MULTILEVEL REGULARITY OF ORBITS OF EXPANDING LORENZ MAPS WITH APPLICATION TO THE COURBAGE-NEKORKIN-VDOVIN MODEL

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Abstract. We discuss the structure and properties of itineraries of periodic orbits for expanding Lorenz maps with nontrivial rotation interval. In particular, it is shown that periodic orbits of such maps are organized in two cascades (called Stern-Brocot and Geller-Misiurewicz cascades), which are closely related to the Farey tree of rational rotation numbers of a given map. The obtained results are illustrated with a reduced Courbage-Nekorkin-Vdovin neuron model allowing us to characterize regularity of periodic spiking patterns in the model.

Introduction

Map-based models represent an important category of models used to describe simplified activity of individual nerve cells or dynamics of larger ensembles of synchronized neurons. In concerns where low computational complexity plays an important role, they can serve as a valuable complement to ODE-based ([10, 14]), or hybrid neuron models ([16, 20]). In discrete models differential equations are replaced by maps, which can lead to highly complex behaviours.

Usually two-dimensional map-based models take the form:

$$x_{n+1} = F(x_n, y_n), \tag{1a}$$

$$y_{n+1} = y_n + \varepsilon G(x_n, y_n), \tag{1b}$$

where x denotes the membrane voltage, y is so-called *adaptation* or *recovery* variable and ε might be a small positive parameter to include separation of time scales between these two variables. For a comprehensive review of map-based neuron models see, e.g., [15] but let us mention that particular examples of widely used 2D discrete models include Chialvo model ([4]), Rulkov model ([21]) or Courbage, Nekorkin and Vdovin (CNV for short) model ([6]):

$$x_{n+1} = f_1(x_n, y_n) = x_n + F(x_n) - y_n - \beta H(x_n - d), \tag{2a}$$

$$y_{n+1} = f_2(x_n, y_n) = y_n + \varepsilon(x_n - J), \tag{2b}$$

where J might stand for a constant external stimulus, $\beta > 0$ and $d > 0$ control the threshold properties of oscillations and $H(x)$ is the usual Heaviside step function:

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

When the CNV model was introduced (see [6]), $F(x)$ was defined as follows:

$$F(x) = \begin{cases} -m_0 x, & \text{if } x \leq J_{\min}, \\ m_1(x - a), & \text{if } J_{\min} \leq x \leq J_{\max}, \\ -m_0(x - 1), & \text{if } x \geq J_{\max}, \end{cases} \tag{3}$$

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where $m_0, m_1 > 0$, $1 > a > 0$, $J_{\min} = \frac{am_1}{m_0+m_1}$, $J_{\max} = \frac{m_0+am_1}{m_0+m_1}$. Due to the piecewise linear form of $F(x)$ the model (2a)–(2b) with F given by (3) will be referred to as the plCNV model (piecewise linear CNV model). However, it is also natural to replace the piecewise linear function F with some cubic polynomial (see [7]):

$$F(x) = \mu x(x-a)(1-x), \tag{4}$$

with $\mu > 0$ and $0 < a < 1$, giving rise to the nonlinear CNV model (denoted nlCNV). Its one dimensional reduction consisting of (2a), while $y_n := \alpha$ is a constant parameter, is referred to as the 1D nlCNV model. The plCNV and nlCNV models, as well as their reductions the 1D plCNV and 1D nlCNV, were recently studied in [2] and [3].

Note that in the CNV model, the map governing the dynamics of the voltage variable x is an expanding Lorenz map (see also Figure 1). Moreover, itineraries of its periodic orbits can be matched with spiking patterns (i.e., the regularity of time series of action potentials generated by a neuron). Therefore majority of this work is devoted to studying various types of periodic orbits of expanding Lorenz maps and their itineraries, which we introduce, respectively, in Sections 2 and 4. Section 1 presents some preliminaries and Section 3 contains a brief summary of Geller and Misiurewicz results from [12]. In turn, Section 5 discusses the structure and hierarchy of two cascades of periodic orbit itineraries, which we call Stern-Brocot and Geller-Misiurewicz cascades. Various levels of regularity of orbit itineraries are summarized in Section 6, while Sections 7 and 8 present relation of these results to the CNV model.

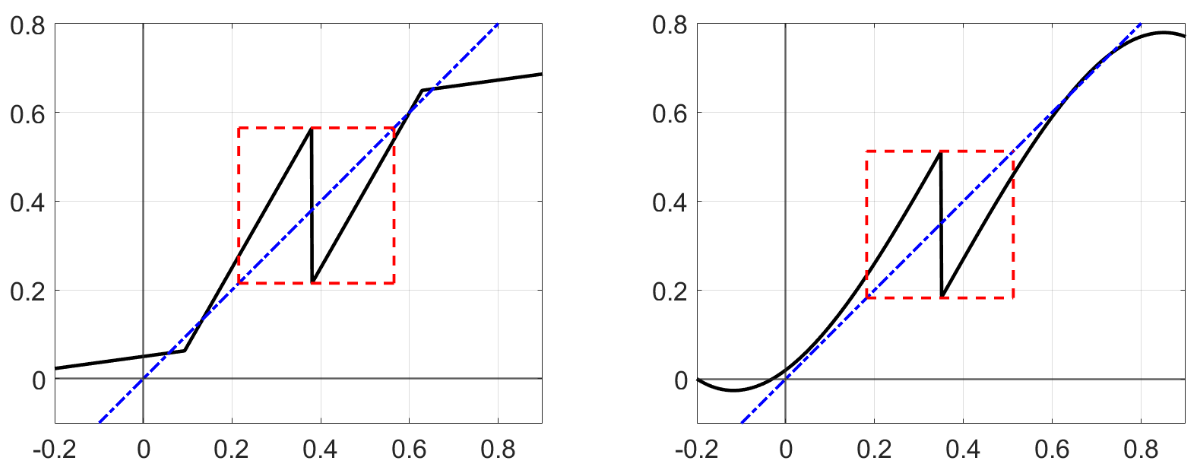

FIGURE 1. Example plots of plCNV (left) and nlCNV (right) functions. Square boxes show invariant intervals. Parameters: $a = 0.2$; $d = 0.38$; $m_0 = 0.864$; $m_1 = 0.8$; $\beta = 0.35$; $y_0 = -0.05$ (left), $\alpha = -0.02$; $\beta = 0.33$; $\mu = 2.5$; $a = 0.1$; $d = 0.35$ (right).

1. LORENZ MAPS

For the convenience of the reader we present here the definitions of expanding Lorenz maps, lift, rotation number, rotation interval and itinerary. For simplicity of notation, we will formulate the definitions and results below for the unit interval $[0,1]$. However, all definitions make sense and all results still hold if we replace the unit interval by $[a,b]$ for fixed a and b and use the linear change of variables (conjugacy).

1.1. **Maps.** An *expanding Lorenz map* is a map $f\colon [0,1] \longrightarrow [0,1]$ satisfying the following three conditions:

- there is a *critical point* $c \in (0,1)$ such that f is continuous and strictly increasing on $[0,c)$ and $(c,1]$,
- $\lim_{x\to c^-} f(x) = f(c^-) = 1$ and $\lim_{x\to c^+} f(x) = f(c^+) = f(c) = 0$,
- f is differentiable for all points not belonging to a finite set $F \subset [0,1]$ and

$$\beta_f := \inf\{f'(x) \mid x \in [0,1) \setminus F\} > 1.$$

Observe that, since $(1-f(0))/c > 1$ and $f(1)/(1-c) > 1$, expanding Lorenz maps satisfy the condition $f(0) < f(1)$. Finally, note that the 1D CNV model map is an expanding Lorenz map both in piecewise linear and nonlinear case.

However, to define our main tools: rotation number and rotation interval we need a more general class of Lorenz maps called Lorenz-like. Namely, a *Lorenz-like map* is a map f of an interval $[0,1]$ to itself, for which there exists a point $c \in (0,1)$ such that

- f is continuous and increasing (not necessarily strictly) on $[0,c)$ and on $(c,1]$,
- $\lim_{x\to c^-} f(x) = 1$ and $\lim_{x\to c^+} f(x) = f(c) = 0$.

Set $I_L = [0,c)$ and $I_R = [c,1]$. If $f(0) \geq f(1)$ ($f(0) < f(1)$) the Lorenz-like map f is called *nonoverlapping* (*overlapping*). By definition, expanding Lorenz maps are overlapping Lorenz-like.

1.2. **Lifts.** We will denote by $\pi\colon \mathbb{R} \longrightarrow [0,1)$ the natural projection $\pi(x) = x \pmod 1$. Let f be a Lorenz-like map. The map $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined (uniquely) by the conditions:

- $F(x) = \begin{cases} f(x) & \text{if } x \in [0,c), \\ f(x)+1 & \text{if } x \in [c,1) \end{cases}$
- $F(x+k) = F(x) + k$ for $x \in [0,1)$ and $k \in \mathbb{Z}$

is called the *lift* of f. Note that $\pi \circ F = f \circ \pi$. Moreover, if f is an expanding Lorenz map then lift of f is continuous and strictly increasing on $(0,1)$ and discontinuous with a negative jump at integers.

Remark 1.1. Observe that, by definition, $F(1) = F(0) + 1 = f(0) + 1$. In consequence, we do not include the value $f(1)$ in the definition of F and we cannot use F in the study of orbits containing 1. However, since for expanding Lorenz maps the only possible periodic orbit containing 1 is the fixed point $x = 1$, it poses no problem for our work.

1.3. **Rotation number and rotation interval.** Let f be a Lorenz-like map. For a point $x \in [0,1]$ and a positive integer n we will denote by $R(x,n)$ the number of integers $i \in \{0,\ldots,n-1\}$ such that $f^i(x) \in I_R$. If the limit

$$\rho(x) = \lim_{n\to\infty} \frac{R(x,n)}{n}$$

exists, we will call it the *rotation number* of x. By definition, $0 \leq \rho(x) \leq 1$ if it exists. Note that if x is a periodic point of f of period q then $\rho(x)$ exists and is equal to $R(x,q)/q$. The following classical result from [19] establishes the uniqueness of the pointwise rotation number.

Theorem 1.2 (Rhodes-Thompson). *If $f(0) \geq f(1)$ (i.e., if the map is nonoverlapping), then all points have the same rotation number.*

In that case we will denote it by $\rho(f)$. If $t \in f(I_L) \cap f(I_R)$, then we define the *water map* at level t by

$$f_t(x) = \begin{cases} \max(t, f(x)) & \text{if } x \in I_L, \\ \min(t, f(x)) & \text{if } x \in I_R. \end{cases}$$

It is obvious that this map is also Lorenz-like and $f_t(0) = f_t(1)$. Consequently, for fixed t, all points have the same rotation number $\rho(f_t)$ for it. It is known that $\rho(f_t)$ is an increasing continuous function of t, and if $f(0) \leq f(1)$, then the set of the rotation numbers for f of all points having rotation number is equal to the interval $\left[\rho(f_{f(0)}), \rho(f_{f(1)})\right]$ (see, e.g. [1] for details). We will call it the *rotation interval* of f and denote it by $\mathrm{Rot}(f)$.

1.4. **Itineraries.** Let f be a Lorenz-like map.

Definition 1.3. The *itinerary* of x under f is the sequence $S(x) = (s_0 s_1 s_2 \dots)$ where

$$s_j = s_j(x) = \begin{cases} 0 & \text{if } f^j(x) \in I_L, \\ 1 & \text{if } f^j(x) \in I_R. \end{cases}$$

2. TPO, NTPO and FPTPO

Unless otherwise stated we assume that f is an expanding Lorenz map. However, the approach presented in this section should also work for general Lorenz-like maps.

Recall that a periodic P orbit of f is called

- a *twist periodic orbit* (TPO for short) if the lift of f restricted to $\pi^{-1}(P)$ is increasing,
- a *nontwist periodic orbit* (NTPO for short) otherwise.

Figure 2 shows examples of twist (left panel) and nontwist (middle panel) periodic orbits for the map $f(x) = 2x \pmod 1$.

Remark 2.1. By definition, fixed points are TPOs. The only possible fixed points for expanding Lorenz maps are $x = 0$ and $x = 1$. Note that the lift is not well-defined at $x = 1$ (see Remark 1.1). However, for expanding Lorenz maps the only periodic orbit containing 1 is the fixed point. In consequence, our definition works also in this case.

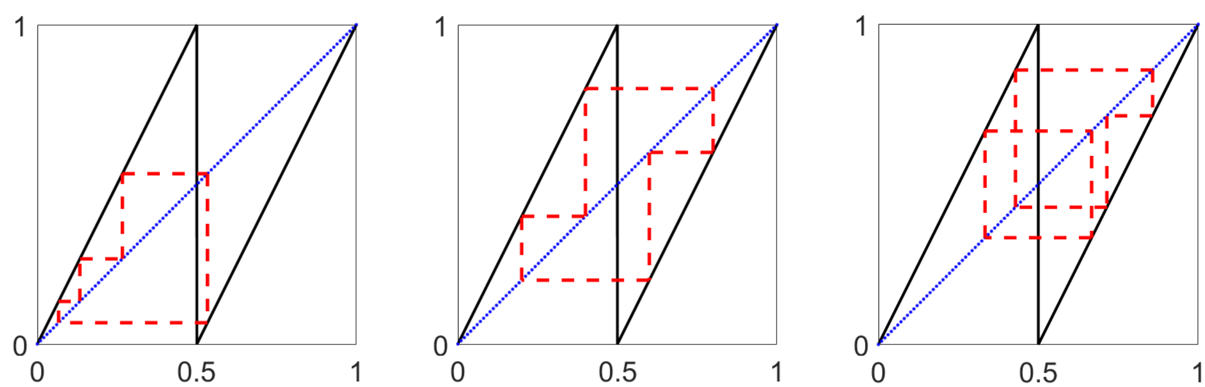

FIGURE 2. TPO (left), NTPO (middle) and FPTPO (right) for the map $f(x) = 2x \pmod 1$.

Let $P = \{x_0 < x_1 < \cdots < x_{q-1}\}$ be a q-periodic orbit of f ($q \geq 1$). For each periodic orbit we will consider its *permutation*, i.e., if $f(x_i) = x_{\sigma(i)}$ for $i = 0, \dots, q-1$, then σ is the permutation of P. Let natural numbers $0 \leq p \leq q$ be coprime. An orbit P is called a (p, q)-*translation* if for $i = 0, \dots, q-1$ we have

$$f(x_i) = x_{i \oplus p}, \quad \text{where } i \oplus p = i + p \pmod q.$$

If, in addition, a (p, q)-*translation* satisfies the conditions $x_{p-1} \leq f(0)$ and $f(1) \leq x_p$, we call it *primary* (this requires $q > p \geq 1$). A periodic orbit P is called a (p, q)-*cycle* if it has a prime period q and its itinerary (starting from x_0) satisfies the formula

$$(*) \qquad s_k = 0 \iff kp \pmod q < q - p.$$

The repetend of this itinerary, i.e., the sequence $\{s_k\}_0^{q-1}$ with s_k satisfying $(*)$ will be called the (p,q)-*word*. We start with some basic observation concerning TPOs.

Proposition 2.2. *Let $P = \{x_0 < x_1 < \cdots < x_{q-1}\}$ be a q-periodic orbit of an expanding Lorenz map f. Then the following conditions are equivalent:*

(1) P is a TPO with rotation number p/q,
(2) P is a (p,q)-translation,
(3) $\sigma(0) \geq \sigma(q-1)$ and p is the number of elements of $I_R \cap P$,
(4) P is a (p,q)-cycle.

Proof of Proposition 2.2. If P contains 1 then $P = \{1\}$ and the assertion holds. So in further considerations we can assume P does not contain 1 and use the lift of f. In general, the reasonings are similar to that for homeomorphisms of the circle (see for instance [17, Sec. 11]). Let us sketch them briefly. *Ad* (1)⟶(2). Set $\pi^{-1}(P) = B = \{\cdots < b_{-1} < b_0 < b_1 < \cdots\}$. By assumption, $F\colon B \longrightarrow B$ is an order preserving bijection. So it must be a "translation" by a fixed value $l \in \mathbb{Z}$, i.e., $F(b_i) = b_{i+l}$. Since rotation number is p/q, we have $F^q(b_0) = b_0 + p$. On the other hand, $F^q(b_0) = b_{ql} = b_0 + l$. This gives $l = p$. So finally $F(b_i) = b_{i+p}$, which implies $f(x_i) = x_{i\oplus p}$. *Ad* (2)⟶(3). By the definition of a (p,q)-translation, $\sigma(0) = p > p-1 = \sigma(q-1)$. *Ad* (3)⟶(1). Let $\pi^{-1}(P) = \{\cdots < b_{-1} < b_0 < b_1 < \cdots\}$ and $b_0 = x_0$. Since, by definition, F is increasing on $[0,1)$, it suffices to show that $F(b_{q-1}) < F(b_q)$. But, by assumption,

$$F(b_{q-1}) = f(x_{q-1}) + 1 = x_{\sigma(q-1)} + 1 < x_{\sigma(0)} + 1 = f(x_0) + 1 = F(b_q).$$

Ad (2)⟶(4). By assumption, $f^k(x_0) = x_{k\odot p}$, where $\odot$ is multiplication mod q. Observe that $I_R \cap P = \{x_{q-p} < \cdots < x_{q-1}\}$ and, in consequence,

$$s_k(x_0) = 0 \iff f^k(x_0) = x_{k\odot p} \notin I_R \iff kp \pmod q < q - p,$$

which establishes the formula $(*)$. *Ad* (4)⟶(1). If P is a (p,q)-cycle then the rotation number of P is p/q and $p/q \in \mathrm{Rot}(f)$. Since there is a unique TPO Q with rotation number p/q and the itinerary of Q satisfies $(*)$, by Proposition 4.1, it must be that $P = Q$. □

Remark 2.3. We emphasize that, by Proposition 2.2, the itinerary of TPO with a given rotation number does not depend on the map f. Moreover, in (3) we allow equality so that Proposition 2.2 also covers the case of fixed points.

Recall that two fractions a/p and b/q between 0 and 1 are called *Farey neighbours* if $bp - aq = 1$. Moreover, we will call them *proper* (resp. *improper*) Farey neighbours if, in addition, $p < q$ (resp. $p > q$). For example, $1/2 < 2/3$ are proper Farey neighbours and $1/3 < 1/2$ are improper ones. Basic properties of Farey neighbours, or more generally, *Farey series* and their relation to the binary *Stern-Brocot tree* are discussed, for example, in [13]. Finally, a *Farey pair of twist periodic orbits* (FPTPO for short) is a set of two twist periodic orbits for the map f with rotation numbers being Farey neighbours. In Figure 2 (right panel) we present *FPTPO* with rotation numbers $1/2 < 2/3$ for the map $f(x) = 2x \pmod 1$.

3. Two theorems of Geller and Misiurewicz

In this section we present two results of Geller and Misiurewicz (see [12]), which can be treated as a starting point for all our further considerations of this paper. To formulate precisely these results we need a bunch of definitions, which will be used only in this section. Moreover, following [12] we will use a different convention here than in the rest of the work. Namely, we will start the numbering from 1 rather than 0.

Recall that a permutation σ of $\{1,\dots,n\}$ is called an *L-permutation* (L for Lorenz) if there is a unique $k \in \{1,\dots,n-1\}$ such that σ is increasing on $\{1,\dots,k\}$ and on $\{k+1,\dots,n\}$. Then the number $(n-k)/n$ is called the *rotation number* of σ. With every L-permutation σ we can associate a Lorenz-like map f which realizes this permutation. It will be called the *canonical model* for σ and it is defined as the *connect the dots* map with the dots being the points $(x, f(x))$ with coordinates as in Table 1.

TABLE 1. *connect the dots* map.

x	0	$\frac{1}{n+1}$	$\dots$	$\frac{k}{n+1}$	$\frac{k+\frac{1}{2}}{n+1}$	$\frac{k+\frac{1}{2}}{n+1}$	$\frac{k+1}{n+1}$	$\dots$	$\frac{n}{n+1}$	1
$f(x)$	$\frac{\sigma(1)}{n+1}$	$\frac{\sigma(1)}{n+1}$	$\dots$	$\frac{\sigma(k)}{n+1}$	1	0	$\frac{\sigma(k+1)}{n+1}$	$\dots$	$\frac{\sigma(n)}{n+1}$	$\frac{\sigma(n)}{n+1}$

A *twist* permutation is a cyclic L-permutation σ of $\{1,\dots,n\}$ such that $\sigma(1) > \sigma(n)$. It is easy to see that if σ is a twist permutation with the rotation number l/n then

$$\sigma(i) = i + l \pmod{n},$$

where $m \pmod{n} = (m-1) \pmod{n} + 1$. Note that this formula implies immediately that $\sigma(1) = \sigma(n)+1$. Finally, an L-permutation σ of $\{1,\dots,p+q\}$ is called an *FL-permutation* (FL for Farey-Lorenz) if it consists of two twist permutations of lengths p and q with rotation numbers a/p and b/q being proper Farey neighbours.

Let us state now two crucial theorems from [12].

Theorem 3.1 ([12, Thm 2]). *Assume that*

- *σ is an FL-permutation consisting of cycles π and κ,*
- *f is a canonical model for σ,*
- *P (resp. Q) is a periodic orbit of f corresponding to π (resp. κ).*

Then

(1) any periodic orbit of f has a periodic itinerary whose repetend is a finite concatenation of repetends of itineraries of P and Q,
(2) for each such concatenation C there is a periodic orbit whose itinerary has C as its repetend.

The next result is not included in this version in [12] but follows immediately from the considerations presented in Section 5 of [12].

Theorem 3.2 ([12, Sec. 5]). *Assume that*

- *f is a Lorenz-like map,*
- *P and Q are twist periodic orbits of f with rotation numbers a/p and b/q being Farey neighbours ($p < q$),*
- *A and B are repetends of itineraries of P and Q respectively.*

Then for each finite concatenation of A and B there is a periodic orbit with itinerary having this concatenation as a repetend.

Remark 3.3. As Example 5.1 in [12] shows the first part of the assertion of Theorem 3.1 is not true for arbitrary Lorenz-like map. Namely, even if the rotation interval of f is equal to $[a/p, b/q]$, some periodic orbits of f may have itineraries with repetends not being concatenation of repetends of itineraries corresponding to a/p and b/q.

Our results presented here are also based on the idea of concatenations of itineraries of orbits introduced in [12], but our methods are different as they are mainly purely arithmetic and combinatorial and we do not rely on the notions of canonical model or Markov graphs used in [12]. The vast majority of our results neither are stated explicitly in [12] nor are simple immediate consequences of results proved therein (for example, our Propositions 5.3 and 5.8, Lemmas A.1, A.2, A.3 and A.4, Proposition B.1 and Lemmas C.1 and C.2 have no counterparts in [12]). Moreover, in our paper we aim at introducing the rotation theory, various types of periodic orbits and their organization in Lorenz-like maps to researchers working in various applications (not only neuroscience) where Lorenz-like maps arise in a natural way. Perhaps our results and methods can be more accessible to application-oriented scientists than those already available in the literature.

4. Itineraries of orbits

This section is devoted to the study of itineraries of orbits of expanding Lorenz maps.

Proposition 4.1. *Assume that f is an expanding Lorenz map. Then different points have different itineraries. In consequence, if S is an infinite periodic binary sequence, then there is at most one periodic orbit with itinerary S.*

Proof. If points $x < y$ had the same itinerary then the length l_k of the interval $\left(f^k(x), f^k(y)\right)$ would satisfy for all k the inequality $l_k \geq \beta_f^k(y-x)$ with $\beta_f > 1$, which is impossible. □

Corollary 4.2. *If x is a periodic point with itinerary S then the prime period of x and the prime period of S are equal.*

Proof. Since the prime period of a point is also a period of its itinerary, the prime period k of its itinerary divides the prime period n of x. Suppose, contrary to our claim, that $n > k$. Then x and $y = f^k(x)$ are different points with the same itinerary. This contradicts Proposition 4.1. □

Note that itineraries of points can be ordered lexicographically, i.e., $(s_0 s_1 s_2 \dots) \prec (t_0 t_1 t_2 \dots)$ if there exists an $i \geq 0$ such that $s_j = t_j$ for all $j < i$ and $s_i = 0$, $t_i = 1$.

The following result follows immediately from monotonicity of f on I_L and I_R and Proposition 4.1.

Proposition 4.3. *The lexicographical order on itineraries corresponds to the normal order on points in $[0,1)$, i.e., $S(x) \prec S(y)$ if and only if $x < y$.*

The following three results provide us with a natural starting point for our further considerations. Let natural numbers $0 \leq p \leq q$ be coprime.

Proposition 4.4. *If $p/q \in \mathrm{Rot}(f)$ then there is a unique TPO of prime period q and rotation number p/q.*

Proof. By [12, Sec. 5] or [1], there is a TPO of prime period q and rotation number p/q. From Proposition 2.2 its itinerary satisfies $(*)$ and hence, by Proposition 4.1, such an orbit is unique. □

In what follows, a string means a finite sequence of symbols.

Proposition 4.5 (Properties of itineraries of TPOs). *Let $S = \{s_k\}_0^\infty$ be an itinerary of a TPO (starting from x_0) with rotation number p/q and $A = \{s_k\}_0^{q-1}$. Then S satisfies the formula $(*)$. In particular,*

1. *S is the infinite repetition of the string A, i.e., $S = AAA\dots$,*
2. *A is aperiodic (indecomposable), i.e., there is no B and $n > 1$ such that $A = B^n$, where B^n denotes the repetition of the string B for n times,*
3. *A starts with 0, i.e., $s_0 = 0$, except the fixed point with $p/q = 1/1$ and $A = 1$,*
4. *A ends with 1, i.e., $s_{q-1} = 1$, except the fixed point with $p/q = 0/1$ and $A = 0$,*
5. *A contains exactly p 1's and $q - p$ 0's.*
6. *if $p/q \leq 1/2$ then $s_k = 1$ implies $s_{k+1} = 0$ for each k, i.e., there are no two consecutive 1's in S,*
7. *if $p/q \geq 1/2$ then $s_k = 0$ implies $s_{k+1} = 1$ for each k, i.e., there are no two consecutive 0's in S.*

Proof. By Proposition 2.2, S satisfies the formula $(*)$ and, in consequence, A is the (p,q)-word. Hence we check at once that (1), (3), (4) and (5) hold. What is left is to show (2) and (6) ((7) has the same justification as (6)). *Ad* (2). It follows from p and q being coprime. Namely, suppose, contrary to our claim, that $A = B^n$ for some $n > 1$. Let m denote the length of B and k denote the number of 1's in B. Then the length of A is $q = n \cdot m$ and $p = n \cdot k$ and, hence, $\gcd(p,q) \geq n > 1$, a contradiction. *Ad* (6). Let $0 \leq k < q-1$ and $z_k = kp \,(\mathrm{mod}\ q)$. If $s_k = 1$ then $q > z_k \geq q-p$. Hence, by assumption, $2q - p \geq q + p > z_k + p \geq q$ and, in consequence,

$$q - p = 2q - p \,(\mathrm{mod}\ q) > z_{k+1} = z_k + p \,(\mathrm{mod}\ q) \geq 0,$$

which gives $s_{k+1} = 0$. □

Let A and B be the repetends of itineraries S and S' corresponding, by the formula $(*)$, to two different rotation numbers. Note that the matching of the strings A and B from their beginning coincides with the matching of the infinite sequences S and S'. However, to match the strings A and B from their end, it seems to be more convenient to use negative indices than to renumber indices. For this purpose, we will introduce *generalized itineraries*, i.e., two-sided infinite sequence ($k \in \mathbb{Z}$) satisfying $(*)$.

Proposition 4.6 (Properties of itineraries of FPTPOs). *Assume that* $0 \leq a/p < b/q \leq 1$ *are Farey neighbours. Let* $\overline{S} = \{s_k\}_{k\in\mathbb{Z}}$ *and* $\overline{S}' = \{s'_k\}_{k\in\mathbb{Z}}$ *be generalized itineraries of a FPTPO (with* s_0 *and* s'_0 *corresponding to* x_0 *and* y_0*, i.e., the first points of the orbits relative to the order on the real line) with rotation numbers* a/p *and* b/q*, respectively. Then*

(1) $s_k = s'_k$ *for* $1-p \leq k \leq q-2$*,*
(2) $s_{q-1} = 0 \neq 1 = s'_{q-1}$*,*
(3) $s_{-p} = 0 \neq 1 = s'_{-p}$*.*

In particular, if $A = \{s_k\}_0^{p-1}$ *and* $B = \{s'_k\}_0^{q-1}$ *then*

(4) *for proper Farey neighbours* ($p < q$) *the shorter string* A *is the initial segment of the longer string* B*, i.e.,* $B = AC$ *for some* C*,*
(5) *for improper Farey neighbours* ($p > q$) *the shorter string* B *is the final segment of the longer string* A*, i.e.,* $A = CB$ *for some* C*.*

The mainly technical proof of (1) is postponed to Appendix A. A trivial verification shows that (2) and (3) are true. In turn, (4) and (5) follow easily from (1).

Remark 4.7. In fact, Proposition 4.6 (1) and Proposition 4.5 (3) and (4) imply that the shorter of the two strings A, B uniquely determines the longer one (assuming we know the length of the longer). Namely, in the case of proper Farey neighbours, to obtain B from A it is enough to take $n = \lceil q/p \rceil$ copies of A, i.e., A^n, then cut an initial segment of the length $q-1$ from A^n and finally append at the end (q-th place) 1. Similar procedure works for improper Farey neighbours. Finally, by Proposition 4.6 (4) and (5), the longer string clearly determines the shorter one (again assuming we know the length of the shorter).

5. Stern-Brocot and Geller-Misiurewicz cascades

5.1. **Periodic orbits.** Let us denote by $\mathrm{Per}(u,w)$ the set of all periodic orbits of f with rotation numbers contained in the interval $[u,w]$.

Proposition 5.1. *If* f *is an expanding Lorenz map, then the set of all periodic orbits of* f*, i.e. the set* $\mathrm{Per}(0,1)$*, is at most countable.*

Proof. Each periodic point has periodic itinerary. Since there are countably infinitely many periodic binary sequences and, by Proposition 4.1, for each periodic binary sequence there is at most one periodic orbit with such itinerary, we are done. □

Remark 5.2. The question naturally arises when the set of all periodic orbits is dense in the interval $[0,1]$. It is closely related to the study of the chaotic behaviour of the map f (recall that the density of the set of periodic points is one of the three main components of the Devaney definition of chaos). We provided partial answers to this question for both β-transformations and general expanding Lorenz maps in our two articles (see [2, Prop. 5 and 6, Thm 5, 6, 7, 10 and 11] and [3, Thm A.8, Cor. A.9]), but in this paper we are mainly interested in types of periodic orbits and properties of their itineraries.

5.2. **Stern-Brocot cascade.** In this subsection we assume that $0 \leq a/p < b/q \leq 1$ are Farey neighbours contained in $\mathrm{Rot}(f)$. Recall that their *mediant* is defined as $(a+b)/(p+q)$. Let us denote by A and B respectively the repetends of the periodic itineraries of TPOs corresponding to the fractions a/p and b/q according to Proposition 4.5. We start with the basic observation.

Proposition 5.3 (mediant *effect*)**.** *The repetend X of the itinerary of the unique TPO corresponding to the mediant of the Farey neighbours $a/p < b/q$ has the form of the concatenation AB.*

Proof. By Proposition 2.2, X is the $(a+b, p+q)$-word. In consequence, by Lemma A.4, X is equal to the concatenation AB. □

Note that since pairs $a/p < (a+b)/(p+q)$ and $(a+b)/(p+q) < b/q$ are again Farey neighbours, the procedure of taking the mediant may be repeated for them. Thus in the next step we get the new concatenation AAB and ABB (see Figure 3). Continuing in this fashion we obtain the countably infinite set of concatenations of the symbols A and B. They are called *Stern-Brocot concatenations* and are discussed in Appendix B, where we provide their natural characterization (Proposition B.1). It is an easy consequence of Proposition B.1 that the set of Stern-Brocot concatenations does not contain all possible finite concatenations of A and B.

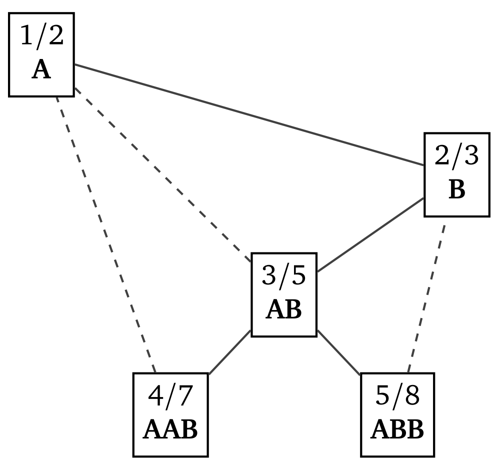

FIGURE 3. The fragment of the Stern-Brocot subtree rooted at 1/2 (solid edges) and the corresponding Stern-Brocot concatenations assuming we start from the Farey neighbours $1/2 < 2/3$ (dashed edges show the remaining Farey neighbours).

In the formulation of the next result we use notions from Appendix B.

Proposition 5.4 (Stern-Brocot cascade)**.** *For each irreducible fraction r/s from $[a/p, b/q]$ there is a unique TPO with the rotation number r/s and the repetend of the itinerary being a Stern-Brocot concatenation of A and B uniquely determined by the position of r/s in relation to the Farey pair $a/p < b/q$ in the Stern-Brocot subarray of fractions hooked at a/p and b/q. In particular, the set of TPOs whose rotation numbers are in $[a/p, b/q]$ and repetends of itineraries are Stern-Brocot concatenations of A and B is countably infinite.*

Proof. By Proposition 4.4, for r/s there is a unique TPO with this rotation number. On the other hand, each irreducible fraction r/s occurs in the Stern-Brocot subarray of fractions hooked at a/p and b/q (see Appendix B for more details). The position of r/s in this subarray of fractions determines (by means of the bijection described in Appendix B) the unique concatenation of A and B corresponding to r/s in the Stern-Brocot array of concatenations. □

The set of TPOs described in Proposition 5.4, i.e., the set of all TPOs with rotation number in $[a/p, b/q]$, will be called the *Stern-Brocot cascade* corresponding to the Farey neighbours $a/p < b/q$ and denoted by $C_{\mathrm{SB}}(a/p, b/q)$.

Remark 5.5. Observe that, by Propositions 4.5 (2) and 5.4, the set $C_{\mathrm{SB}}(a/p, b/q)$ contains periodic orbits with arbitrarily long aperiodic repetends of itineraries.

5.3. **Geller-Misiurewicz cascade.** In turn, in this subsection we assume that $a/p < b/q$ are **proper** Farey neighbours contained in $\mathrm{Rot}(f)$. We need this assumption to apply the basic version of Geller-Misiurewicz construction, because the improper case requires some translation of results via conjugacy (see the end of Section 2 in [12]), which we want to avoid here. Let again A and B be the repetends of the periodic itineraries of TPOs corresponding to the fractions a/p and b/q. The following observation is crucial for our considerations.

Proposition 5.6 (FPTPO effect)**.** *For each finite binary sequence C being a finite concatenation of A and B, there is a unique periodic orbit (not necessarily a TPO), which itinerary has the repetend equal to C.*

Proof. Existence part was proved in [12, Sec. 5] for Lorenz-like maps (not necessarily expanding Lorenz). Uniqueness follows from Proposition 4.1. □

Remark 5.7. The set of all periodic orbits (not necessarily TPOs) whose rotation numbers are in $[a/p, b/q]$ and repetends of itineraries are finite concatenations of A and B will be called the *Geller-Misiurewicz cascade* and denoted by $C_{\mathrm{GM}}(a/p, b/q)$. Obviously,

$$C_{\mathrm{SB}}(a/p, b/q) \subset C_{\mathrm{GM}}(a/p, b/q) \subset \mathrm{Per}(a/p, b/q).$$

In consequence, by Propositions 5.1 and 5.4, the set $C_{\mathrm{GM}}(a/p, b/q)$ is also countably infinite.

The next result explains how many NTPOs we can expect in $C_{\mathrm{GM}}(a/p, b/q)$.

Proposition 5.8. *For each irreducible fraction r/s from $(a/p, b/q)$ and for each $k > 1$ there is an NTPO belonging to $C_{GM}(a/p, b/q)$ with rotation number r/s and prime period ks. In consequence, the set $C_{GM}(a/p, b/q)$ contains countably infinitely many NTPOs.*

Proof. Recall that, by Proposition 5.4, for $r/s \in (a/p, b/q)$ we have a unique TPO with rotation number r/s and the repetend of the itinerary being Stern-Brocot concatenation of A and B. Let us denote this concatenation by $C(A, B)$. Now if for $k > 1$ we replace in $C(A, B)$ each instance of A with A^k (i.e., the repetition of A for k times) and each instance B with B^k we obtain a new concatenation of A and B, which will be denoted by $C(A^k, B^k)$. Since, by Proposition 4.5 (2), $C(A, B)$ is aperiodic (with respect to A and B), $C(A^k, B^k)$ is also aperiodic (with respect to A and B) from Lemma C.1. Hence, by Lemma C.2, $C(A^k, B^k)$ is aperiodic with respect to 0 and 1. Finally, if P is a periodic orbit with the repetend of the itinerary equal to $C(A^k, B^k)$ (such a periodic orbit exists from Proposition 5.6) then P has rotation number r/s and period ks, because periods of periodic orbits and their itineraries are equal (see Corollary 4.2). □

We end this section with two remarks.

Remark 5.9. Note that the proof of Proposition 5.8 is primarily based on the following useful observation: inserting (m, n)-words corresponding to Farey neighbours into an aperiodic pattern of two symbols gives a new aperiodic binary string (for the precise formulation, see Lemma C.2).

Remark 5.10. However, we do not claim that there are no other itineraries of periodic orbits with rotation number in $(a/p, b/q)$ than those from $C_{\rm GM}(a/p, b/q)$. Namely, note that the set $\mathrm{Per}(a/p, b/q) \setminus C_{\rm GM}(a/p, b/q)$ may be nonempty as the following example shows. Consider the map $f(x) = 2x \pmod 1$ with $\mathrm{Rot}(f) = [0, 1]$. The overbar below means that the given symbol or set of symbols is repeated ad infinitum. It is easy to see that $3/5$ is a mediant of the Farey neighbours $1/2 < 2/3$ and the repetends of the corresponding itineraries have the form $A = 01$ for $1/2$ and $B = 011$ for $2/3$. Now note that the point $x_0 = 0.\overline{00111}\ldots$ is 5-periodic with rotation number $3/5$ and the repetend of its itinerary is simply $C = 00111$. However, C is not a concatenation of A and B, because no two consecutive 0's occur in any concatenation of A and B. In consequence, in this case the set $\mathrm{Per}(1/2, 2/3) \setminus C_{\rm GM}(1/2, 2/3)$ is nonempty.

6. Four levels of regularity of itineraries of orbits in the Geller-Misiurewicz cascade

This section only summarizes the previous results and does not contain new ones. Assume that f is an expanding Lorenz map and that $a/p < b/q$ are proper Farey neighbours contained in $\mathrm{Rot}(f)$. Under the above assumption we can observe multilevel regularity of infinitely many orbits of the map f. We emphasise that this regularity is evident in the structure of itineraries of periodic orbits. Let us try to explain it in detail by focusing on the Geller-Misiurewicz cascade.

6.1. **First level – abundance of periodicity.** Let us start with the most basic observation. By Proposition 4.4, Remark 5.7 and Proposition 5.8, our dynamical system has countably infinitely many twist periodic orbits and countably infinitely many nontwist periodic orbits. More precisely, for each irreducible fraction $r/s \in (a/p, b/q)$ there is exactly one TPO and countably infinitely many different NTPOs with the same rotation number r/s. However, we should emphasize that this abundance of periodicity is not visible from the numerical point of view, because all periodic orbits are repelling ($f' > 1$).

6.2. **Second level – concatenations.** More interestingly, the repetends of itineraries of countably infinitely many periodic orbits (both twist and nontwist) are of a particular form. Namely, by Remark 5.7, the repetends of itineraries of orbits from $C_{\rm GM}(a/p, b/q)$ are always concatenations of only two fixed binary strings A and B. In other words, there is a simple algebraic operation (concatenation), which from two periodic orbits with rotation numbers that are proper Farey neighbours immediately "produce" countably infinitely many different periodic orbits whose itineraries are arbitrary finite concatenations of the repetends of the itineraries of the two initial orbits. Furthermore, exactly this way, i.e., by means of concatenation operation, we construct a unique TPO and infinitely many NTPOs in the proof of Proposition 5.8. Although, it is also worth pointing out that in our dynamical system there may also be NTPOs with rotation number in $(a/p, b/q)$, whose itineraries are not concatenation of A and B (see Remark 5.10).

6.3. **Third level – (m, n)-words.** Equally interesting is the fact that the strings A and B, from which we build concatenations, are themselves of a very special form. Namely, by Proposition 2.2, A is the (a, p)-word and B is the (b, q)-word. In consequence, both A and B have all specific and distinctive properties described in Proposition 4.5.

6.4. **Fourth level – matching.** What is also not insignificant is that the strings A and B are closely related, because they come from Farey neighbours. More precisely, in our case (proper Farey neighbours), the string A is the initial substring of B (see Proposition 4.6 (4)). In fact, we even obtain better matching. Namely, by Proposition 4.6 (1),

(1) the infinite sequence $AAA\ldots$ and the string B always match up to the penultimate symbol in B (corresponding to the index $i = q - 2$) and mismatch on the last one ($i = q - 1$ counting from 0),

(2) denoting by A' the string obtained from A by removing the first symbol, the string A' is a final substring of B.

Consequently, A uniquely determines B and, of course, vice versa (see Remark 4.7).

Let us see the above relationships on an example. Consider the proper Farey neighbours $3/5 < 8/13$. The corresponding $(3,5)$-word and $(8,13)$-word are respectively $A = 01011$ and $B = 0101101011011$. Now note that

- A is the initial segment of B:
  01011
  0101101011011
- twelve initial elements of A^3 match B without the last element (the last element of B is 1):
  010110101101
  0101101011011
- A without the first element, which is 0, matches the ending segment of B:
  1011
  0101101011011

6.5. **Additional level of regularity in the Stern-Brocot cascade.** In the previous subsections we have analyzed the itineraries of orbits from $C_{\mathrm{GM}}(a/p, b/q)$. However, a new additional regularity appears in the set $C_{\mathrm{SB}}(a/p, b/q)$, which is not present in the set $C_{\mathrm{GM}}(a/p, b/q)$. Namely, not only the strings A and B are (m,n)-words (see Subsection 6.3) but all Stern-Brocot concatenations of A and B, which, by definition, correspond bijectively to periodic orbits from $C_{\mathrm{SB}}(a/p, b/q)$, are also (m,n)-words. And in addition, in a double meaning, namely,

- as usual binary strings, which can be concluded from the general description of itineraries for TPOs (see Proposition 2.2),
- as strings of A and B with A treated as 0 and B treated as 1, which follows from the characterization of Stern-Brocot concatenations (see Proposition B.1).

7. Rotation intervals of the CNV model in numerical simulations

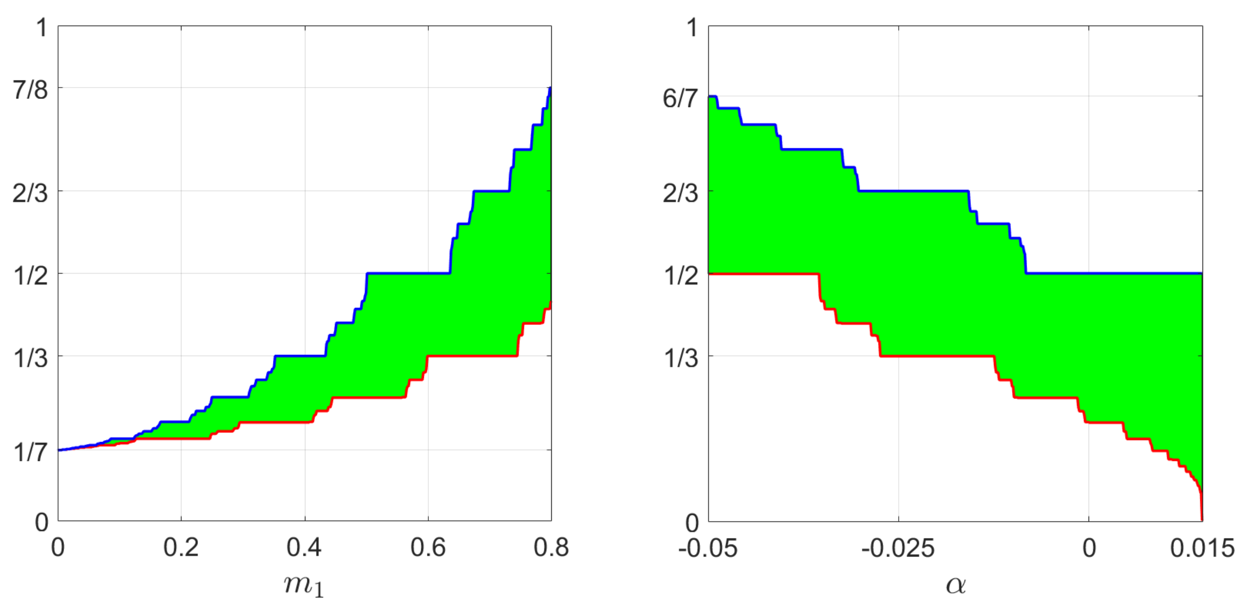

FIGURE 4. Parameter dependence of rotation interval (in green): plCNV with respect to m_1 (left) and nlCNV with respect to α (right). Other values: $a = 0.2$; $d = 0.38$; $\beta = 0.35$; $m_0 = 0.86$; $y_0 = -0.05$ (left), $\beta = 0.33$; $\mu = 2.5$; $a = 0.1$; $d = 0.35$ (right).

In [5, Thm 5.1] the authors prove that if an expanding Lorenz map f has a primary (p,q)-translation, then $\mathrm{Rot}(f) = \{p/q\}$. Using this result they also provide examples of expanding Lorenz maps such that their rotations intervals are degenerate (i.e., are singletons). Namely, if $f(x) = \beta x + \alpha \,(\mathrm{mod}\ 1)$ then for $\beta = \sqrt{2}$ and $\alpha = (2-\sqrt{2})/2$ we have $\mathrm{Rot}(f) = \{1/2\}$ ([18, Ex 6.1]) and for $\beta = 9\sqrt[5]{2}/10$ and $\alpha = \sqrt[5]{2}/3$ we have $\mathrm{Rot}(f) = \{2/5\}$ ([5, Ex 3.8]). It is to be expected (as suggested by the authors of [5]) that the inverse implication is also true, i.e., expanding Lorenz maps without any primary (p,q)-translation have nondegenerate rotation intervals.

Moreover, our numerical simulations (see Figure 4) show that the situation in which the rotation interval degenerates to singleton is rather not typical for both the plCNV and the nlCNV model in a broad range of parameters. The left panel of Figure 4 shows the dependence of the rotation interval of the plCNV map on the parameter m_1, which controls the slope $q = 1 + m_1$ of the β-transformation. When m_1 is close to 0, the interval rotation seems to degenerate to a single point. Moreover, in general, the larger m_1 the bigger the interval, but the expansion of the interval is not simply monotonic. In turn, the right panel of Figure 4 presents dependence of the rotation interval of the nlCNV map on α (the main parameter of this model). Note that the rotation interval moves down with the increase of α, but remains quite large for all values of α.

If the rotation interval of the CNV model map is nondegenerate, then we can always find (proper) Farey neighbours in it and apply our results from previous sections. Furthermore, the "best" choice of Farey neighbours is to find the smallest denominators $p < q$ of the fractions from the rotation interval and choose from all fractions with those denominators two neighbouring (in the line order) fractions with different denominators (see [12, Sec. 5] for details).

8. Regularity of spiking patterns in CNV model

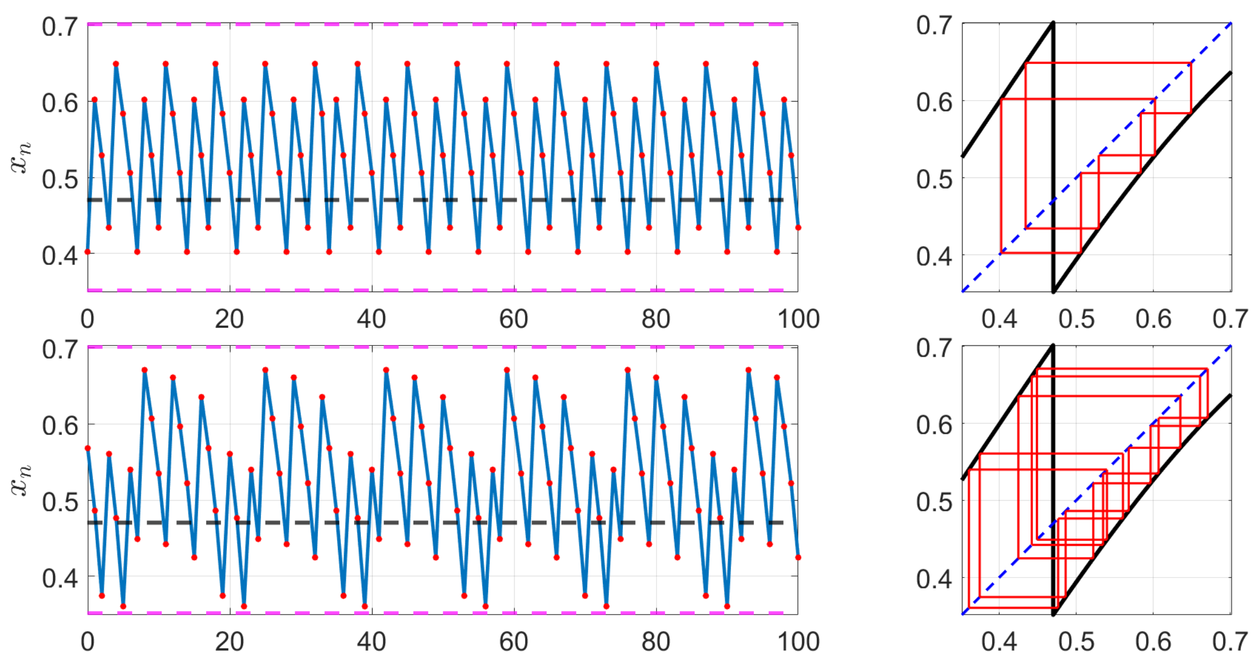

Figure 5. A period 7 TPO (top) and period 17 NTPO (bottom) and their corresponding time series in 1D nlCNV model (left). Common parameter values: $\alpha = -0.082$, $\beta = 0.35$, $\mu = 1.62$, $a = 0.1$, $d = 0.47$. Initial points (from the top): 0.402; 0.568.

The itineraries of periodic orbits can be identified with spike patterns fired by the 1D CNV model, as we showed in our previous works [2, 3] (see in particular Propositions 6.1 and 6.2 in [3]). Moreover, under some conditions and with ε small, periodic spike trains observed in the 1D CNV model persist in the full 2D model.

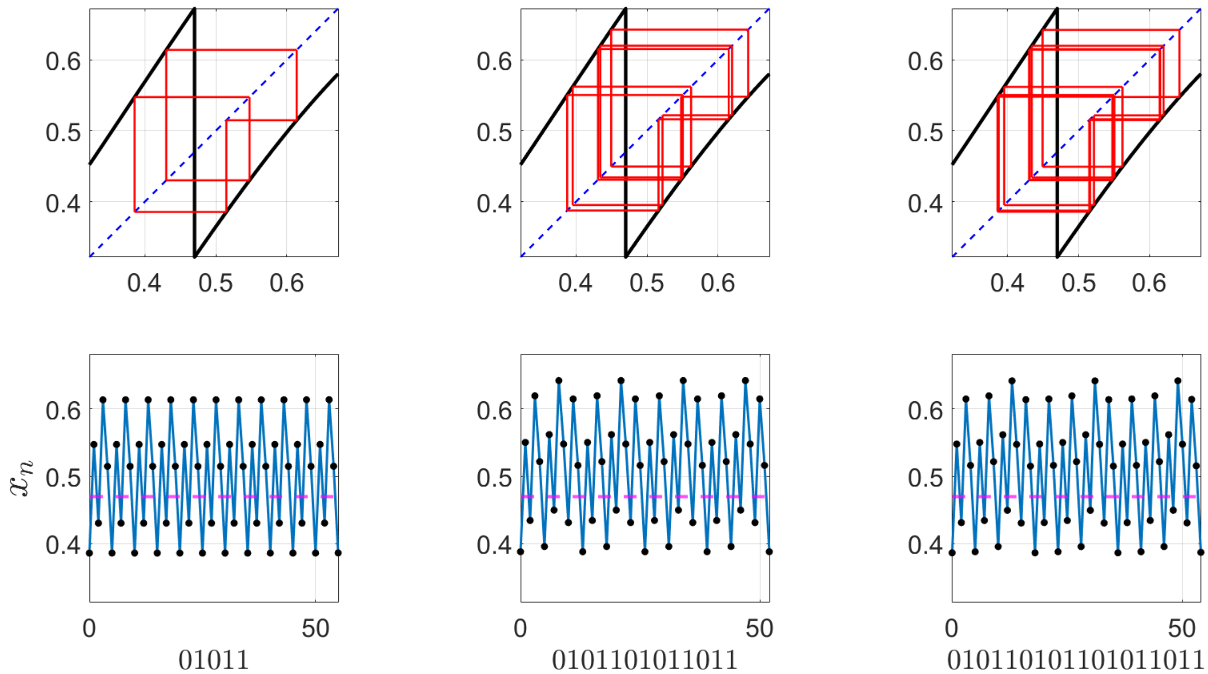

FIGURE 6. The *mediant effect* in the nlCNV system: creation of TPO with rotation number 11/18 (right panel) from FPTPO corresponding to rotation numbers $3/5 < 8/13$ (left and middle panels) combined with concatenation of the repetends of their itineraries (bottom panel). Parameter values: $\alpha = -0.05$, $\beta = 0.35$, $\mu = 1.65$, $a = 0.1$, $d = 0.47$.

In this work we extend these results by showing the existence of multilevel regularity in expanding Lorenz maps, thus also in the 1D nlCNV model, and exemplify TPO and NTPO firing patterns. In particular, Figure 5 presents these two types of periodic orbits (TPO with rotation number 5/7 and NTPO with rotation number 12/17, respectively) in the cobweb diagrams for 1D CNV map and corresponding spike-trains. Figure 6 illustrates the mediant effect in the nlCNV model, based on the abstract example discussed at the end of Subsection 6.4, where FPTPO with rotations numbers 3/5 and 8/13 give rise to a TPO with rotation number 11/18. Additionally, Figure 7 presents four time series corresponding to various periodic orbits, all with rotation number 12/17. We remark that the itinerary of periodic orbit encodes increasing and decreasing parts of the trajectory, and if, for example, one assumes that the maxima of the voltage correspond to spikes, then it actually determines the spike pattern fired. Moreover, in the case of TPO its rotation number allows us to uniquely deduce i ts itinerary and thus to determine the corresponding spike train. In particular, TPO with rotation number of the form of $1/q$ is composed of $q-1$ consecutive points on the left of the discontinuity point followed by one point to the right of it and thus such an orbit yields a spike train where the neuron voltage monotonically increases until it quite rapidly drops down which resembles emitting a spike. Thus in the related spike train we observe clearly separated spikes without smaller intermediate oscillations. On the other hand, there can be multiple NTPOs with a given rotation number (and thus multiple corresponding spike-trains). The works [2, 3] discuss in more detail the connection between orbit itineraries and spike patterns fired and present more examples of various spike trains.

The strength of the presented methods relies on enabling identifying periodic orbit itineraries existing in the model (or in general Lorenz-like maps) without actually finding those orbits. We note that numerical computation of periodic orbits can be very hard, if not impossible, especially in the case of expanding Lorenz-like maps (like in the CNV model) due to the instability of these orbits. Since periodic orbits are responsible for regular tonic spiking and bursting, which can be important from the point of view of information encoding in neurons, knowing the structure of periodic orbit itineraries is desirable. Moreover, the ability of a neuron to adjust its response (e.g. by modifying periodic or bursting pattern fired depending on perturbing the initial condition or the input as in case of α parameter in 1D CNV model) can be useful in solving various information processing tasks and storing temporal information in neuronal codes both at the level of a single cell as well as in neural networks ([8, 9, 11, 22]).

In the view of the above, it is essential to indicate complexity of the set of various spiking patterns displayed by a seemingly simple neuron model such as 1D CNV. In this model constant value of $y_n = \alpha$ can be interpreted as a constant internal stimulus. Regularity and periodicity of spike-trains (resulting from periodic orbit itineraries) is a measure of spike-timing precision in a response to a given stimulus. Simultaneously, the methods presented here can also be applied to other models, higher dimensional or hybrid neuron models, where sometimes spike-trains can be decoded via iterations of 1D map of Lorenz-type. A particular example is the work [20] which considers a bidimensional spiking neuron model yielding so-called adaptation map which can be Lorenz-like map. Iterates of the adaptation map and itineraries of its orbits are directly connected with signatures of mixed-mode oscillations which in turn contribute to the precision, timing, and robustness of neuronal spiking (see references therein). Thus, the results and methods developed here for the CNV model can be directly applied to obtain new results of the model investigated in [20] as well as in other models.

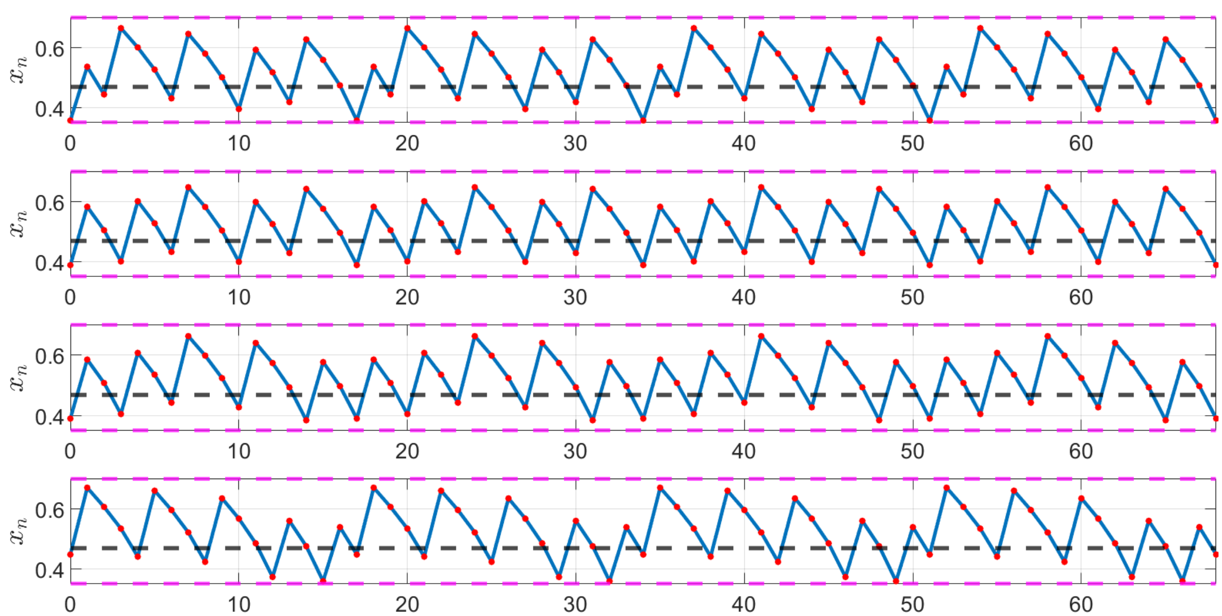

FIGURE 7. Four different time series of voltage traces corresponding to periodic orbits of 1D nlCNV model with period 17 and rotation number $\frac{12}{17}$ (TPO - second from the top, the remaining are NTPOs). Common parameter values: $\alpha = -0.082$, $\beta = 0.35$, $\mu = 1.62$, $a = 0.1$, $d = 0.47$. Initial points (from the top): 0.358; 0.389; 0.391; 0.449.

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Appendix A. Proof of Proposition 4.6 (1)

To prove Proposition 4.6 (1) we need some additional definitions and lemmas. Assume that $0 \leq a/p < b/q \leq 1$ are Farey neighbours and $\overline{S} = \{s_k\}_{k\in\mathbb{Z}}$ and $\overline{S}' = \{s'_k\}_{k\in\mathbb{Z}}$ are two periodic binary sequences corresponding to these Farey neighbours by the formulas

$$s_k = 0 \iff ka \pmod{p} < p - a \iff kaq \pmod{pq} < pq - aq,$$
$$s'_k = 0 \iff kb \pmod{q} < q - b \iff kbp \pmod{pq} < pq - bp.$$

We will abbreviate $x \pmod{n}$ to $(x)_n$. Let us introduce the temporary notation: $u_k = (ka)_p$, $v_k = (kb)_q$, $x_k = u_k q = (kaq)_{pq}$ and $y_k = v_k p = (kbp)_{pq}$. The next lemma follows immediately from definitions.

Lemma A.1. *For $k = 0, 1, \ldots, q-1$ we have $y_k = x_k + k$.*

Note that the equality from Lemma A.1 is not true for $k = q$. Now let us formulate the two main results of this appendix.

Lemma A.2. *Let $\lambda = pq - aq$ and $\mu = pq - bp$. Then for $k = 0, 1, \ldots, q-2$ we have $x_k < \lambda \iff y_k < \mu$.*

Moreover, it is easy to check that the equivalence is false for $k = q - 1$.

Proof. Assume that $x_k < \lambda$. Then $u_k < p - a$, i.e., $u_k \leq p - a - 1$ and hence

$$y_k = x_k + k = u_k q + k \leq (p - a - 1)q + q - 2 = pq - aq - 2 = \mu - 1 < \mu.$$

Similarly, assume that $y_k < \mu$, i.e., $v_k < q - b$. Then

$$x_k = y_k - k = v_k p - k < (q - b)p - k = pq - bp - k = \lambda - 1 - k < \lambda. \qquad \square$$

Lemma A.3. *Let $\lambda = aq$ and $\mu = bp$. Then for $k = 1, \ldots, q-1$ we have $x_k < \lambda \iff y_k < \mu$.*

Proof. The reasoning is nearly identical as in the previous proof. If $x_k < \lambda$ then $u_k < a$, i.e., $u_k \leq a - 1$ and

$$y_k = x_k + k = u_k q + k \leq (a - 1)q + k = aq - q + k < \mu,$$

because $k \leq q - 1$. In turn, if $y_k < \mu$ then $v_k < b$ and

$$x_k = y_k - k = v_k p - k < bp - k = aq + 1 - k \leq \lambda,$$

because $k \geq 1$. Moreover, the equivalence does not hold for $k = q$. $\square$

Proof of Proposition 4.6 (1). We will consider two cases.

Case I: $\boxed{0 \leq k \leq q - 2.}$ It follows immediately from Lemma A.2. Namely,

$$s_k = 0 \iff x_k < pq - aq \iff y_k < pq - bp \iff s'_k = 0.$$

Case II: $\boxed{1 - p \leq k \leq -1.}$ This case is a little harder. We start with a change of variables: $a' = p - a$ and $b' = q - b$. Note that $a/p < b/q$ implies $b'/q < a'/p$. Now applying Lemma A.3 to $b'/q < a'/p$ we see at once that

$$kb'p \pmod{pq} < b'p \iff ka'q \pmod{pq} < a'q \quad \text{for } 1 \leq k \leq p - 1.$$

Since $kb'p \pmod{pq} = -kbp \pmod{pq}$, $ka'q \pmod{pq} = -kaq \pmod{pq}$, $b'p = pq - bp$ and $a'q = pq - aq$, we finally obtain

$$s'_{-k} = 0 \iff -kbp \pmod{pq} < pq - bp \iff -kaq \pmod{pq} < pq - aq \iff s_{-k} = 0$$

for $1 \leq k \leq p - 1$, which is our claim. $\square$

Recall that for $0 \le p \le q$ coprime the (p,q)-*word* is a binary string of length q uniquely determined by the formula $(*)$ (see Section 2). In the remainder of this section we assume that $0 \le a/p < b/q \le 1$ are Farey neighbours. Let AB denote the concatenation of strings A and B.

Lemma A.4 (mediant lemma)**.** *If A is the (a,p)-word and B is the (b,q)-word, then AB is the $(a{+}b, p{+}q)$-word.*

Proof. Let C be the $(a+b, p+q)$-word. Note that the equality

$$s_k = 0 \iff s'_k = 0 \quad \text{for } 1-p \le k \le q-2, \tag{5}$$

obtained for the Farey pair $0 \le a/p < b/q \le 1$ in the proof of Proposition 4.6, can be applied to the Farey pairs $a/p < (a+b)/(p+q)$ and $(a+b)/(p+q) < b/q$. Applying (5) to $a/p < (a+b)/(p+q)$ we see that A covers the first p symbols of C. Similarly, applying (5) to $(a+b)/(p+q) < b/q$ we deduce that B covers the last q symbols of C. But C consists of $p+q$ symbols, so $C = AB$. □

Corollary A.5. *If A is the (a,p)-word and AB is the $(a+b, p+q)$-word, then B is the (b,q)-word.*

Proof. Let X be the (b,q)-word. By Lemma A.4, $AX = AB$ and, in consequence, $X = B$. □

Appendix B. Stern-Brocot arrays

Assume that $0 \le a/p < b/q \le 1$ are Farey neighbours. Their *mediant* is defined as $(a+b)/(p+q)$. It is easy to check that both $a/p < (a+b)/(p+q)$ and $(a+b)/(p+q) < b/q$ are also Farey neighbours. Following [13], let us define the *Stern-Brocot array of fractions* as the infinite multi-array of the form

$$\begin{array}{lllll} \frac{0}{1}, & \frac{1}{1} & & & \\ \frac{0}{1}, & \frac{1}{2}, & \frac{1}{1} & & \\ \frac{0}{1}, & \frac{1}{3}, & \frac{1}{2}, & \frac{2}{3}, & \frac{1}{1} \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

i.e., we start with two fractions $(\frac{0}{1}, \frac{1}{1})$ and then in each step we insert a mediant between any adjacent fractions from the previous step. It is proved in [13] that

- in each step we obtain a finite sequence, in which consecutive fractions are Farey neighbours,
- all possible irreducible fractions $0 \le m/n \le 1$ occur in this multi-array.

Of course, we can carry out the analogous construction starting from an arbitrary Farey pair $0 \le a/p < b/q \le 1$ and obtain this way the *Stern-Brocot subarray of fractions hooked at $a/p < b/q$*, which has the form

$$\begin{array}{lll} \frac{a}{p}, & \frac{b}{q} & \\ \frac{a}{p}, & \frac{a+b}{b+q}, & \frac{b}{q} \\ \dots & \dots & \dots \end{array}$$

Finally, replacing the operation of taking the mediant by the operation of concatenation of strings we obtain the *Stern-Brocot array of concatenations*. Namely, if we start with two symbols A and B, we get the multi-array

$$\begin{array}{lllll} A, & B & & & \\ A, & AB, & B & & \\ A, & AAB, & AB, & ABB, & B \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

Note that all these array structures can be represented as infinite binary trees (see [13] for details). Moreover, the arrangement of the Stern-Brocot arrays of fractions and concatenations establishes a bijective correspondence between them. We will call an arbitrary concatenation, which appears in the above Stern-Brocot array, a *Stern-Brocot concatenation*. As the following result shows they are closely related to (m,n)-words.

Proposition B.1. *The string X is a Stern-Brocot concatenation in the position corresponding to the position of the irreducible fraction $0 \leq m/n \leq 1$ if and only if X is the (m,n)-word with 0 replaced by A and 1 replaced by B.*

Proof. The induction is on the levels of the construction. If $k = 1$ then $X = A$ ($(0,1)$-word) or $X = B$ ($(1,1)$-word). Assume that the proposition is true for the level k. If X appears on the level $k+1$ then it is either an element from the level k (so we are done) or a concatenation of two consecutive strings from the level k corresponding to some Farey pair $a/p < b/q$. Hence, by Lemma A.4, X is the $(a+b, p+q)$-word with 0 replaced by A and 1 replaced by B. □

Appendix C. Two Lemmas on aperiodicity

The following two lemmas are needed in the proof of Proposition 5.8.

Lemma C.1 (insertions of repetitions)**.** *Let $k \geq 1$ be natural. Assume that the string X consisting of the symbols A and B is aperiodic and the string X' is obtained from X by simultaneously replacing A with A^k and B with B^k. Then X' is also aperiodic with respect to A and B.*

Proof. On the contrary, suppose that X' is periodic, i.e., $X' = Z^n$ for some $n > 1$. Note that Z must contain both A and B. Without loss of generality we can assume that X' starts from A. Let $Z = UVW$, where U and W contain only A and V starts and ends with B (W may be empty). Since Z is the beginning block of X', we have $U = (A^k)^l$ for some $l \geq 1$. Similarly, since Z is the ending block of X', $W = (A^k)^m$ for some $m \geq 0$. Also the blocks of A in V between consecutive B's are the repetition of A^k from the definition of X'. Hence Z is in fact a concatenation of A^k and B^k and substituting A for each A^k and B for each B^k in Z we obtain Y such that $X = Y^n$, which is impossible. □

Lemma C.2 (insertions of 0-1)**.** *Assume that*

- $0 \leq a/p < b/q \leq 1$ *are Farey neighbours,*
- *A is the (a,p)-word and B is the (b,q)-word,*
- *the string X of the symbols A and B is aperiodic with respect to A and B,*
- *the string X' of the symbols 0 and 1 is obtained from X by inserting in place of A and B their zero-one expansions ((m,n)-words).*

Then X' is aperiodic with respect to 0 and 1.

Proof. The proof is by induction on k (i.e., $p, q \leq k$). For $k = 1$ we must have $a/p = 0/1$ and $b/q = 1/1$. Hence $A = 0$ and $B = 1$ and, in consequence $X = X'$, which immediately gives our assertion. Assume that the lemma is true for k. We will prove it for $p, q \leq k+1$. Let $p < q$ (the opposite case is analogous). Then $B = AC$ and, by Corollary A.5, C is the $(b-a, q-p)$-word. Let $\overline{X}$ be the string formed from X by replacing all instances of B with AC. We show that $\overline{X}$ is aperiodic with respect to A and C. Conversely, suppose that $\overline{X} = \overline{Y}^n$ for some $n > 1$. Note that B's in X correspond bijectively to C's in $\overline{X}$. Moreover, since $\overline{Y}$ starts with A and each occurrence of C in $\overline{Y}$ is preceded by A, if we replace in $\overline{Y}$ each AC by B, we obtain Y such that $X = Y^n$, a contradiction. Hence $\overline{X}$ is aperiodic with respect to A and C, a/p and $(b-a)/(q-p)$ are Farey neighbours and $p, q-p \leq k$. By the induction assumption, the binary string $\overline{X}'$ obtained from $\overline{X}$ by replacing A and C with the corresponding (m,n)-words is aperiodic with respect to 0 and 1. But, obviously, $\overline{X}' = X'$, which completes the proof. □

Remark C.3. Note that, in general, it is not true that inserting aperiodic binary strings into an aperiodic pattern produces an aperiodic binary result. For example, consider the binary strings $A = 1001$, $B = 00100$ and the pattern $X = AB$.

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