

Mountain pass type periodic solutions for Euler-Lagrange equations in anisotropic Orlicz-Sobolev space

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Abstract

Using the Mountain Pass Theorem, we establish the existence of periodic solution for Euler-Lagrange equation. Lagrangian consists of kinetic part (an anisotropic G-function), potential part $K - W$ and a forcing term. We consider two situations: G satisfying $\Delta_2 \cap \nabla_2$ at infinity and globally. We give conditions on the growth of the potential near zero for both situations.

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1. Introduction

We consider the second order system

$$\begin{cases} \frac{d}{dt}L_v(t, u(t), \dot{u}(t)) = L_x(t, u(t), \dot{u}(t)) & \text{for a.e. } t \in (-T, T) \\ u(-T) = u(T) \end{cases} \quad (\text{ELT})$$

where $L: [-T, T] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is given by

$$L(t, x, v) = G(v) + V(t, x) + \langle f(t), x \rangle.$$

We assume that G is a differentiable G-function (in the sense of Trudinger [1]) and V satisfies suitable growth conditions. If $G(v) = \frac{1}{p}|v|^p$ then the equation (ELT) reduces to p-laplacian. More general case is $G(v) = \phi(|v|)$, where ϕ is convex and nonnegative. In the above cases, G depends on norm $|v|$ and its

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growth is the same in all directions (isotropic). In this paper we consider the situation when the growth of G is different in different directions (anisotropic) e.g. $G(x, y) = |x|^p + |y|^q$.

Existence of periodic solutions for the problem (ELT) was investigated in many papers, e.g.: [2] (anisotropic case), [3] (isotropic case), [4] $((p, q)$ -laplacian), [5, 6] (p) -laplacian, [7] (laplacian) and many others.

This paper is motivated by [8, 9, 10], where the existence of homoclinic solution of $\frac{d}{dt}L_v(t, u(t), \dot{u}(t)) = L_x(t, u(t), \dot{u}(t))$ is investigated (see also [11, 12]). In all these papers an intermediate step is to show, using the Mountain Pass Theorem, that corresponding periodic problem has a solution.

We want to adapt methods from [8] to anisotropic Orlicz-Sobolev space setting. It turns out, that the mountain pass geometry of action functional is strongly depended on Simonenko indices p_G and q_G (see section 2). To show that the action functional satisfies the Palais-Smale condition we need index q_G^∞ . Similar observation can be found in [13, 14, 15, 16] where the existence of solutions to elliptic systems via the Mountain Pass Theorem is considered. In [14] authors deal with an anisotropic problem. The isotropic case is considered in [13, 15, 16].

We assume that:

(A₁) $G: \mathbb{R}^N \rightarrow [0, \infty)$ is a continuously differentiable G -function (i.e. G is convex, even, $G(0) = 0$ and $G(x)/|x| \rightarrow \infty$, as $|x| \rightarrow \infty$) satisfying Δ_2 and ∇_2 condition,

(A₂) $V(t, x) = K(t, x) - W(t, x)$, where $K, W \in \mathbf{C}^1([-T, T] \times \mathbb{R}^N, \mathbb{R})$,

(A₃) there exist $a \in \mathbf{L}^1([-T, T], \mathbb{R})$, $b > 1$ and $\rho_0 > 0$ such that

$$V(t, x) \geq bG(x) - a(t) \quad \text{for } |x| \leq \rho_0, t \in [-T, T],$$

(A₄) there exist $b_1 > 0$ and $p > 1$ satisfying $|\cdot|^p \prec G$, such that

$$\liminf_{|x| \rightarrow \infty} \frac{K(t, x)}{|x|^p} \geq b_1 \text{ uniformly in } t \in [-T, T]$$

and

$$\liminf_{|x| \rightarrow \infty} \frac{W(t, x)}{\max\{K(t, x), G(x)\}} > 3 \text{ uniformly in } t \in [-T, T],$$

(A₅) there exist $\nu \in \mathbb{R}$, $\mu > q_G^\infty + \nu$ and $\kappa \in \mathbf{L}^1([-T, T], [0, \infty))$ such that

$$\langle V_x(t, x), x \rangle \leq (q_G^\infty + \nu)K(t, x) - \mu W(t, x) + \kappa(t) \quad \text{for } (t, x) \in [-T, T] \times \mathbb{R}^N,$$

(A₆) $\int_{-T}^T V(t, 0) dt = 0$,

(A₇) $f \in \mathbf{L}^{G^*}([-T, T], \mathbb{R}^N)$.

Assumptions (A₃), (A₄) and (A₅) are essential for the Mountain Pass Theorem. We need (A₃) to show that there exists $\alpha > 0$ such that functional

$$\mathcal{J}(u) = \int_{-T}^T G(\dot{u}) + V(t, u) + \langle f, u \rangle dt \quad (\mathcal{J})$$

is greater than α on the boundary of some ball (see lemma 3.4). To do this we need to control behavior of V near zero.

Condition (A₄) allows us to control the growth of V at infinity. The first condition, together with (A₅), is used to show that the Palais-Smale condition is satisfied. The latter condition is used to show that functional \mathcal{J} is negative far from zero. Assumption (A₅) is a modification of the well known Ambrosetti-Rabinowitz condition.

Let us denote by $C_{\infty, \mathbf{W}^1 \mathbf{L}^G}$ an embedding constant for $\mathbf{W}^1 \mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ and define

$$\rho := \frac{\rho_0}{C_{\infty, \mathbf{W}^1 \mathbf{L}^G}}. \quad (\rho)$$

Now we can formulate our main theorems.

Theorem 1.1. *Let $L: [-T, T] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies (A₁)–(A₇). Assume that G satisfies Δ_2 and ∇_2 globally, and*

$$\int_{-T}^T G^*(f(t)) + a(t) dt < \min\{1, b-1\} \begin{cases} (\rho/2)^{q_G}, & \rho \leq 2 \\ (\rho/2)^{p_G}, & \rho > 2 \end{cases} \quad (1)$$

Then (ELT) possesses a periodic solution.

The assumption that G satisfies Δ_2 and ∇_2 globally can be relaxed if we assume that $\rho \geq 2$. In this case we need a stronger assumption on f and a .

Theorem 1.2. *Let $L: [-T, T] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies (A₁)–(A₇). Assume that $\rho \geq 2$ and*

$$\int_{-T}^T G^*(f(t)) + a(t) dt < \min\{1, b-1\}(\rho/2) \quad (2)$$

Then (ELT) possesses a periodic solution.

Theorem 1.1 generalizes Lemma 3.1 from [8]. Actually, assumption (1) has the same form as (H₅) in [8], since in p-laplacian case $p_G = q_G = q_G^\infty = p$. Note that p-laplacian satisfies Δ_2 and ∇_2 globally. To the best of the authors knowledge there is no analogue of Theorem 1.2 in the literature.

Now we give two examples of potentials suitable for our setting.

Example 1.3. Consider the functions

$$G(x, y) = x^2 + (x - y)^4, \quad K(t, x, y) = (2 + \sin t)G(x, y) + |x^2 + y^2|^2 \cos^2 t$$

$$W(t, x, y) = \frac{|x^2 + y^2|^{5/2}(e^{t^2(x^2+y^2-1)} - 1)}{t^2 + 1} + \sin t.$$

G is differentiable G -function satisfying Δ_2 and ∇_2 globally. Here $V = K - W$ satisfies (A_2) – (A_6) , where $p_G = 2$, $q_G^\infty = q_G = 4$, $\mu = 5$, $a(t) = \sin t$, $b = 2$, $\kappa(t) \geq 5 \sin t$. On the other hand K does not satisfy assumption (H_1) and W does not satisfy assumption (H_2) from [8].

The next example shows that our results generalize Lemma 7 from [9].

Example 1.4. Set

$$V(t, x) = c(t)G(x) - \lambda d(t)F(x),$$

where F is convex function satisfying Δ_2 globally, $G \prec\prec F$, the functions $c(t)$, $d(t)$ are continuously differentiable, even on \mathbb{R} , $0 < c \leq c(t) \leq C$, $0 < d \leq d(t) \leq D$, $tc'(t) > 0$ for $t \neq 0$ and $tc'(t) < 0$ for $t \neq 0$. Then V satisfies conditions (A_2) – (A_7) .

Theorems 1.1 and 1.2 assert the existence of periodic solutions for

$$\frac{d}{dt} \nabla G(\dot{u}) - c(t) \nabla G(u) + \lambda d(t) \nabla F(u) = f(t),$$

$$u(-T) = u(T) = 0$$

which is a generalization of the problem (2) from [9].

2. Some facts about G -functions and Orlicz-Sobolev spaces

Assume that $G: \mathbb{R}^N \rightarrow [0, \infty)$ satisfies assumption (A_1) . We say that

- G satisfies the Δ_2 condition if

$$\exists_{K_1 > 2} \exists_{M_1 \geq 0} \forall_{|x| \geq M_1} G(2x) \leq K_1 G(x), \quad (\Delta_2)$$

- G satisfies the ∇_2 condition if

$$\exists_{K_2 > 1} \exists_{M_2 \geq 0} \forall_{|x| \geq M_2} G(x) \leq \frac{1}{2K_2} G(K_2 x). \quad (\nabla_2)$$

- G satisfies Δ_2 (resp. ∇_2) globally if $M_1 = 0$ (resp. $M_2 = 0$).

Functions $G_1(x) = |x|^p$, $G_2(x) = |x|^{p_1} + |x|^{p_2}$ satisfy Δ_2 and ∇_2 globally. If G does not satisfy Δ_2 globally, then it could decrease very fast near zero. For example,

$$G(x) = \begin{cases} |x|^2 e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies Δ_2 but does not satisfy Δ_2 globally. For more details about Δ_2 condition in case of N-function we refer the reader to [17].

Since G is differentiable and convex,

$$G(x) - G(x - y) \leq \langle \nabla G(x), y \rangle \leq G(x + y) - G(x) \quad \text{for all } x, y \in \mathbb{R}^N. \quad (3)$$

A function $G^*(y) = \sup_{x \in \mathbb{R}^N} \{ \langle x, y \rangle - G(x) \}$ is called the Fenchel conjugate of G . As an immediate consequence of the definition we have the Fenchel inequality:

$$\forall_{x, y \in \mathbb{R}^N} \quad \langle x, y \rangle \leq G(x) + G^*(y).$$

Now we briefly recall the notion of anisotropic Orlicz space. For more details we refer the reader to [18] and [19]. The Orlicz space associated with G is defined to be

$$\mathbf{L}^G = \{u: [-T, T] \rightarrow \mathbb{R}^N: \int_{-T}^T G(u) dt < \infty\}.$$

The space \mathbf{L}^G equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \lambda > 0: \int_{-T}^T G\left(\frac{u}{\lambda}\right) dt \leq 1 \right\}$$

is a reflexive Banach space. We have the Hölder inequality

$$\int_I \langle u, v \rangle dt \leq 2\|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^*}} \quad \text{for every } u \in \mathbf{L}^G \text{ and } v \in \mathbf{L}^{G^*}.$$

Let us denote by

$$\mathbf{W}^1 \mathbf{L}^G := \{u \in \mathbf{L}^G : \dot{u} \in \mathbf{L}^G\}$$

an anisotropic Orlicz-Sobolev space of vector valued functions with norm

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}.$$

We introduce the following subset of $\mathbf{W}^1 \mathbf{L}^G$

$$\mathbf{W}_T^1 \mathbf{L}^G := \{u \in \mathbf{W}^1 \mathbf{L}^G : u(-T) = u(T)\}.$$

We will also consider an equivalent norm given by

$$\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G} = \inf \left\{ \lambda > 0: \int_{-T}^T G\left(\frac{u}{\lambda}\right) + G\left(\frac{\dot{u}}{\lambda}\right) dt \leq 1 \right\}.$$

The following proposition will be crucial to Lemma 3.4.

Proposition 2.1.

$$\frac{1}{2}\|u\|_{\mathbf{W}^1 \mathbf{L}^G} \leq \|u\|_{1, \mathbf{W}^1 \mathbf{L}^G} \leq 2\|u\|_{\mathbf{W}^1 \mathbf{L}^G}$$

The proof for isotropic case can be found in [20, Proposition 9, p.177]. It remains the same for anisotropic case.

Functional $R_G(u) := \int_{-T}^T G(u) dt$ is called modular.

Proposition 2.2. [21, Proposition 2.7] $R_G(u)$ is coercive on \mathbf{L}^G in the following sense:

$$\lim_{\|u\|_{\mathbf{L}^G} \rightarrow \infty} \frac{R_G(u)}{\|u\|_{\mathbf{L}^G}} = \infty.$$

Define the Simonenko indices for G-function

$$p_G = \inf_{|x|>0} \frac{\langle x, \nabla G(x) \rangle}{G(x)}, \quad q_G = \sup_{|x|>0} \frac{\langle x, \nabla G(x) \rangle}{G(x)}, \quad q_G^\infty = \limsup_{|x| \rightarrow \infty} \frac{\langle x, \nabla G(x) \rangle}{G(x)}.$$

It is obvious that $p_G \leq q_G^\infty \leq q_G$. Moreover, if G satisfies Δ_2 and ∇_2 globally, then $1 < p_G$ and $q_G < \infty$. The following results is crucial to Lemma 3.4.

Proposition 2.3. Let G satisfies Δ_2 and ∇_2 globally.

1. If $\|u\|_{\mathbf{L}^G} \leq 1$, then $\|u\|_{\mathbf{L}^G}^{q_G} \leq R_G(u)$.
2. If $\|u\|_{\mathbf{L}^G} > 1$, then $\|u\|_{\mathbf{L}^G}^{p_G} \leq R_G(u)$.

The proof can be found in appendix. More information about indices for isotropic case can be found in [22], [23] and [13]. For relations between Luxemburg norm and modular for anisotropic spaces we refer the reader to [19, Examples 3.8 and 3.9].

For, respectively, continuous and compact embeddings we will use the symbols \hookrightarrow and $\hookrightarrow\hookrightarrow$.

Let G_1 and G_2 be G-functions. Define

$$G_1 \prec G_2 \iff \exists_{M \geq 0} \exists_{K > 0} \forall_{|x| \geq M} G_1(x) \leq G_2(Kx). \quad (4)$$

The relation \prec allows to compare growth rate of functions G_1 and G_2 .

It is well known that if $G_1 \prec G_2$, then $\mathbf{L}^{G_2} \hookrightarrow \mathbf{L}^{G_1}$. Let $u \in \mathbf{W}^1 \mathbf{L}^G$, $A_G : \mathbb{R}^N \rightarrow [0, \infty)$ be the greatest convex radial minorant of G (see [2]). Then

$$\|u\|_{\mathbf{L}^\infty} \leq C_{\infty, \mathbf{W}^1 \mathbf{L}^G} \|u\|_{\mathbf{W}^1 \mathbf{L}^G},$$

where $C_{\infty, \mathbf{W}^1 \mathbf{L}^G} = A_G^{-1} \left(\frac{1}{2T} \right) \max\{1, 2T\}$.

The following proposition will be used in the proof of Lemma 3.2.

Proposition 2.4. (cf. [8]) For any $1 < p \leq q < \infty$, such that $|\cdot|^p \prec G(\cdot) \prec |\cdot|^q$,

$$\int_{-T}^T |u|^p dt \geq C_{\infty, \mathbf{W}^1 \mathbf{L}^G}^{p-q} C_{G,q}^{-q} \|u\|_{\mathbf{W}^1 \mathbf{L}^G}^{p-q} \|u\|_{\mathbf{L}^G}^q$$

for $u \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$, where $C_{G,q}$ is an embedding constant from $\mathbf{L}^q \hookrightarrow \mathbf{L}^G$.

Proof. Let $u \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$. Since $G \prec |\cdot|^q$,

$$\int_{-T}^T |u|^q dt = \|u\|_{\mathbf{L}^q}^q \geq C_{G,q}^{-q} \|u\|_{\mathbf{L}^G}^q.$$

From Hölder's inequality and embedding $\mathbf{W}^1 \mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ we obtain

$$\begin{aligned} \int_{-T}^T |u|^q dt &= \int_{-T}^T |u|^p |u|^{q-p} dt \leq \\ &\leq \|u\|_{\mathbf{L}^\infty}^{q-p} \int_{-T}^T |u|^p dt \leq (C_{\infty, \mathbf{W}^1 \mathbf{L}^G} \|u\|_{\mathbf{W}^1 \mathbf{L}^G})^{q-p} \int_{-T}^T |u|^p dt. \end{aligned}$$

□

3. Proof of the main results

Let $\mathcal{J} : \mathbf{W}_T^1 \mathbf{L}^G \rightarrow \mathbb{R}$ be given by

$$\mathcal{J}(u) = \int_{-T}^T G(\dot{u}) + K(t, u) - W(t, u) + \langle f, u \rangle dt. \quad (\mathcal{J})$$

From (A_1) , (A_2) and [19, Thm. 5.5] we have $\mathcal{J} \in C^1$ and

$$\mathcal{J}'(u)\varphi = \int_{-T}^T \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt + \int_{-T}^T \langle V_x(t, u) + f(t), \varphi \rangle dt. \quad (\mathcal{J}')$$

It is standard to prove that critical points of \mathcal{J} are solutions of (ELT).

Our proof is based on the well-known Mountain Pass Theorem (see [24]).

Theorem 3.1. Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfies the following conditions:

1. $I(0) = 0$,
2. I satisfies Palais-Smale condition,
3. there exist $\rho > 0$, $e \in X$ such that $\|e\|_X > \rho$ and $I(e) < 0$,

4. there exists $\alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$.

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where $\Gamma = \{g \in C([0,1], X); \ g(0) = 0, \ g(1) = e\}$.

We divide the proof into sequence of lemmas.

Lemma 3.2. \mathcal{J} satisfies the Palais-Smale condition, i.e. every sequence $\{u_n\} \subset \mathbf{W}_T^1 \mathbf{L}^G$ such that $\{\mathcal{J}(u_n)\}$ is bounded and $\mathcal{J}'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ contains a convergent subsequence.

Proof. From (A_5) and (\mathcal{J}') we get

$$\begin{aligned} \int_{-T}^T \mu W(t, u) - (q_G^\infty + \nu) K(t, u) dt &\leq \\ &\leq -\mathcal{J}'(u)u + \int_{-T}^T \langle \nabla G(\dot{u}), \dot{u} \rangle dt + \int_{-T}^T \langle f(t), u \rangle + \kappa(t) dt. \end{aligned} \quad (5)$$

From the definition of the functional we obtain

$$\begin{aligned} \mu \int_{-T}^T G(\dot{u}) dt + (\mu - q_G^\infty - \nu) \int_{-T}^T K(t, u) dt &= \\ &= \mu \mathcal{J}(u) + \int_{-T}^T \mu W(t, u) - (q_G^\infty + \nu) K(t, u) dt - \mu \int_{-T}^T \langle f(t), u \rangle dt. \end{aligned}$$

Applying (5), the Hölder inequality and (A_7) we have

$$\begin{aligned} \mu \int_{-T}^T G(\dot{u}) dt + (\mu - q_G^\infty - \nu) \int_{-T}^T K(t, u) dt &\leq \\ &\leq \mu \mathcal{J}(u) - \mathcal{J}'(u)u + \int_{-T}^T \langle \nabla G(\dot{u}), \dot{u} \rangle dt + C_\kappa + (1 - \mu)C_f \|u\|_{\mathbf{W}^1 \mathbf{L}^G}, \end{aligned} \quad (6)$$

where $C_\kappa = \int_{-T}^T \kappa(t) dt$ and $C_f = 2\|f\|_{\mathbf{L}^{G^*}}$. By the definition of q_G^∞ , there exists $M > 0$ such that

$$\langle x, \nabla G(x) \rangle \leq (q_G^\infty + \nu) G(x), \text{ for } |x| > M.$$

Hence

$$\begin{aligned} \int_{-T}^T \langle \nabla G(\dot{u}), \dot{u} \rangle dt &= \int_{\{|\dot{u}| > M\}} \langle \nabla G(\dot{u}), \dot{u} \rangle dt + \int_{\{|\dot{u}| \leq M\}} \langle \nabla G(\dot{u}), \dot{u} \rangle dt \leq \\ &\leq (q_G^\infty + \nu) \int_{\{|\dot{u}| > M\}} G(\dot{u}) dt + C_{\nabla G} \leq (q_G^\infty + \nu) \int_{-T}^T G(\dot{u}) dt + C_{\nabla G}, \end{aligned} \quad (7)$$

where $C_{\nabla G} = \max_{|x| \leq M} 2T M \nabla G(x)$. Applying (7) we can rewrite (6) as

$$\begin{aligned} (\mu - q_G^\infty - \nu) \int_{-T}^T G(\dot{u}) + K(t, u) dt &\leq \\ &\leq \mu \mathcal{J}(u) - \mathcal{J}'(u)u + C_\kappa + (1 - \mu)C_f \|u\|_{\mathbf{W}^1 \mathbf{L}^G} + C_{\nabla G}. \end{aligned} \quad (8)$$

From (A_4) , given any $0 < \varepsilon_1 < b_1$, there exists $\delta_1 \geq 0$ such that

$$K(t, x) \geq (b_1 - \varepsilon_1)|x|^p - \delta_1, \quad \text{for } x \in \mathbb{R}^N \text{ and } t \in [-T, T].$$

By Proposition 2.4 we obtain

$$\begin{aligned} \int_{-T}^T K(t, u) dt &\geq \int_{-T}^T (b_1 - \varepsilon_1)|u|^p dt - 2T\delta_1 \geq \\ &\geq (b_1 - \varepsilon_1)C_{\infty, \mathbf{W}^1 \mathbf{L}^G}^{p-q} C_{G,q}^{-q} \frac{\|u\|_{\mathbf{L}^G}^q}{\|u\|_{\mathbf{W}^1 \mathbf{L}^G}^{q-p}} - 2T\delta_1, \end{aligned} \quad (9)$$

for any q such that $G \prec |\cdot|^q$. Finally, applying (9) to (8) we obtain

$$\begin{aligned} (\mu - q_G^\infty - \nu) \left(\int_{-T}^T G(\dot{u}) dt + (b_1 - \varepsilon_1)C_{\infty, \mathbf{W}^1 \mathbf{L}^G}^{p-q} C_{G,q}^{-q} \frac{\|u\|_{\mathbf{L}^G}^q}{\|u\|_{\mathbf{W}^1 \mathbf{L}^G}^{q-p}} \right) + \\ - (\mu - 1)C_f \|u\|_{\mathbf{W}^1 \mathbf{L}^G} + \mathcal{J}'(u)u \leq \mu \mathcal{J}(u) + C_\kappa + C_{\nabla G} + C_{\delta_1}, \end{aligned} \quad (10)$$

where $C_{\delta_1} = 2T\delta_1(\mu - q_G^\infty - \nu)$.

Let $\{u_n\} \subset \mathbf{W}_T^1 \mathbf{L}^G$ will be a Palais-Smale sequence for \mathcal{J} . There exist $C_J, C_{J'} > 0$ such that

$$\mathcal{J}(u_n) \leq C_J, \quad \mathcal{J}'(u_n)u_n \geq -C_{J'} \|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}$$

Without loss of generality we can assume, that $\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G} > 0$. Substituting u_n into (10) we obtain

$$\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G} \left(\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}} + \frac{\|u_n\|_{\mathbf{L}^G}^q}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}^{1+q-p}} - C' \right) \leq C'', \quad (11)$$

where $C', C'' > 0$ are suitable constants independent of n .

We show that $\{u_n\}$ is bounded. On the contrary, suppose that there exists a subsequence of u_n (still denoted u_n) such that $\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G} \rightarrow \infty$. Consider three cases.

1. Let $\|u_n\|_{\mathbf{L}^G} \rightarrow \infty$ and $\|\dot{u}_n\|_{\mathbf{L}^G} \rightarrow \infty$ (again, w.l.o.g. $\|\dot{u}_n\|_{\mathbf{L}^G} > 0$). From Proposition 2.2 we have that

$$\begin{aligned} \frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}} + \frac{\|u_n\|_{\mathbf{L}^G}^q}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}^{1+q-p}} &= \\ &= \frac{R_G(\dot{u}_n)}{\|\dot{u}_n\|_{\mathbf{L}^G}} \frac{\|\dot{u}_n\|_{\mathbf{L}^G}}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}} + \left(\frac{\|u_n\|_{\mathbf{L}^G}}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}} \right)^{1+q-p} \|u_n\|_{\mathbf{L}^G}^{p-1} \rightarrow \infty. \end{aligned}$$

2. Let $\|\dot{u}_n\|_{\mathbf{L}^G} \rightarrow \infty$ and $\|u_n\|_{\mathbf{L}^G}$ is bounded. Then

$$\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}} = \frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{L}^G} + \|\dot{u}_n\|_{\mathbf{L}^G}} = \frac{\frac{R_G(\dot{u}_n)}{\|\dot{u}_n\|_{\mathbf{L}^G}}}{\frac{\|u_n\|_{\mathbf{L}^G}}{\|\dot{u}_n\|_{\mathbf{L}^G}} + 1} \rightarrow \infty \text{ as } \|\dot{u}_n\|_{\mathbf{L}^G} \rightarrow \infty.$$

3. Let $\|u_n\|_{\mathbf{L}^G} \rightarrow \infty$ and $\|\dot{u}_n\|_{\mathbf{L}^G}$ is bounded. Since $p > 1$, we have

$$\frac{R_G(\dot{u}_n)}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}} + \frac{\|u_n\|_{\mathbf{L}^G}^q}{\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G}^{1+q-p}} \geq \frac{\|u_n\|_{\mathbf{L}^G}^q}{(\|u_n\|_{\mathbf{L}^G} + \|\dot{u}_n\|_{\mathbf{L}^G})^{1+q-p}} \rightarrow \infty.$$

Therefore, in view of (11), $\{u_n\}$ is bounded in $\mathbf{W}_T^1 \mathbf{L}^G$.

It follows from reflexivity of $\mathbf{W}_T^1 \mathbf{L}^G$ and embeddings $\mathbf{W}_T^1 \mathbf{L}^G \hookrightarrow \mathbf{L}^G$, $\mathbf{W}_T^1 \mathbf{L}^G \hookrightarrow \mathbf{W}^{1,1}$ that there exists $u \in \mathbf{W}_T^1 \mathbf{L}^G$ and a subsequence of u_n (still denoted u_n) such that $u_n \rightarrow u$ in \mathbf{L}^G . Moreover, $\dot{u}_n \rightarrow \dot{u}$ in \mathbf{L}^1 and hence pointwise a.e.

Since $\{u_n\}$ is a Palais-Smale sequence, we have

$$0 \leftarrow \mathcal{J}'(u_n)(u_n - u) = \int_{-T}^T \langle \nabla G(\dot{u}_n), \dot{u}_n - \dot{u} \rangle dt + \int_{-T}^T \langle V_x(t, u_n) + f(t), u_n - u \rangle dt.$$

Since $\int_{-T}^T \langle V_x(t, u_n) + f(t), u_n - u \rangle dt \rightarrow 0$ we can deduce that

$$\int_{-T}^T \langle \nabla G(\dot{u}_n), \dot{u}_n - \dot{u} \rangle dt \rightarrow 0.$$

From (3) we obtain

$$\int_{-T}^T G(\dot{u}_n) dt \leq \int_{-T}^T G(\dot{u}) dt + \int_{-T}^T \langle \nabla G(\dot{u}_n), (\dot{u}_n - \dot{u}) \rangle dt$$

Hence

$$\limsup_{n \rightarrow +\infty} \int_{-T}^T G(\dot{u}_n) dt \leq \int_{-T}^T G(\dot{u}) dt.$$

On the other hand, by Fatou's Theorem we have

$$\liminf_{n \rightarrow +\infty} \int_{-T}^T G(\dot{u}_n) dt \geq \int_{-T}^T G(\dot{u}) dt.$$

Combining these inequalities we get that

$$\int_{-T}^T G(\dot{u}_n) dt \rightarrow \int_{-T}^T G(\dot{u}) dt.$$

Therefore $\dot{u}_n \rightarrow \dot{u}$ in \mathbf{L}^G by [19, Lemma 3.16]. \square

We next prove that \mathcal{J} is negative for some point outside $B_\rho(0)$.

Lemma 3.3. *There exist $e \in \mathbf{W}_T^1 \mathbf{L}^G$ such that $\|e\|_{\mathbf{W}^1 \mathbf{L}^G} > \rho$ and $\mathcal{J}(e) < 0$.*

Proof. By assumption (A_4) , there exist $\varepsilon_0, r > 0$ such that

$$W(t, x) \geq (3 + \varepsilon_0) \max\{K(t, x), G(x)\} \quad \text{for } |x| > r.$$

This gives

$$K(t, x) - W(t, x) \leq -(2 + \varepsilon_0)G(x) \quad \text{for } |x| > r. \quad (12)$$

Fix $v \in \mathbb{R}^N$. For $\xi > T + 1$ define $e: [-T, T] \rightarrow \mathbb{R}^N$ by

$$e(t) = \xi \left(1 - \frac{|t|}{T+1} \right) v.$$

Direct computation shows

$$\dot{e}(t) = \begin{cases} -\frac{\xi}{T+1} v, & t \in (0, T], \\ \frac{\xi}{T+1} v, & t \in [-T, 0). \end{cases}$$

Since $\|e\|_{\mathbf{L}^\infty} = \xi > T + 1$ and $\|\dot{e}\|_{\mathbf{L}^\infty} = \xi/(T+1) > 1$, we can choose ξ such that both (12) and $\|e\|_{\mathbf{W}^1 \mathbf{L}^G} \geq \rho$ hold. From (\mathcal{J}) , the Fenchel inequality and (12) we have

$$\begin{aligned} \mathcal{J}(e) &\leq \int_{-T}^T G(\dot{e}) + K(t, e) - W(t, e) + G(e) + G^*(f) dt \leq \\ &\leq \int_{-T}^T G(\dot{e}) - G(e) - \varepsilon_0 G(e) + G^*(f) dt. \end{aligned}$$

Since $1 - \frac{|t|}{T+1} \geq \frac{1}{T+1}$ for $t \in [-T, T]$, we have

$$\int_{-T}^T G(\dot{e}) - G(e) dt = \int_{-T}^T G\left(\frac{\xi}{T+1}v\right) - G\left(\xi\left(1 - \frac{|t|}{T+1}\right)v\right) dt \leq 0.$$

Choosing ξ large enough we get

$$\mathcal{J}(e) \leq \int_{-T}^T -\varepsilon_0 G(e) + G^*(f) dt < 0.$$

□

In order to show that \mathcal{J} satisfies the fourth assumption of the Mountain Pass Theorem, we first provide some estimates for $R_G(\dot{u}) + R_G(u)$ on $\partial B_\rho(0)$.

If Δ_2 and ∇_2 are satisfied globally then we can use Proposition 2.3 to estimate $R_G(\dot{u}) + R_G(u)$ from below by $(\rho/2)^r$, $r > 1$, for any $\rho > 0$. If G does not satisfies Δ_2 and ∇_2 globally then we cannot use Proposition 2.3 (for explanation see Remark A.2). In this case we use equivalent norm and Proposition 2.1 but we obtain only that $R_G(\dot{u}) + R_G(u) \geq \rho/2$. Moreover, we are forced to assume $\rho > 2$.

Let $u \in \mathbf{W}_T^1 \mathbf{L}^G$ be such that $\|u\|_{\mathbf{W}^1 \mathbf{L}^G} = \rho$. Set $\rho_1 = \|u\|_{\mathbf{L}^G}$, $\rho_2 = \|\dot{u}\|_{\mathbf{L}^G}$, $\rho_1 + \rho_2 = \rho$. Assuming that G satisfies Δ_2 and ∇_2 globally we get, by Proposition 2.3, the following estimates:

1. If $\rho_1, \rho_2 \leq 1$ then $R_G(\dot{u}) + R_G(u) \geq \|\dot{u}\|_{\mathbf{L}^G}^{qG} + \|u\|_{\mathbf{L}^G}^{qG}$. Hence

$$R_G(\dot{u}) + R_G(u) \geq 2^{1-qG} (\|\dot{u}\|_{\mathbf{L}^G} + \|u\|_{\mathbf{L}^G})^{qG} \geq (\rho/2)^{qG}, \quad (13)$$

since $\rho_1^{qG} + \rho_2^{qG} \geq 2^{1-qG} (\rho_1 + \rho_2)^{qG}$.

2. If $\rho_1 \leq 1$, $\rho_2 \geq 1$ then $(\rho_1 + \rho_2)^{pG} \leq (2\rho_2)^{pG} \leq 2^{pG} (\rho_1^{qG} + \rho_2^{pG})$. Hence

$$R_G(\dot{u}) + R_G(u) \geq \|\dot{u}\|_{\mathbf{L}^G}^{pG} + \|u\|_{\mathbf{L}^G}^{qG} \geq (\rho/2)^{pG}. \quad (14)$$

3. If $\rho_1 \geq 1$, $\rho_2 \leq 1$ then $(\rho_1 + \rho_2)^{pG} \leq (2\rho_1)^{pG} \leq 2^{pG} (\rho_1^{pG} + \rho_2^{qG})$. Thus

$$R_G(\dot{u}) + R_G(u) \geq \|\dot{u}\|_{\mathbf{L}^G}^{qG} + \|u\|_{\mathbf{L}^G}^{pG} \geq (\rho/2)^{pG}. \quad (15)$$

4. If $\rho_1, \rho_2 \geq 1$ then

$$R_G(\dot{u}) + R_G(u) \geq \|\dot{u}\|_{\mathbf{L}^G}^{pG} + \|u\|_{\mathbf{L}^G}^{pG} \geq (\rho/2)^{pG}. \quad (16)$$

From the other hand, Proposition 2.1 implies

$$\inf \left\{ \lambda > 0: \int_{-T}^T G\left(\frac{u}{\lambda}\right) + G\left(\frac{\dot{u}}{\lambda}\right) dt \leq 1 \right\} \geq \frac{1}{2}\rho.$$

Therefore

$$\int_{-T}^T G\left(\frac{2u}{\rho}\right) + G\left(\frac{2\dot{u}}{\rho}\right) dt \geq 1$$

and consequently,

$$R_G(u) + R_G(\dot{u}) \geq \frac{\rho}{2}, \quad (17)$$

provided $\rho > 2$.

Lemma 3.4. Assume that either (1) or (2) holds. There exists positive constant α such that $\mathcal{J}|_{\partial B_\rho(0)} \geq \alpha$.

Proof. From the definition of ρ and embedding $\mathbf{W}_T^1 \mathbf{L}^G \hookrightarrow \mathbf{L}^\infty$ we have

$$|u(t)| \leq \|u\|_{\mathbf{L}^\infty} \leq C_{\infty, \mathbf{W}^1 \mathbf{L}^G} \|u\|_{\mathbf{W}^1 \mathbf{L}^G} = C_{\infty, \mathbf{W}^1 \mathbf{L}^G} \rho = \rho_0 \quad \text{for } t \in [-T, T].$$

From (A_3) and the Fenchel inequality we obtain

$$\begin{aligned} \mathcal{J}(u) &\geq \int_{-T}^T G(\dot{u}) + bG(u) - a(t) + \langle f, u \rangle dt \geq \\ &\geq \min\{1, b-1\}(R_G(\dot{u}) + R_G(u)) - R_{G^*}(f) - \int_{-T}^T a(t) dt. \end{aligned}$$

Assume that (1) holds. If $\rho \leq 2$ then (13), (14) and (15) yields

$$\mathcal{J}(u) \geq \min\{1, b-1\}(\rho/2)^{q_G} - R_{G^*}(f) - \int_{-T}^T a(t) dt =: \alpha.$$

If $\rho > 2$, then by (14), (15) and (16) we get

$$\mathcal{J}(u) \geq \min\{1, b-1\}(\rho/2)^{p_G} - R_{G^*}(f) - \int_{-T}^T a(t) dt > 0 =: \alpha.$$

From (1) it follows that in both cases $\alpha > 0$.

Assume that (2) holds. From (17) we obtain

$$\begin{aligned} \mathcal{J}(u) &\geq \min\{1, b-1\}(R_G(\dot{u}) + R_G(u)) - R_{G^*}(f) - \int_{-T}^T a(t) dt \geq \\ &\geq \min\{1, b-1\}(\rho/2) - R_{G^*}(f) - \int_{-T}^T a(t) dt =: \alpha. \end{aligned}$$

From (2) we have $\alpha > 0$. □

Now we are in position to prove our main theorems. Note that by (A_6) and $G(0) = 0$ we have $\mathcal{J}(0) = 0$. From Lemmas 3.2, 3.3 and 3.4 we have that \mathcal{J} satisfies all assumptions of the Mountain Pass Theorem. Hence there exists a critical point $u \in \mathbf{W}_T^1 \mathbf{L}^G$ of \mathcal{J} and (ELT) have periodic solution.

Actually, we can show that any solution to (ELT) is more regular (cf. Corollary 16.16 in [25]).

Proposition 3.5. If $u \in \mathbf{W}_T^1 \mathbf{L}^G$ is a solution of (ELT), then $u \in \mathbf{W}^1 \mathbf{L}^\infty$.

Proof. Let $u \in \mathbf{W}^1 \mathbf{L}^G$ be a solution of (ELT). Then

$$\nabla G(\dot{u}(t)) = \int_{-T}^t \nabla V(t, u(t)) dt + C$$

and there exists $M > 0$ such that $|\nabla G(\dot{u}(t))| \leq M < \infty$. From the other hand

$$G(\dot{u}(t)) \leq \langle \nabla G(\dot{u}(t)), \dot{u}(t) \rangle \leq M|\dot{u}(t)|.$$

Since $\frac{G(v)}{|v|} \rightarrow \infty$ as $|v| \rightarrow \infty$, we obtain $|\dot{u}(t)|$ is bounded. \square

Remark 3.6. If G is strictly convex then one can show that if $u \in \mathbf{W}_T^1 \mathbf{L}^G$ is a solution of (ELT), then $u \in \mathbf{C}^1$.

Remark 3.7. Theorem 1.1 remains true if we change assumption (1) to

$$\int_{-T}^T G^*(f(t)) + a(t) dt < \min\{1, b-1\} \begin{cases} 2(\rho/2)^{q_G}, & \rho \leq 2^{1-1/(q_G-p_G)} \\ (\rho/2)^{p_G} & \rho > 2^{1-1/(q_G-p_G)}. \end{cases} \quad (18)$$

Estimate in the first case is better than (1) but it is taken on smaller set. In the second case estimate is the same as in (1) but can be taken on bigger set.

Remark 3.8. In the proof of Lemma 3.4 we can use the Hölder inequality instead of the Fenchel inequality to estimate $\int_{-T}^T \langle f, u \rangle dt$. It allows us to take $b > 0$ if $\rho \leq 1$.

Appendix A.

Assume that G satisfies ∇_2 globally. It is easy to show that G^* satisfies Δ_2 globally with $K_1^* = 2K_2$.

Since $G \in \mathbf{C}^1$ and is convex, we have

$$K_1 G(x) \geq G(2x) \geq G(2x) - G(x) \geq \langle x, \nabla G(x) \rangle \quad \text{for all } x \in \mathbb{R}^N.$$

Let $y \in \mathbb{R}^N$ and $s \in \partial G^*(y)$, where ∂G^* denotes the subdifferential of G^* . Since G^* satisfies Δ_2 globally, we have

$$K_1^* G^*(y) \geq G^*(2y) \geq G^*(2y) - G^*(y) \geq \langle s, y \rangle \quad \text{for all } y \in \mathbb{R}^N$$

Let $x \in \mathbb{R}^N$. Then $x \in \partial G^*(\nabla G(x))$ and $G(x) + G^*(\nabla G(x)) = \langle x, \nabla G(x) \rangle$. It follows that

$$G(x) = \langle x, \nabla G(x) \rangle - G^*(\nabla G(x)) \leq \left(1 - \frac{1}{K_1^*}\right) \langle x, \nabla G(x) \rangle.$$

Finally

$$\frac{2K_2}{2K_2-1} \leq \frac{\langle x, \nabla G(x) \rangle}{G(x)} \leq K_1 \quad \text{for all } x \in \mathbb{R}^N. \quad (\text{A.1})$$

Since $K_2 > 1$, we have $\frac{2K_2}{2K_2-1} > 1$ and from (A.1) we obtain

$$p_G > 1 \quad \text{and} \quad q_G < \infty.$$

For any $x \in \mathbb{R}^N$ and $\lambda \geq 1$ we have

$$\log G(\lambda x) - \log G(x) = \int_1^\lambda \frac{\langle \nabla G(\lambda x), x \rangle}{G(\lambda x)} d\lambda \leq \int_1^\lambda \frac{q_G}{\lambda} d\lambda = \log \lambda^{q_G}.$$

Thus

$$G(\lambda x) \leq \lambda^{q_G} G(x) \quad \text{for all } x \in \mathbb{R}^N, \lambda \geq 1. \quad (\text{A.2})$$

Similarly, we get

$$G(\lambda x) \geq \lambda^{p_G} G(x) \quad \text{for all } x \in \mathbb{R}^N, \lambda \geq 1. \quad (\text{A.3})$$

Lemma A.1. *Let $u \in \mathbf{L}^G$.*

1. *If $\|u\|_{\mathbf{L}^G} < 1$ then $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{q_G}$.*
2. *If $\|u\|_{\mathbf{L}^G} > 1$ then $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{p_G}$.*

Proof. For any $0 < \beta < \|u\|_{\mathbf{L}^G} < 1$ we have $R_G\left(\frac{u}{\beta}\right) \geq 1$. From (A.2) we obtain that $G\left(\frac{x}{\beta}\right) \leq \left(\frac{1}{\beta}\right)^{q_G} G(x)$ for all $x \in \mathbb{R}^N$. Hence

$$R_G(u) \geq \beta^{q_G} R_G\left(\frac{u}{\beta}\right) \geq \beta^{q_G}.$$

Letting $\beta \uparrow \|u\|_{\mathbf{L}^G}$ gives $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{q_G}$.

For any $1 < \beta < \|u\|_{\mathbf{L}^G}$ we have $R_G\left(\frac{u}{\beta}\right) > 1$. Then from (A.3) we obtain that $G(x) \geq \beta^{p_G} G\left(\frac{x}{\beta}\right)$ for all $x \in \mathbb{R}^N$. Hence

$$R_G(u) \geq \beta^{p_G} R_G\left(\frac{u}{\beta}\right) \geq \beta^{p_G}.$$

Letting $\beta \uparrow \|u\|_{\mathbf{L}^G}$ gives $R_G(u) \geq \|u\|_{\mathbf{L}^G}^{p_G}$. □

Remark A.2. *If G satisfies Δ_2 and ∇_2 (not globally), estimates similar to (A.2) and (A.3) can be obtained for sufficiently large $|x|$. However, even if $\|u\|_{\mathbf{L}^G}$ is large it does not necessarily mean that $|u(t)|$ is large. Hence we cannot use these estimates to obtain result similar to Lemma A.1.*

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