathcal{J}\left(u_{n}\right) \leq C_{J}, \quad \mathcal{J}^{\prime}\left(u_{n}\right) u_{n} \geq-C_{J^{\prime}}\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}
$$

Without loss of generality we can assume, that $\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}>0$. Substituting $u_{n}$ into (10) we obtain

$$
\begin{equation*}
\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}\left(\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}}+\frac{\left\|u_{n}\right\|_{\mathbf{L}^{G}}^{q}}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}^{+q-p}}-C^{\prime}\right) \leq C^{\prime \prime} \tag{11}
\end{equation*}
$$

where $C^{\prime}, C^{\prime \prime}>0$ are suitable constants independent of $n$.
We show that $\left\{u_{n}\right\}$ is bounded. On the contrary, suppose that there exists a subsequence of $u_{n}$ (still denoted $u_{n}$ ) such that $\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}} \rightarrow \infty$. Consider three cases.

1. Let $\left\|u_{n}\right\|_{\mathbf{L}^{G}} \rightarrow \infty$ and $\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}} \rightarrow \infty$ (again, w.l.o.g. $\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}>0$ ). From Proposition 2.2 we have that

$$
\begin{aligned}
\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}} & +\frac{\left\|u_{n}\right\|_{\mathbf{L}^{G}}^{q}}{\left\|u_{n}\right\|_{\mathbf{W}^{1}+\mathbf{L}^{G}}^{1+q-p}}= \\
& =\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}} \frac{\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}}+\left(\frac{\left\|u_{n}\right\|_{\mathbf{L}^{G}}}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}}\right)^{1+q-p}\left\|u_{n}\right\|_{\mathbf{L}^{G}}^{p-1} \rightarrow \infty .
\end{aligned}
$$

2. Let $\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}} \rightarrow \infty$ and $\left\|u_{n}\right\|_{\mathbf{L}^{G}}$ is bounded. Then

$$
\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}}=\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|u_{n}\right\|_{\mathbf{L}^{G}}+\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}}=\frac{\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}}}{\frac{\left\|u_{n}\right\|_{\mathbf{L}^{G}}}{\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}}+1} \rightarrow \infty \text { as }\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}} \rightarrow \infty .
$$

3. Let $\left\|u_{n}\right\|_{\mathbf{L}^{G}} \rightarrow \infty$ and $\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}$ is bounded. Since $p>1$, we have

$$
\frac{R_{G}\left(\dot{u}_{n}\right)}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}}+\frac{\left\|u_{n}\right\|_{\mathbf{L}^{G}}^{q}}{\left\|u_{n}\right\|_{\mathbf{W}^{1} \mathbf{L}^{G}}^{1+p}} \geq \frac{\left\|u_{n}\right\|_{\mathbf{L}^{G}}^{q}}{\left(\left\|u_{n}\right\|_{\mathbf{L}^{G}}+\left\|\dot{u}_{n}\right\|_{\mathbf{L}^{G}}\right)^{1+q-p}} \rightarrow \infty .
$$

Therefore, in view of (11), $\left\{u_{n}\right\}$ is bounded in $\mathbf{W}_{T}^{1} \mathbf{L}^{G}$.
It follows from reflexivity of $\mathbf{W}_{T}^{1} \mathbf{L}^{G}$ and embeddings $\mathbf{W}_{T}^{1} \mathbf{L}^{G} \hookrightarrow \hookrightarrow \mathbf{L}^{G}$, $\mathbf{W}_{T}^{1} \mathbf{L}^{G} \hookrightarrow \hookrightarrow \mathbf{W}^{1,1}$ that there exists $u \in \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ and a subsequence of $u_{n}$ (still denoted $u_{n}$ ) such that $u_{n} \rightarrow u$ in $\mathbf{L}^{G}$. Moreover, $\dot{u}_{n} \rightarrow \dot{u}$ in $\mathbf{L}^{1}$ and hence pointwise a.e.

Since $\left\{u_{n}\right\}$ is a Palais-Smale sequence, we have
$0 \leftarrow \mathcal{J}^{\prime}\left(u_{n}\right)\left(u_{n}-u\right)=\int_{-T}^{T}\left\langle\nabla G\left(\dot{u}_{n}\right), \dot{u}_{n}-\dot{u}\right\rangle d t+\int_{-T}^{T}\left\langle V_{x}\left(t, u_{n}\right)+f(t), u_{n}-u\right\rangle d t$.
Since $\int_{-T}^{T}\left\langle V_{x}\left(t, u_{n}\right)+f(t), u_{n}-u\right\rangle d t \rightarrow 0$ we can deduce that

$$
\int_{-T}^{T}\left\langle\nabla G\left(\dot{u}_{n}\right), \dot{u}_{n}-\dot{u}\right\rangle d t \rightarrow 0 .
$$

From (3) we obtain

$$
\int_{-T}^{T} G\left(\dot{u}_{n}\right) d t \leq \int_{-T}^{T} G(\dot{u}) d t+\int_{-T}^{T}\left\langle\nabla G\left(\dot{u}_{n}\right),\left(\dot{u}_{n}-\dot{u}\right)\right\rangle d t
$$

Hence

$$
\limsup _{n \rightarrow+\infty} \int_{-T}^{T} G\left(\dot{u}_{n}\right) d t \leq \int_{-T}^{T} G(\dot{u}) d t .
$$

On the other hand, by Fatou's Theorem we have

$$
\liminf _{n \rightarrow+\infty} \int_{-T}^{T} G\left(\dot{u}_{n}\right) d t \geq \int_{-T}^{T} G(\dot{u}) d t .
$$

Combining these inequalities we get that

$$
\int_{-T}^{T} G\left(\dot{u}_{n}\right) d t \rightarrow \int_{-T}^{T} G(\dot{u}) d t .
$$

Therefore $\dot{u}_{n} \rightarrow \dot{u}$ in $\mathbf{L}^{G}$ by [19, Lemma 3.16].
We next prove that $\mathcal{J}$ is negative for some point outside $B_{\rho}(0)$.
Lemma 3.3. There exist $e \in \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ such that $\|e\|_{\mathbf{W}^{1} \mathbf{L}^{G}}>\rho$ and $\mathcal{J}(e)<0$.
Proof. By assumption $\left(A_{4}\right)$, there exist $\varepsilon_{0}, r>0$ such that

$$
W(t, x) \geq\left(3+\varepsilon_{0}\right) \max \{K(t, x), G(x)\} \quad \text { for }|x|>r \text {. }
$$

This gives

$$
\begin{equation*}
K(t, x)-W(t, x) \leq-\left(2+\varepsilon_{0}\right) G(x) \quad \text { for }|x|>r . \tag{12}
\end{equation*}
$$

Fix $v \in \mathbb{R}^{N}$. For $\xi>T+1$ define $e:[-T, T] \rightarrow \mathbb{R}^{N}$ by

$$
e(t)=\xi\left(1-\frac{|t|}{T+1}\right) v .
$$

Direct computation shows

$$
\dot{e}(t)= \begin{cases}-\frac{\xi}{T+1} v, & t \in(0, T], \\ \frac{\xi}{T+1} v, & t \in[-T, 0) .\end{cases}
$$

Since $\|e\|_{\mathbf{L}^{\infty}}=\xi>T+1$ and $\|\dot{e}\|_{\mathbf{L}^{\infty}}=\xi /(T+1)>1$, we can choose $\xi$ such that both (12) and $\|e\|_{\mathbf{W}^{1} \mathbf{L}^{G}} \geq \rho$ hold. From (J), the Fenchel inequality and (12) we have

$$
\begin{aligned}
\mathcal{J}(e) \leq \int_{-T}^{T} G(\dot{e})+K(t, e)-W(t, e) & +G(e)+G^{*}(f) d t \leq \\
\leq & \int_{-T}^{T} G(\dot{e})-G(e)-\varepsilon_{0} G(e)+G^{*}(f) d t .
\end{aligned}
$$

Since $1-\frac{|t|}{T+1} \geq \frac{1}{T+1}$ for $t \in[-T, T]$, we have

$$
\int_{-T}^{T} G(\dot{e})-G(e) d t=\int_{-T}^{T} G\left(\frac{\xi}{T+1} v\right)-G\left(\xi\left(1-\frac{|t|}{T+1}\right) v\right) d t \leq 0 .
$$

Choosing $\xi$ large enough we get

$$
\mathcal{J}(e) \leq \int_{-T}^{T}-\varepsilon_{0} G(e)+G^{*}(f) d t<0
$$

In order to show that $\mathcal{J}$ satisfies the fourth assumption of the Mountain Pass Theorem, we first provide some estimates for $R_{G}(\dot{u})+R_{G}(u)$ on $\partial B_{\rho}(0)$.

If $\Delta_{2}$ and $\nabla_{2}$ are satisfied globally then we can use Proposition 2.3 to estimate $R_{G}(\dot{u})+R_{G}(u)$ from below by $(\rho / 2)^{r}, r>1$, for any $\rho>0$. If $G$ does not satisfies $\Delta_{2}$ and $\nabla_{2}$ globally then we cannot use Proposition 2.3 (for explanation see Remark A.2). In this case we use equivalent norm and Proposition 2.1 but we obtain only that $R_{G}(\dot{u})+R_{G}(u) \geq \rho / 2$. Moreover, we are forced to assume $\rho>2$.

Let $u \in \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ be such that $\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}=\rho$. Set $\rho_{1}=\|u\|_{\mathbf{L}^{G}}, \rho_{2}=\|\dot{u}\|_{\mathbf{L}^{G}}$, $\rho_{1}+\rho_{2}=\rho$. Assuming that $G$ satisfies $\Delta_{2}$ an $\nabla_{2}$ globally we get, by Proposition 2.3, the following estimates:

1. If $\rho_{1}, \rho_{2} \leq 1$ then $R_{G}(\dot{u})+R_{G}(u) \geq\|\dot{u}\|_{\mathbf{L}^{G}}^{q_{G}}+\|u\|_{\mathbf{L}^{G}}^{q_{G}}$. Hence

$$
\begin{equation*}
R_{G}(\dot{u})+R_{G}(u) \geq 2^{1-q_{G}}\left(\|\dot{u}\|_{\mathbf{L}^{G}}+\|u\|_{\mathbf{L}^{G}}\right)^{q_{G}} \geq(\rho / 2)^{q_{G}} \tag{13}
\end{equation*}
$$

since $\rho_{1}{ }^{q_{G}}+\rho_{2}^{q_{G}} \geq 2^{1-q_{G}}\left(\rho_{1}+\rho_{2}\right)^{q_{G}}$.
2. If $\rho_{1} \leq 1, \rho_{2} \geq 1$ then $\left(\rho_{1}+\rho_{2}\right)^{p_{G}} \leq\left(2 \rho_{2}\right)^{p_{G}} \leq 2^{p_{G}}\left(\rho_{1}^{q_{G}}+\rho_{2}^{p_{G}}\right)$. Hence

$$
\begin{equation*}
R_{G}(\dot{u})+R_{G}(u) \geq\|\dot{u}\|_{\mathbf{L}^{G}}^{p_{G}}+\|u\|_{\mathbf{L}^{G}}^{q_{G}} \geq(\rho / 2)^{p_{G}} . \tag{14}
\end{equation*}
$$

3. If $\rho_{1} \geq 1, \rho_{2} \leq 1$ then $\left(\rho_{1}+\rho_{2}\right)^{p_{G}} \leq\left(2 \rho_{1}\right)^{p_{G}} \leq 2^{p_{G}}\left(\rho_{1}^{p_{G}}+\rho_{2}^{q_{G}}\right)$. Thus

$$
\begin{equation*}
R_{G}(\dot{u})+R_{G}(u) \geq\|\dot{u}\|_{\mathbf{L}^{G}}^{q_{G}}+\|u\|_{\mathbf{L}^{G}}^{p_{G}} \geq(\rho / 2)^{p_{G}} . \tag{15}
\end{equation*}
$$

4. If $\rho_{1}, \rho_{2} \geq 1$ then

$$
\begin{equation*}
R_{G}(\dot{u})+R_{G}(u) \geq\|\dot{u}\|_{\mathbf{L}^{G}}^{p_{G}}+\|u\|_{\mathbf{L}^{G}}^{p_{G}} \geq(\rho / 2)^{p_{G}} . \tag{16}
\end{equation*}
$$

From the other hand, Proposition 2.1 implies

$$
\inf \left\{\lambda>0: \int_{-T}^{T} G\left(\frac{u}{\lambda}\right)+G\left(\frac{\dot{u}}{\lambda}\right) d t \leq 1\right\} \geq \frac{1}{2} \rho .
$$

Therefore

$$
\int_{-T}^{T} G\left(\frac{2 u}{\rho}\right)+G\left(\frac{2 \dot{u}}{\rho}\right) d t \geq 1
$$

and consequently,

$$
\begin{equation*}
R_{G}(u)+R_{G}(\dot{u}) \geq \frac{\rho}{2} \tag{17}
\end{equation*}
$$

provided $\rho>2$.

Lemma 3.4. Assume that either (1) or (2) holds. There exists positive constant $\alpha$ such that $\left.\mathcal{J}\right|_{\partial B_{\rho}(0)} \geq \alpha$.

Proof. From the definition of $\rho$ and embedding $\mathbf{W}_{T}^{1} \mathbf{L}^{G} \hookrightarrow \mathbf{L}^{\infty}$ we have

$$
|u(t)| \leq\|u\|_{\mathbf{L}^{\infty}} \leq C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G}}\|u\|_{\mathbf{W}^{1} \mathbf{L}^{G}}=C_{\infty, \mathbf{W}^{1} \mathbf{L}^{G} \rho}=\rho_{0} \quad \text { for } t \in[-T, T] .
$$

From $\left(A_{3}\right)$ and the Fenchel inequality we obtain

$$
\begin{aligned}
\mathcal{J}(u) \geq \int_{-T}^{T} G(\dot{u})+ & b G(u)-a(t)+\langle f, u\rangle d t \geq \\
& \geq \min \{1, b-1\}\left(R_{G}(\dot{u})+R_{G}(u)\right)-R_{G^{*}}(f)-\int_{-T}^{T} a(t) d t
\end{aligned}
$$

Assume that (1) holds. If $\rho \leq 2$ then (13), (14) and (15) yields

$$
\mathcal{J}(u) \geq \min \{1, b-1\}(\rho / 2)^{q_{G}}-R_{G^{*}}(f)-\int_{-T}^{T} a(t) d t=: \alpha
$$

If $\rho>2$, then by (14), (15) and (16) we get

$$
\mathcal{J}(u) \geq \min \{1, b-1\}(\rho / 2)^{p_{G}}-R_{G^{*}}(f)-\int_{-T}^{T} a(t) d t>0=: \alpha .
$$

From (1) it follows that in both cases $\alpha>0$.
Assume that (2) holds. From (17) we obtain

$$
\begin{aligned}
& \mathcal{J}(u) \geq \min \{1, b-1\}\left(R_{G}(\dot{u})+R_{G}(u)\right)-R_{G^{*}}(f)-\int_{-T}^{T} a(t) \geq \\
& \geq \min \{1, b-1\}(\rho / 2)-R_{G^{*}}(f)-\int_{-T}^{T} a(t) d t=: \alpha .
\end{aligned}
$$

From (2) we have $\alpha>0$.
Now we are in position to prove our main theorems. Note that by $\left(A_{6}\right)$ and $G(0)=0$ we have $\mathcal{J}(0)=0$. From Lemmas 3.2, 3.3 and 3.4 we have that $\mathcal{J}$ satisfies all assumptions of the Mountain Pass Theorem. Hence there exists a critical point $u \in \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ of $\mathcal{J}$ and (ELT) have periodic solution.

Actually, we can show that any solution to (ELT) is more regular (cf. Corollary 16.16 in [25]).
Proposition 3.5. If $u \in \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ is a solution of (ELT), then $u \in \mathbf{W}^{1} \mathbf{L}^{\infty}$.

Proof. Let $u \in \mathbf{W}^{1} \mathbf{L}^{G}$ be a solution of (ELT). Then

$$
\nabla G(\dot{u}(t))=\int_{-T}^{t} \nabla V(t, u(t)) d t+C
$$

and there exists $M>0$ such that $|\nabla G(\dot{u}(t))| \leq M<\infty$. From the other hand

$$
G(\dot{u}(t)) \leq\langle\nabla G(\dot{u}(t)), \dot{u}(t)\rangle \leq M|\dot{u}(t)| .
$$

Since $\frac{G(v)}{|v|} \rightarrow \infty$ as $|v| \rightarrow \infty$, we obtain $|\dot{u}(t)|$ is bounded.
Remark 3.6. If $G$ is strictly convex then one can show that if $u \in \mathbf{W}_{T}^{1} \mathbf{L}^{G}$ is a solution of (ELT), then $u \in \mathbf{C}^{1}$.

Remark 3.7. Theorem 1.1 remains true if we change assumption (1) to

$$
\int_{-T}^{T} G^{*}(f(t))+a(t) d t<\min \{1, b-1\} \begin{cases}2(\rho / 2)^{q_{G}}, & \rho \leq 2^{1-1 /\left(q_{G}-p_{G}\right)}  \tag{18}\\ (\rho / 2)^{p_{G}} & \rho>2^{1-1 /\left(q_{G}-p_{G}\right)}\end{cases}
$$

Estimate in the first case is better than (1) but it is taken on smaller set. In the second case estimate is the same as in (1) but can be taken on bigger set.

Remark 3.8. In the proof of Lemma 3.4 we can use the Hölder inequality instead of the Fenchel inequality to estimate $\int_{-T}^{T}\langle f, u\rangle d t$. It allows us to take $b>0$ if $\rho \leq 1$.

## Appendix A.

Assume that $G$ satisfies $\nabla_{2}$ globally. It is easy to show that $G^{*}$ satisfies $\Delta_{2}$ globally with $K_{1}^{*}=2 K_{2}$.

Since $G \in \mathbf{C}^{1}$ and is convex, we have

$$
K_{1} G(x) \geq G(2 x) \geq G(2 x)-G(x) \geq\langle x, \nabla G(x)\rangle \quad \text { for all } x \in \mathbb{R}^{N} .
$$

Let $y \in \mathbb{R}^{N}$ and $s \in \partial G^{*}(y)$, where $\partial G^{*}$ denotes the subdifferential of $G^{*}$. Since $G^{*}$ satisfies $\Delta_{2}$ globally, we have

$$
K_{1}^{*} G^{*}(y) \geq G^{*}(2 y) \geq G^{*}(2 y)-G^{*}(y) \geq\langle s, y\rangle \quad \text { for all } y \in \mathbb{R}^{N}
$$

Let $x \in \mathbb{R}^{N}$. Then $x \in \partial G^{*}(\nabla G(x))$ and $G(x)+G^{*}(\nabla G(x))=\langle x, \nabla G(x)\rangle$. It follows that

$$
\begin{gathered}
G(x)=\langle x, \nabla G(x)\rangle-G^{*}(\nabla G(x)) \leq\left(1-\frac{1}{K_{1}^{*}}\right)\langle x, \nabla G(x)\rangle . . . ~
\end{gathered}
$$

Finally

$$
\begin{equation*}
\frac{2 K_{2}}{2 K_{2}-1} \leq \frac{\langle x, \nabla G(x)\rangle}{G(x)} \leq K_{1} \quad \text { for all } x \in \mathbb{R}^{N} \tag{A.1}
\end{equation*}
$$

Since $K_{2}>1$, we have $\frac{2 K_{2}}{2 K_{2}-1}>1$ and from (A.1) we obtain

$$
p_{G}>1 \quad \text { and } \quad q_{G}<\infty
$$

For any $x \in \mathbb{R}^{N}$ and $\lambda \geq 1$ we have

$$
\log G(\lambda x)-\log G(x)=\int_{1}^{\lambda} \frac{\langle\nabla G(\lambda x), x\rangle}{G(\lambda x)} d \lambda \leq \int_{1}^{\lambda} \frac{q_{G}}{\lambda} d \lambda=\log \lambda^{q_{G}}
$$

Thus

$$
\begin{equation*}
G(\lambda x) \leq \lambda^{q_{G}} G(x) \quad \text { for all } x \in \mathbb{R}^{N}, \lambda \geq 1 \tag{A.2}
\end{equation*}
$$

Similarly, we get

$$
\begin{equation*}
G(\lambda x) \geq \lambda^{p_{G}} G(x) \quad \text { for all } x \in \mathbb{R}^{N}, \lambda \geq 1 \tag{A.3}
\end{equation*}
$$

Lemma A.1. Let $u \in \mathbf{L}^{G}$.

1. If $\|u\|_{\mathbf{L}^{G}}<1$ then $R_{G}(u) \geq\|u\|_{\mathbf{L}^{G}}^{q_{G}}$.
2. If $\|u\|_{\mathbf{L}^{G}}>1$ then $R_{G}(u) \geq\|u\|_{\mathbf{L}^{G}}^{p_{G}}$.

Proof. For any $0<\beta<\|u\|_{\mathbf{L}^{G}}<1$ we have $R_{G}\left(\frac{u}{\beta}\right) \geq 1$. From (A.2) we obtain that $G\left(\frac{x}{\beta}\right) \leq\left(\frac{1}{\beta}\right)^{q_{G}} G(x)$ for all $x \in \mathbb{R}^{N}$. Hence

$$
R_{G}(u) \geq \beta^{q_{G}} R_{G}\left(\frac{u}{\beta}\right) \geq \beta^{q_{G}}
$$

Letting $\beta \uparrow\|u\|_{\mathbf{L}^{G}}$ gives $R_{G}(u) \geq\|u\|_{\mathbf{L}^{G}}^{q_{G}}$.
For any $1<\beta<\|u\|_{\mathbf{L}^{G}}$ we have $R_{G}\left(\frac{u}{\beta}\right)>1$. Then from (A.3) we obtain that $G(x) \geq \beta^{p_{G}} G\left(\frac{x}{\beta}\right)$ for all $x \in \mathbb{R}^{N}$. Hence

$$
R_{G}(u) \geq \beta^{p_{G}} R_{G}\left(\frac{u}{\beta}\right) \geq \beta^{p_{G}}
$$

Letting $\beta \uparrow\|u\|_{\mathbf{L}^{G}}$ gives $R_{G}(u) \geq\|u\|_{\mathbf{L}^{G}}^{p_{G}}$.
Remark A.2. If $G$ satisfies $\Delta_{2}$ and $\nabla_{2}$ (not globally), estimates similar to (A.2) and (A.3) can be obtained for sufficiently large $|x|$. However, even if $\|u\|_{\mathbf{L}^{G}}$ is large it does not necessarily mean that $|u(t)|$ is large. Hence we cannot use these estimates to obtain result similar to Lemma A.1.

## References

[1] N. S. Trudinger, An imbedding theorem for $H_{0}(G, \Omega)$ spaces, Studia Math. 50 (1974) 17-30.
[2] S. Acinas, F. Mazzone, Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, preprint on ArXiv at https://arxiv.org/abs/1708.06657.
[3] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, E. Schwindt, Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, Nonlinear Anal. 125 (2015) 681-698.
[4] D. Paşca, Z. Wang, On periodic solutions of nonautonomous second order Hamiltonian systems with $(q, p)$-Laplacian, Electron. J. Qual. Theory Differ. Equ. (2016) Paper No. 106, 9.
[5] B. Xu, C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems, J. Math. Anal. Appl. 333 (2) (2007) 1228-1236.
[6] S. Ma, Y. Zhang, Existence of infinitely many periodic solutions for ordinary p-Laplacian systems, J. Math. Anal. Appl. 351 (1) (2009) 469-479.
[7] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989.
[8] A. Daouas, Existence of homoclinic orbits for unbounded time-dependent $p$-Laplacian systems, Electron. J. Qual. Theory Differ. Equ. (2016) Paper No. 88, 12.
[9] S. Tersian, On symmetric positive homoclinic solutions of semilinear p-laplacian differential equations, Bound Value Probl 2012 (1) (2012) 121.
[10] M. Izydorek, J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems, J. Diff. Eq. 219 (2) (2005) 375-389.
[11] V. Coti Zelati, P. H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, J. Amer. Math. Soc. 4 (4) (1991) 693-727.
[12] X. Lv, S. Lu, Homoclinic solutions for ordinary p-laplacian systems., Appl Math Comput. 218 (9) (2012) 5682-5692.
[13] P. Clément, B. Pagter, G. Sweers, F. Thélin, Existence of solutions to a semilinear elliptic system through orlicz-sobolev spaces, Mediterr. J. Math. 1 (3) (2004) 241-267.
[14] G. Barletta, A. Cianchi, Dirichlet problems for fully anisotropic elliptic equations, Proc. Royal Soc. Ed. 147 (1) (2017) 2560.
[15] P. Clément, M. Garcí a Huidobro, R. Manásevich, K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, Calc. Var. Partial Differential Equations 11 (1) (2000) 3362.
[16] V. K. Le, Nontrivial solutions of mountain pass type of quasilinear equations with slowly growing principal parts, Differential Integral Equations 15 (7) (2002) 839-862.
[17] M. A. Krasnoselskiŭ, J. B. Rutickiĭ, Convex functions and Orlicz spaces, P. Noordhoff Ltd., Groningen, 1961.
[18] G. Schappacher, A notion of Orlicz spaces for vector valued functions, Appl. Math. 50 (4) (2005) 355-386.
[19] M. Chmara, J. Maksymiuk, Anisotropic Orlicz-Sobolev spaces of vector valued functions and Lagrange equations, J. Math. Anal. Appl. 456 (1) (2017) 457-475.
[20] V. D. Radulescu, D. Repovs, Partial Differential Equations with Variable Exponents Variational Methods and Qualitative Analysis, Chapman and Hall/CRC, 2015.
[21] V. K. Le, On second order elliptic equations and variational inequalities with anisotropic principal operators, Topol. Methods Nonlinear Anal. 44 (1) (2014) 41-72.
[22] I. B. Simonenko, Interpolation and extrapolation of linear operators in Orlicz spaces, Mat. Sb. (N.S.) 63 (105) (1964) 536-553.
[23] L. Maligranda, Orlicz spaces and interpolation, Vol. 5 of Seminários de Matemática [Seminars in Mathematics], 1989.
[24] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (4) (1973) $349-381$.
[25] F. Clarke, Functional analysis, calculus of variations and optimal control, Vol. 264, Springer, London, 2013.


[^0]:    * corresponding author

    Email addresses: magdalena.chmara@pg.edu.pl (M. Chmara), jakub.maksymiuk@pg.edu.pl (J. Maksymiuk)

