Discrete Applied Mathematics 182 (2015) 61–72

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

New potential functions for greedy independence and coloring[☆]

Piotr Borowiecki[a,*], Dieter Rautenbach[b]

[a] Department of Algorithms and System Modeling, Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

[b] Institut für Optimierung und Operations Research, Universität Ulm, D-89069 Ulm, Germany

ARTICLE INFO

Article history:
Received 19 January 2013
Received in revised form 30 August 2013
Accepted 18 December 2013
Available online 4 January 2014

Keywords:
Independent set
Graph coloring
Greedy algorithm
Grundy number
Vertex degree

ABSTRACT

A potential function f_G of a finite, simple and undirected graph $G = (V, E)$ is an arbitrary function $f_G : V(G) \to \mathbb{N}_0$ that assigns a nonnegative integer to every vertex of a graph G. In this paper we define the iterative process of computing the *step potential function* q_G such that $q_G(v) \le d_G(v)$ for all $v \in V(G)$. We use this function in the development of new Caro–Wei-type and Brooks-type bounds for the independence number $\alpha(G)$ and the Grundy number $\Gamma(G)$. In particular, we prove that $\Gamma(G) \le Q(G) + 1$, where $Q(G) = \max\{q_G(v) \mid v \in V(G)\}$ and $\alpha(G) \ge \sum_{v \in V(G)} (q_G(v)+1)^{-1}$. This also establishes new bounds for the number of colors used by the algorithm `Greedy` and the size of an independent set generated by a suitably modified version of the classical algorithm `GreedyMAX`.

1. Introduction

In this paper we investigate applications of certain potential functions in the development of new Caro–Wei-type and Brooks-type bounds for the two classical problems of discrete optimization, the maximum independent set problem (`MIS`) and the problem of vertex coloring (`COLORING`). Both problems are considered for simple, finite and undirected graphs $G = (V, E)$ with the vertex set V, edge set E and of order $n = |V(G)|$. For a given graph G we say that a set of vertices I, $I \subseteq V(G)$ is *independent* if it does not contain any pair of adjacent vertices. The `MIS` problem is to find an independent set with the aim of maximizing its cardinality. The *independence number* $\alpha(G)$ of a graph G is defined as the largest cardinality of an independent set in G. The `COLORING` problem is closely related to `MIS` and can be defined as the problem of finding a partition of the vertex set of a graph into the minimum number of independent sets. The least number of sets in such a partition is called the *chromatic number* $\chi(G)$ of a graph G. Alternatively, coloring can also be viewed as a function $c : V(G) \to \mathbb{N}$ such that $c(u) \ne c(v)$ for all $uv \in E(G)$.

Both `MIS` and `COLORING` has gained a significant interest in theoretical investigations and in the context of various applications ranging from distributed computing, data mining and database design to image processing, frequency assignment and scheduling. In the context of applications a graph G is often a conflict graph with vertices representing the appropriate objects and edges uv to express that objects represented by u and v are in conflict. The goal in the `MIS` problem is to select as many desirable objects as possible while never selecting the conflicting ones. In `COLORING` we aim at partitioning all vertices of a graph into the smallest number of sets consisting of nonconflicting objects.

☆ A preliminary version of Section 2.1 appeared in Proc. SOFSEM 2011, LNCS 6543 (2011).

* Corresponding author. Tel.: +48 58 347 19 56; fax: +48 58 347 17 66.
E-mail addresses: pborowie@eti.pg.gda.pl (P. Borowiecki), dieter.rautenbach@uni-ulm.de (D. Rautenbach).

0166-218X/$ – see front matter © 2014 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.dam.2013.12.011

Both problems are known to be $\mathbb{NP}$-hard [18] and hard to approximate within $n^{1-\varepsilon}$ under a widely believed assumption that $\mathbb{ZPP} \neq \mathbb{NP}$ [16,27]. Various exponential-time exact algorithms were proposed in the literature, e.g., for MIS the algorithm of Robson [40] runs in $O(2^{0.25n})$ time using an exponential space. A slightly slower, but much simpler $O(2^{0.288n})$-time algorithm was proposed by Fomin et al. [17]. Currently the fastest $O(2.2461^n)$-time polynomial space algorithm for computing the chromatic number of a graph is the algorithm developed by Björklund et al. [4]. In view of the computational hardness a lot of effort was put into establishing the bounds on $\alpha(G)$ and $\chi(G)$ as well as into analysis of approximation algorithms (see e.g., [5,22,38]) and exploring the boundary between hard and polynomially solvable cases (see e.g., [2,23,25,29,30,35]). Since the subject is too wide to be surveyed in a short paper, we refer the reader interested in particular aspects of MIS and COLORING to [31,33,34]. In the sequel we focus on the two polynomial-time heuristics. We develop new bounds for the performance of the classical algorithm Greedy for COLORING and a suitably adapted version of the algorithm GreedyMAX for the MIS problem. We start our investigations with the following well-known bounds: the Brooks bound [10]

$$\chi(G) \leq \Delta(G) + 1, \tag{1}$$

and the bound discovered independently by Caro [11] and Wei [45] (known as the Caro–Wei bound)

$$\alpha(G) \geq \mathrm{CW}(G, d_G) = \sum_{v \in V(G)} \frac{1}{d_G(v) + 1}, \tag{2}$$

where $d_G(v)$ is the degree of a vertex v in the graph G, while $\Delta(G) = \max_{v \in V(G)} d_G(v)$. The Caro–Wei bound extends the classical result of Turán [44] $\alpha(G) \geq n/(\bar{d}(G) + 1)$ stated in terms of graph's order and its average degree $\bar{d}(G)$. It has also been proved using probabilistic methods, e.g., by Alon and Spencer [3] and Selkow [41], while Griggs [20] as well as Chvátal and McDiarmid [14] showed that GreedyMAX (selecting a vertex of maximum degree, deleting it with all incident edges from the graph and iterating this process on the resulting graph until no edge remains) outputs an independent set of size at least $\mathrm{CW}(G, d_G)$. Several modifications of bounds (1) and (2), given in terms of various invariants are known (see e.g., [6,42,47,8,20,26,24,37,41], to mention just a few). Our new Brooks-type and Caro–Wei-type bounds are stated in terms of the carefully constructed step potential function that is the key notion of this paper.

Namely, a *potential function* f_G of a graph $G = (V, E)$ is an arbitrary function $f_G : V(G) \to \mathbb{N}_0$ that assigns to every vertex v of G some nonnegative integer value $f_G(v)$. The most natural example of a potential function can be obtained by choosing $f_G(v) = d_G(v)$. Several potential functions were introduced and analyzed in [7,8]. A function p_G defined by Borowiecki and Göring [7] will be also used in this paper, while in Section 2, after introduction of step sequences, we define an iterative process of calculating the *step potential function* q_G. The potential functions considered in this paper satisfy

$$q_G(v) \leq p_G(v) \leq d_G(v)$$

for every vertex $v \in V(G)$. In particular, if $Q(G)$ and $P(G)$ denote the maxima of q_G and p_G, taken over $V(G)$, respectively, then $Q(G) \leq P(G) \leq \Delta(G)$ holds for every graph G. Since potential function f_G will be usually clear from the context, we simply call $f_G(v)$ the *potential* of a vertex v.

If c is a coloring in which for every two colors i, j, with $i < j$, every vertex colored j has a neighbor colored i, then c is called a *Grundy coloring*. The largest number of colors for which there exists a Grundy coloring of G is called the *Grundy number* $\Gamma(G)$ of a graph G. The notion of the Grundy number is usually attributed to Christen and Selkow [12], and it is well known that Grundy colorings are exactly the colorings produced by the algorithm Greedy, which colors every vertex with the smallest possible color. Consequently,

$$\Gamma(G) \leq \Delta(G) + 1. \tag{3}$$

When a graph G and an integer k are part of the input, the problem of deciding whether $\Gamma(G) \geq k$ is known to be $\mathbb{NP}$-complete [19,36] and it remains so, even if we consider bipartite graphs [28] or their complements [46]. On the other hand, by the finite basis theorem of Gyárfás et al. [21], the problem is polynomially solvable when k is fixed (see also [9] for the results on Grundy k-critical graphs). In Section 3 we strengthen (3) and the bound of Zaker [47] by proving that

$$\Gamma(G) \leq Q(G) + 1. \tag{4}$$

Moreover, we argue that for almost all graphs $Q(G) < \Delta(G)$. The graphs for which $\chi(G) = Q(G) + 1$ constitute a nontrivial class of graphs optimally colorable by Greedy. We show that deciding whether $\chi(G) \leq Q(G)$ is $\mathbb{NP}$-complete, when $Q(G) \geq 3$.

Since the independence number of G is at least $\alpha(G) \geq n/\chi(G)$, by (4) we immediately obtain $\alpha(G) \geq \sum_{v \in V(G)} (Q(G) + 1)^{-1}$. An even stronger bound is proved in Section 4. Namely,

$$\alpha(G) \geq \mathrm{CW}(G, q_G) = \sum_{v \in V(G)} \frac{1}{q_G(v) + 1}. \tag{5}$$

In the same section we prove that every graph G contains a vertex critical with respect to q_G and that the `GreedyMAX`-type algorithm that in each step removes such a vertex, outputs an independent set of order at least $\mathrm{CW}(G, q_G)$. This strengthens Caro–Wei bound (2) as well as the corresponding result on $\mathrm{CW}(G, p_G)$ proved in [7].

In contrast to the bounds of Caro, Wei and Brooks, which strongly rely on counting the neighbors of a vertex, the potential functions give us a closer insight into the properties of the neighborhoods. The main advantage of such an approach is that the potential of a vertex lets us incorporate both quantitative and qualitative aspects in algorithms' performance analysis.

2. Step potential of a graph

Throughout the paper we use the notions of a finite multiset of integers and a finite nondecreasing sequence of integers interchangeably. If A is a finite multiset of integers such that $a_1 < a_2 < \cdots < a_p$ are the distinct elements of A and m_i is the multiplicity of a_i in A for $i \in \{1, \dots, p\}$, then A corresponds to the finite nondecreasing sequence

$$(\underbrace{a_1, \dots, a_1}_{m_1}, \underbrace{a_2, \dots, a_2}_{m_2}, \dots, \underbrace{a_p, \dots, a_p}_{m_p}).$$

Conversely, if $S = (s_1, s_2, \dots, s_k)$ is a finite nondecreasing sequence of integers such that

$$\begin{aligned} s_1 &= \cdots = s_{m_1} < s_{m_1+1} = \cdots = s_{m_1+m_2} \\ &< s_{m_1+m_2+1} = \cdots = s_{m_1+m_2+m_3} \\ &< \cdots \\ &< s_{m_1+\cdots+m_{p-1}+1} = \cdots = s_k \end{aligned}$$

and $m_p = k - (m_1 + \cdots + m_{p-1})$, then S corresponds to the finite multiset whose *distinct* elements are $s_{m_1} < s_{m_1+m_2} < \cdots < s_{m_1+\cdots+m_p}$ such that m_i is the multiplicity of $s_{m_1+\cdots+m_i}$ for all $i \in \{1, \dots, p\}$. If A and B are finite multisets of integers, then the cardinality $|A|$ of A is the sum of the multiplicities of all elements in A, while $B \subseteq A$ means that for every element x of B, the multiplicity of x in A is at least the multiplicity of x in B. Using these conventions we write "$S \subseteq A$", where S is a finite nondecreasing sequence of integers and A is a finite multiset of integers, meaning that $B \subseteq A$, where B is the multiset corresponding to S, in the sense explained above.

2.1. Step sequences

Let $A \subset \mathbb{N}$ be a multiset. We say that the nondecreasing sequence $S = (s_1, \dots, s_k)$, $S \subseteq A$ is a *step sequence in* A if for each $i \in \{1, \dots, k\}$ it holds that $s_i \geq i$. A step sequence of length k is called a *k-step sequence*. We say that a k-step sequence is *maximal in* A if there does not exist $k_1 > k$ such that A contains a k_1-step sequence. A maximal k-step sequence with the largest sum of elements is called *maximum in* A and it is denoted by $S^A_{\max}$, while $S^A_{\min}$ stands for a maximal k-step sequence with the smallest sum, and is called *minimum in* A. We say that a k-step sequence S is *saturating in* A if $k = |A|$. Otherwise, we say that S is *nonsaturating*. It follows by the maximality that for every nonsaturating maximal k-step sequence S there exists at least one element $s_i \in S$ such that $s_i = i$. An element $s_i \in S$ for which $s_i = i$ is called a *blocking element* in S.

Lemma 1. *Let S and S' be arbitrary k-step sequences maximal in A and let b, b' be the values of the largest blocking elements in S and S', respectively. Then $b = b'$.*

Proof. The lemma clearly holds whenever S and S' are saturating in A. Let $X = A \setminus S$, $M = \{s \in S \mid s \leq b\}$ and $L = S \setminus M$. The sets X', M' and L' for S' are defined analogously.

Assume that $b > b'$. If there existed an $a \in L$ such that $a \notin L'$, then S' would not be maximal in A (consider a $(k+1)$-step sequence obtained from S' by inserting a just after b'). Hence $L \subseteq L'$. Let $C = L' \setminus L$. If C were not empty, then since b' is the largest blocking element in S', C would have to contain an element $c > b$. Obviously $C \subseteq X \cup M$ and if $X \cup M$ contained an element $c > b$, S would not be maximal in A (consider a $(k+1)$-step sequence obtained from S by inserting c just after b). Therefore, $b' \geq b$.

Since by symmetry it follows that $b' \leq b$, we finally get $b' = b$. □

In what follows we use $b^{\max}_A$ to denote the value of the largest blocking element for step sequences maximal in A. We also use $e^{\max}_A$ for the value of the largest element of $X = A \setminus S^A_{\min}$, when $X \neq \emptyset$, and we assume that $e^{\max}_A = 0$ if $X = \emptyset$.

Example 1. If $A = \{1, 2, 3, 3, 3, 5, 5, 8\}$, then $S^A_{\max} = (3, 3, 3, 5, 5, 8)$, $b^{\max}_A = 5$, while $S^A_{\min} = (1, 2, 3, 5, 5, 8)$, $A \setminus S^A_{\min} = \{3, 3\}$ and consequently $e^{\max}_A = 3$.

Now, we prove the two complementary lemmas on extending and shortening of step sequences.

Lemma 2. *If $A' = A \cup \{t\}$, then every $(k+1)$-step sequence S' maximal in A' contains t if and only if A contains a maximal k-step sequence S such that exactly one of the following conditions is satisfied:*

(a) *S contains at least one blocking element and $t > b^{\max}_A$,*
(b) *S does not contain blocking elements.*

Proof. ($\Rightarrow$) Let S' be a $(k+1)$-step sequence maximal in A' and let $t \in S'$. Moreover, let sequence $S' = (s'_1, \ldots, s'_i, t, s'_{i+2}, \ldots, s'_k, s'_{k+1})$ be such that t has the smallest index, say $i+1$. Consequently, $i \le s'_i < i+1 \le t$. Hence, elements of S' satisfy $s'_i = i$, $t \ge i+1$ and $s'_j \ge j$ for $j \in \{i+2, \ldots, k+1\}$. Now, consider the sequence $S = (s_1, s_2, \ldots, s_i, s_{i+1}, \ldots, s_{k-1}, s_k)$, with $s_j = s'_j$ for $j \in \{1, \ldots, i\}$, and $s_j = s'_{j+1}$ when $j \in \{i+1, \ldots, k\}$. Observe that all elements of S satisfy the following conditions: $s_j \ge j$ for $j \in \{1, \ldots, i\}$ with $s_i = i$, and $s_j = s'_{j+1} \ge j+1$ implying $s_j > j$ when $j \in \{i+1, \ldots, k\}$. It follows that s_i is the largest blocking element in S and $t > b_A^{\max}$. If t has index 1 in S', then S does not contain blocking elements. Maximality of S follows from the maximality of S'.

($\Leftarrow$) Assume that A contains a maximal k-step sequence S containing no blocking elements. Then for every element $s_i \in S,\ i \in \{1, \ldots, k\}$ we have $s_i - i \ge 1$. Hence, every sequence S' obtained from S by inserting t, independently of the index of t, is a $(k+1)$-step sequence in A'. Now, let $s_j \in S$ be the largest blocking element. Then similarly, for any $s_i \in S,\ i \in \{j+1, \ldots, k\}$ we have $s_i - i \ge 1$. Hence, any sequence $S' = (s_1, \ldots, s_j, t, s_{j+1}, \ldots, s_k)$ is a $(k+1)$-step sequence in A'. □

Example 2. Let $A = \{1, 2, 3, 3, 3, 5, 5, 8\}$ and recall that by Example 1 the maximum length of a step sequence in A is 6. Assume that $A' = A \cup \{t\}$ and consider $t > b_A^{\max} = 5$, e.g., $t = 6$. Then A' contains a maximal 7-step sequence $S_{\max}^{A'} = (3, 3, 3, 5, 5, 6, 8)$ but for all $t \le b_A^{\max}$ any 6-step sequence is maximal in A'.

Lemma 3. *Let A contain a maximal k-step sequence S. If $A' = A \setminus \{t\}$, then A' contains a maximal $(k-1)$-step sequence if and only if $t > e_A^{\max}$.*

Proof. Let S be any k-step sequence that is saturating in A. Then $e_A^{\max} = 0$ and for every t, $e_A^{\max} < t$. On the other hand, removal of any element t from S directly results in the $(k-1)$-step sequence S' maximal in A'. Now, assume that S is any k-step sequence that is nonsaturating in A.

($\Rightarrow$) On the contrary assume that A' was obtained by removal of $t \le e_A^{\max}$ from sequence $S = (s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_k)$. Let $S' = S \setminus \{t\}$. If $t \notin S_{\min}^A$, then $S_{\min}^A \subseteq A'$, a contradiction. If $t \in S_{\min}^A$, then there exists an element $t' \in A \setminus S_{\min}^A$, $t' = e_A^{\max}$ such that $S' = (s_1, \ldots, s_{i-1}, t', s_{i+1}, \ldots, s_k)$ is a k-step sequence in A, a contradiction.

($\Leftarrow$) It is enough to see that all elements $t > e_A^{\max}$ must belong to every k-step sequence S maximal in A or in other words no element taken from $A \setminus S$ can replace t without decreasing k. □

Example 3. Let $A = \{1, 2, 3, 3, 3, 5, 5, 8\}$ and recall that by Example 1, $e_A^{\max} = 3$. Assume that $A' = A \setminus \{t\}$ and consider $t > e_A^{\max}$, e.g., $t = 5$. Then a 5-step sequence $S_{\min}^{A'} = (1, 2, 3, 5, 8)$ is maximal in A', while for any $t \le e_A^{\max}$, e.g., $t = 2$, a 6-step sequence still exists, e.g., $S_{\max}^{A'} = (3, 3, 3, 5, 5, 8)$.

2.2. Step potential function

In order to define the step potential function of an n-vertex graph G we consider the iterative process that starts with the initial vector $\mathbf{q}^{(0)} = (d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$. In the process of calculating subsequent vectors $\mathbf{q}^{(j)} = (q^{(j)}(v_1), q^{(j)}(v_2), \ldots, q^{(j)}(v_n))$ every element $q^{(j)}(v_i),\ i \in \{1, \ldots, n\}$ is determined as the maximum k for which there exists a k-step sequence in $\{q^{(j-1)}(u) \mid u \in N_G(v_i)\}$, where $N_G(v_i)$ is the set of the neighbors of a vertex v_i. The process continues until $\mathbf{q}^{(t)} = \mathbf{q}^{(t-1)}$ for some $t > 0$. As we will prove later the process is always finite.

Definition 1. A function $q_G : V(G) \to \{q^{(t)}(v_1), \ldots, q^{(t)}(v_n)\}$ such that $q_G(v_i) = q^{(t)}(v_i),\ i \in \{1, \ldots, n\}$ is called the *step potential function* of a graph G, while $q_G(v_i)$ is called the *step potential of a vertex* v_i. The maximum value of q_G on $V(G)$ is called the *step potential of a graph* G and we denote it by $Q(G)$.

Observe that, according to the definition of the iterative process, finding a maximal k-step sequence in $\{q^{(j-1)}(u) \mid u \in N_G(v_i)\}$, during the jth iteration of the process, can be realized in polynomial time, e.g., by greedily qualifying subsequent neighbors of a vertex v_i in nondecreasing order of $q^{(j-1)}(u)$. See Algorithm 1 for a pseudocode of the algorithm that efficiently computes the step potential function of a given graph G.

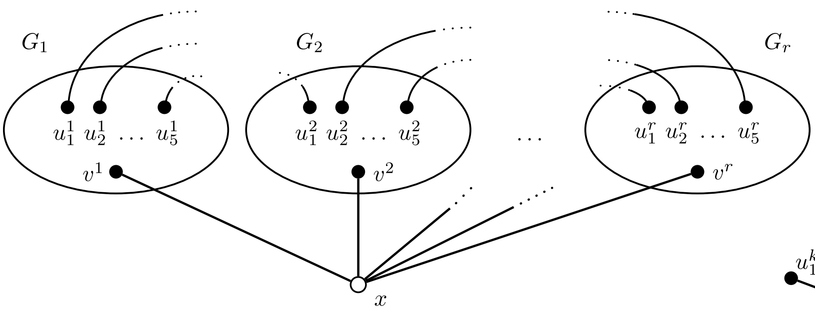
Example 4. Let us consider an example in Fig. 1, which presents the execution of Algorithm 1 for a graph constructed by taking an even number r of disjoint copies of a gadget G_k, adding a vertex x joined with the central vertex v^k of each gadget and $5r/2$ edges between vertices $u_i^{k_1}, u_j^{k_2}$ of the gadgets G_{k_1} and G_{k_2}, so that all vertices $u_i^k,\ i \in \{1, \ldots, 5\}$ of every gadget G_k, $k \in \{1, \ldots, r\}$ have degree two. Now, let us sketch the main calculations that take place during the iterative process. After the first step of the process $q^{(1)}(x) = 6$, since x has r neighbors of degree 6. Similarly, for every gadget G_k, $q^{(1)}(w_i^k) = 3$ for all $i \in \{1, \ldots, 5\}$. This in turn, according to the values of $\mathbf{q}^{(1)}$ in the neighborhood of v^k results in $q^{(2)}(v^k) = 4$, after the second step of the process. Finally, for the vertex x we get $q^{(3)}(x) = 4$.

Proposition 1. *Let G be an arbitrary graph. Then the iterative process of calculating the step potential function q_G converges in a finite number t of steps. Moreover, for each $j \in \{0, \ldots, t\}$ and for every vertex v of G it holds that $\delta(G) \le q^{(j)}(v) \le \Delta(G)$, where $\delta(G) = \min_{v \in V(G)} d_G(v)$.*

Algorithm 1 Calculating the step potential function of a graph

`Input:` G – a simple undirected graph
`Output:` $\mathbf{q}$ – a vector of values of the step potential function

`Begin`
  $\mathbf{x} \leftarrow (d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$
  `repeat`
    $\mathbf{q} \leftarrow \mathbf{x}$
    `for` $i = 1, \ldots, n$ `do`
      find a k-step sequence that is maximal in $\{q(u) \mid u \in N_G(v_i)\}$
      $x(v_i) \leftarrow k$
    `end`
  `until` $\mathbf{q} = \mathbf{x}$
`End`



j	$q^{(j)}(x)$	$q^{(j)}(v^k)$	$q^{(j)}(u_i^k)$	$q^{(j)}(w_i^k)$	$q^{(j)}(z_i^k)$
0	r	6	2	4	2
1	6	5	2	3	2
2	5	4	2	3	2
3	4	4	2	3	2

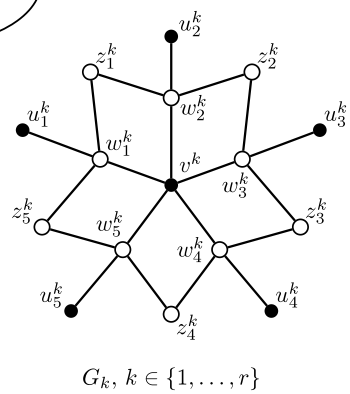


Fig. 1. An example of calculating the step potential function of a graph.

Proof. An upper bound $q_G(v) \leq \Delta(G)$ follows directly from the definition of the iterative process. For a lower bound observe that initially $q^{(0)}(v) = d_G(v) \geq \delta(G)$ holds for all vertices. Suppose that x is the first vertex that after jth iteration satisfied $q^{(j)}(x) = k$, for some $k < \delta(G)$. Since $d_G(x) \geq \delta(G)$, the vertex x has at least $r \geq \delta(G)$ neighbors $\{u_1, \ldots, u_r\}$ and for all of them $q^{(j-1)}(u_i) \geq \delta(G)$. Hence $q^{(j)}(x) \geq \delta(G) > k$, a contradiction. □

Clearly, the vector $\mathbf{q}^{(t)}$ is uniquely determined for every graph G. It is also not hard to see that the functions $q_G^{(j)}$ defined by vectors $\mathbf{q}^{(j)}$, $j \in \{0, \ldots, t\}$, are monotone with respect to taking subgraphs.

Proposition 2. *If $H \subseteq G$, then for each $j \in \{0, \ldots, t\}$ and every vertex $v \in V(H)$ it holds that*

$$q_H^{(j)}(v) \leq q_G^{(j)}(v).$$

Proof. We prove the statement by induction on j. For $j = 0$ it follows immediately from $d_H(v) \leq d_G(v)$. Now, let $j \geq 1$. If $q_H^{(j)}(v) = k$, then $v_1, \ldots, v_k$ are distinct neighbors of v in H with $q_H^{(j-1)}(v_i) \geq i$ for each $i \in \{1, \ldots, k\}$. Hence, by induction, $q_G^{(j-1)}(v_i) \geq i$ for each $i \in \{1, \ldots, k\}$, which implies $q_G^{(j)}(v) \geq k$. □

Szekeres and Wilf [43] proved that whenever λ is a real valued function on the family of all simple graphs such that for every simple graph G it holds that $\delta(G) \leq \lambda(G)$ and λ is monotone with respect to taking induced subgraphs of G, then $\chi(G) \leq \lambda(G) + 1$. Consequently, by Propositions 1 and 2 we directly obtain the following bound:

$$\chi(G) \leq Q(G) + 1. \quad (6)$$

We further strengthen this bound in the next section.

In subsequent sections we will also refer to the function p_G, introduced in [7], which we call here the *simple step potential* function. The function p_G is equivalent to the function $q_G^{(1)}$ defined by the vector $\mathbf{q}^{(1)}$. The maximum value of p_G on $V(G)$, denoted by $P(G)$, will be called the *simple step potential of a graph G*. Naturally, for every vertex $v \in V(G)$, it holds that

$$q_G(v) \leq p_G(v) \leq d_G(v).$$

In what follows we will be interested in the properties of both functions q_G and p_G. The motivation comes from the fact that Caro–Wei and Brooks-type bounds, in which degrees of vertices are replaced with their potentials, are provably not worse than the original bounds. Moreover, they seem to be better for almost all graphs, even for p_G.

Theorem 1. *For almost all graphs* $P(G) < \Delta(G)$.

Proof. By the definition of the simple potential p_G, every vertex v has a neighbor u such that $p_G(u) \geq p_G(v)$. Hence, every graph G containing at least one edge has a pair of adjacent vertices v_1, v_2 for which $p_G(v_1) = p_G(v_2) = P(G)$. In particular, if $P(G) = \Delta(G)$, then $p_G(v_i) = d_G(v_i) = \Delta(G),\ i \in \{1, 2\}$. Hence, both vertices v_1, v_2 have degree $\Delta(G)$. However, the result of Erdős and Wilson [15] says that almost all graphs have only one vertex of maximum degree. □

3. Step potential and the Grundy number

An interesting variation on the notion of the maximum degree of a graph was introduced by Stacho [42]. Namely,

$$\Delta_2(G) = \max_{v \in V(G)} \max_{u \in N^{\leq}(v)} d_G(u),$$

where $N^{\leq}(v) = \{u \in N_G(v) \mid d_G(u) \leq d_G(v)\}$. Clearly, $\Delta_2(G) \leq \Delta(G)$. In the same paper Stacho used Δ_2 to prove a new Brooks-type bound on the chromatic number.

Theorem 2 (*Stacho [42]*). *For every graph G we have*

$$\chi(G) \leq \Delta_2(G) + 1. \tag{7}$$

Stacho's result was further improved by Zaker [47]. Motivated by computational hardness of determining the Grundy number, Zaker proved the following theorem.

Theorem 3 (*Zaker [47]*). *For every graph G we have*

$$\Gamma(G) \leq \Delta_2(G) + 1. \tag{8}$$

In order to compare $\Delta_2(G)$ and $P(G)$ we prove the following two propositions.

Proposition 3. *For every graph G it holds that*

$$P(G) \leq \Delta_2(G).$$

Proof. For an arbitrary vertex v let $d_2(v) = \max_{u \in N^{\leq}(v)} d_G(u)$ and let $k = p_G(v)$. From the definition of the simple step potential it follows that $k \leq d_G(v)$ and there exists $u \in N_G(v)$ such that $d_G(u) \geq k$. Following degree condition in the definition of d_2 we have to investigate two cases. If $d_G(u) \leq d_G(v)$, then $d_2(v) \geq d_G(u) \geq k$. On the other hand, if $d_G(u) > d_G(v)$, then $d_2(u) \geq d_G(v) \geq k$. Thus, for every vertex v we have $d_2(v) \geq p_G(v)$ or v has a neighbor u such that $d_2(u) \geq p_G(v)$. □

Proposition 4. *For every integer* $\eta > 0$ *there exists a connected graph G such that*

$$\Delta_2(G) - P(G) > \eta.$$

Proof. Consider the following two graphs: $H = K_2 + rK_k$, i.e., a join of r independent copies of a complete graph K_k with the complete graph K_2, and the graph H' obtained from two stars $K_{1,k}$ by joining their centers by an edge. Clearly, $\Delta_2(H) = n - 1$ and $P(H) = k + 2$, while $\Delta_2(H') = k + 1$ and $P(H') = 2$. □

The following bound is a strengthening of (6)–(8). Also, note that by Theorem 1 for almost all graphs our new bound is better than (3).

Theorem 4. *For every graph G it holds that*

$$\Gamma(G) \leq Q(G) + 1. \tag{9}$$

Proof. We say that a vertex v is *terminal* if for a Grundy coloring c for all $u \in N_G(v)$ it holds that $c(u) < c(v)$. Otherwise v is *nonterminal*. Let $k = \Gamma(G)$ and let H be an induced Grundy k-critical subgraph of G, i.e., a subgraph such that $\Gamma(H) = k$, but for every $v \in V(H)$, $\Gamma(H - v) < k$. The color classes $(V_1, \ldots, V_k)$ of any Grundy k-coloring of H satisfy the following conditions:

(a) V_k consists of a singleton terminal vertex v_k,
(b) for every $i \in \{1, \dots, k\}$, every vertex in V_i has a neighbor in every $V_j, j \in \{1, \dots, i\}$,
(c) $V_1 \cup \dots \cup V_{k-1}$ contains only nonterminal vertices.

Claim 1. *For every $r \in \mathbb{N}_0$,*

(1) $q_H^{(r)}(v_k) \geq k-1$ *and*
(2) $q_H^{(r)}(v) \geq i$ *for every vertex* $v \in V_i,\ i \in \{1, \dots, k-1\}$.

Proof. The proof is by induction on r. For $r = 0$ we have $q_H^{(0)}(v_k) = d_H(v_k) \geq k-1$ for the terminal vertex v_k and $q_H^{(0)}(v) = d_H(v) \geq i$ for every $i \in \{1, \dots, k-1\}$ and $v \in V_i$ (note that in the latter case all vertices are nonterminal). Now, assume that $r \geq 1$. If $v \in V_i$ for some $i \in \{1, \dots, k-2\}$, then let $u_j \in V_j$ be a neighbor of v in H for $j \in \{1, \dots, i-1\} \cup \{i+1\}$. By induction, $q_H^{(r-1)}(u_j) \geq j$ for every $j \in \{1, \dots, i-1\} \cup \{i+1\}$, which implies that $q_H^{(r)}(v) \geq i$. If $v \in V_{k-1}$, then let $u_j \in V_j \cap N_H(v)$ for $j \in \{1, \dots, k-2\}$. Since v is nonterminal and $c(v) = k-1$, v_k must be a neighbor of v. By induction, $q_H^{(r-1)}(u_j) \geq j$ for every $j \in \{1, \dots, k-2\}$ and $q_H^{(r-1)}(v_k) \geq k-1$, which implies that $q_H^{(r)}(v) \geq k-1$. Finally, if $v = v_k$, then let $u_j \in V_j \cap N_H(v)$ for $j \in \{1, \dots, k-1\}$. By induction, $q_H^{(r-1)}(u_j) \geq j$ for every $j \in \{1, \dots, k-1\}$, which implies $q_H^{(r)}(v) \geq k-1$. This completes the proof of the claim. □

By the claim, $Q(H) \geq q_H(v_k) \geq k-1$ and hence by Proposition 2 we get $\Gamma(G) \leq Q(G) + 1$. □

By the theorem of Brooks $\chi(G) = \Delta(G) + 1$ holds if and only if some component of G is a $(\Delta(G)+1)$-clique or $\Delta(G) = 2$ and G is not bipartite. Considering an analogous equality for a step potential, observe that whenever $\chi(G) = Q(G) + 1$, then by (9) it holds that $\Gamma(G) = \chi(G)$, i.e., a graph G is optimally colorable by `Greedy`. Consequently, if there exists an ordering of vertices for which `Greedy` produces a nonoptimal coloring, then $\chi(G) \leq Q(G)$. In contrast to Brooks' graphs, which can be recognized in polynomial time, it was recently proved by Zhu [48] that determining whether a graph G has the chromatic number smaller than its coloring number is $\mathbb{NP}$-complete, while the reduction proposed by Stacho [42] can be used to prove $\mathbb{NP}$-completeness of deciding whether $\chi(G) \leq Q(G)$. We include the adapted proof to keep the paper self-contained.

Theorem 5. *If $Q(G) \geq 3$, then it is $\mathbb{NP}$-complete to determine whether $\chi(G) \leq Q(G)$.*

Proof. Let $k \geq 3$ be an integer and let G be an arbitrary graph of order n. Given a graph G one can construct in polynomial time a graph G' such that $Q(G') = k$, and such that $\chi(G') \leq k$ if and only if $\chi(G) \leq k$. To construct G' proceed as follows:

(a) Set $V(G') = \bigcup_{uv \in E(G)} C_{uv} \cup W$, where each set C_{uv} corresponds to an edge $uv \in E(G)$, $|C_{uv}| = k$, while every vertex $w_i \in W$ corresponds to the appropriate vertex $v_i \in V(G),\ i \in \{1, \dots, n\}$.
(b) Add edges between vertices of $V(G')$ so that each C_{uv} induces a clique of order k, each vertex $x \in C_{uv}$ is a neighbor of either u or v, and both u and v have at least one neighbor in C_{uv}.

Observe that each vertex $x \in C_{uv}$ has $k-1$ neighbors of degree k and one neighbor of degree at least 1. Hence, $q_{G'}(x) = k$. On the other hand, no vertex $w \in W$ has a neighbor of degree greater than k and hence $q_{G'}(w) \leq k$. Consequently, $Q(G') = k$. To see that given a k-coloring c of G one can obtain a k-coloring of G', color all $w_i \in W$ with the same colors as the corresponding $v_i \in V(G)$ and since there are at least two vertices $x_1, x_2 \in C_{uv}$ such that $ux_1 \notin E(G')$ and $vx_2 \notin E(G')$ use color $c(v)$ for x_2 and $c(u)$ for x_1. The rest of uncolored vertices of C_{uv} can be colored with the remaining $k-2$ colors. Conversely, independently of k-coloring of G', any vertices $w_1, w_2 \in W$ that correspond to the edge $v_1v_2 \in E(G)$ must be colored with different colors because all of k colors are already used in $C_{v_1v_2}$. This results in a k-coloring of G. □

It is also worth pointing out that Reed's conjecture [39], which asks whether $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G)+1) + \frac{1}{2}\omega(G) \rceil$, holds for all graphs G with $\chi(G) = Q(G) + 1$, even if we consider a stronger statement of the conjecture, i.e., with $Q(G)$ in place of $\Delta(G)$ (see also Section 5).

4. Step potential and the independence number

In what follows we need to distinguish several types of vertices. Let $f_G : V(G) \to \mathbb{N}_0$ be a potential function such that $f_G(v) \leq d_G(v)$ for all $v \in V(G)$. Then, a vertex v for which $f_G(v) = d_G(v)$ is called *saturated*, while it is called *nonsaturated*, when $f_G(v) < d_G(v)$.

Let $H = G - x$ be a graph obtained from G by deletion of a vertex x with all incident edges, and let $C_G(x)$ be a subset of neighbors of the vertex x in a graph G such that for each $u \in C_G(x)$ it holds that $f_H(u) = f_G(u)$. We also need $D_G(x) = N_G(x) \setminus C_G(x)$. Less formally $D_G(x)$ is a subset of the neighbors of x whose potentials decrease after deletion of x. Observe that for the step potential function whenever $u \in N_G(x)$ is saturated, then $u \in D_G(x)$. Hence, if $u \in C_G(x)$, then u is nonsaturated. We will use this fact in subsequent sections.

As well as the above-mentioned types of vertices, we will also need critical vertices that turn out to be crucial for the statement and analysis of `GreedyMAX`-type algorithms.

Algorithm 2 `GreedyMAX-type` (G, f_G, I)

```
Input : G – a simple graph
        f_G – a potential function
Output : I – a maximal independent set in G

Begin
    i ← 0;  G_i ← G;  I ← ∅
    While E(G_i) ≠ ∅ do
        select vertex x_i that is critical with respect to f_G
        G_{i+1} ← G_i − x_i
        i ← i + 1
    I ← V(G_i)
End
```

Definition 2. Let G be a graph and let $f_G : V(G) \to \mathbb{N}_0$ be a potential function. A vertex x is said to be *critical* with respect to f_G if

(a) f_G has a local maximum at x, i.e., for each $u \in N_G(x)$, $f_G(u) \le f_G(x)$, and
(b) $|D_G(x)| \ge f_G(x)$.

4.1. *GreedyMAX-type algorithms*

Every GreedyMAX-type algorithm selects a vertex x that is critical with respect to some potential function f_G, deletes x from the graph together with all incident edges and iterates this process on the resulting graph until no edge remains. Repeating deletions naturally defines the sequence of vertices $(x_0, \ldots, x_{r-1})$ as well as the sequence of graphs $(G_0, \ldots, G_r)$ such that $G_{i+1} = G_i - x_i$, where r is the number of iterations.

The pseudocode of GreedyMAX-type algorithm that for a given graph G and an appropriate potential function f_G calculates a maximal independent set I is presented as Algorithm 2. Naturally, in order to define a particular GreedyMAX-type algorithm, one has to specify an appropriate selection rule that depends on f_G and allows us to choose a critical vertex. We refer the readers to [7] for the description of several selection rules that were used for the simple step potential function p_G (a slightly different but more general selection rule was also given in [8]). The GreedyMAX-type algorithms analyzed in [7, 8] always return an independent set I with $|I| \ge \mathrm{CW}(G, p_G)$. Before discussing the properties of vertices that are critical with respect to q_G, we prove a stronger version of Theorem 4 from [8]. The main advantage of this result is that it broadens a family of potential functions and critical vertices that are suitable for GreedyMAX-type algorithms.

Theorem 6. *Let $\mathcal{G}$ be an induced hereditary class of graphs and let $f_G : V(G) \to \mathbb{N}_0$ be the potential function of $G \in \mathcal{G}$. If for every $G \in \mathcal{G}$ there exists a vertex x, critical with respect to f_G, and restriction of f_G to $V(G) \setminus \{x\}$ is an upper bound for f_{G-x}, then GreedyMAX-type algorithm applied to G returns an independent set I satisfying*

$$|I| \ge \mathrm{CW}(G, f_G) = \sum_{v \in V(G)} \frac{1}{f_G(v) + 1}. \tag{10}$$

Proof. To prove (10) by induction we first observe that the assertion trivially holds for edgeless graphs. Let x be a vertex critical with respect to f_G and let $H = G - x$. We are going to argue that

$$\mathrm{CW}(H, f_H) - \mathrm{CW}(G, f_G) \ge 0.$$

Let $U = V(G) \setminus (N_G(x) \cup \{x\})$. Hence, for the subgraph H we write

$$\mathrm{CW}(H, f_H) = \sum_{v \in C_G(x)} \frac{1}{f_H(v) + 1} + \sum_{v \in D_G(x)} \frac{1}{f_H(v) + 1} + \sum_{v \in U} \frac{1}{f_H(v) + 1},$$

while for G we have

$$\mathrm{CW}(G, f_G) = \frac{1}{f_G(x) + 1} + \sum_{v \in C_G(x)} \frac{1}{f_G(v) + 1} + \sum_{v \in D_G(x)} \frac{1}{f_G(v) + 1} + \sum_{v \in U} \frac{1}{f_G(v) + 1}.$$

By assumption for every $v \in C_G(x)$, $f_H(v) = f_G(v)$, while for every $v \in U$, $f_H(v) \le f_G(v)$. Suppose that, in the worst case, for all $v \in D_G(x)$, $f_H(v) = f_G(v) - 1$. Then

$$\mathrm{CW}(H, f_H) - \mathrm{CW}(G, f_G) \ge \sum_{v \in D_G(x)} \left(\frac{1}{f_G(v)} - \frac{1}{f_G(v) + 1} \right) - \frac{1}{f_G(x) + 1}.$$

Since f_G has a local maximum at x, for every $v \in D_G(x), f_G(v) \leq f_G(x)$ and it remains to consider

$$\sum_{v \in D_G(x)} \frac{1}{f_G(x)(f_G(x)+1)} - \frac{1}{f_G(x)+1} \geq 0,$$

$$\frac{|D_G(x)|}{f_G(x)(f_G(x)+1)} - \frac{1}{f_G(x)+1} \geq 0,$$

which finally gives $|D_G(x)| \geq f_G(x)$. □

As we have already mentioned, a simple step potential function p_G is closely related to d_G. However, for the step potential q_G the relationship between degrees and the corresponding values of q_G is much harder to follow. Even the fact that every graph G contains a vertex that is critical with respect to q_G is not obvious. This is proved in the next section.

4.2. Analysis of `GreedyMax`-type algorithm that selects vertices critical with respect to the step potential

Let $A(v) = \{q_G(u) | u \in N_G(v)\}$ be a multiset of potentials of the neighbors of a vertex v and let $e^{\max}(v)$ denote $e^{\max}_{A(v)}$. A set $R(v) \subseteq N_G(v)$ is said to *realize* $q_G(v)$ if $\{q_G(u) | u \in R(v)\}$ contains a $q_G(v)$-step sequence and $|R(v)| = q_G(v)$. We begin with the proof of a basic property of vertices that have the same values of the step potential function.

Lemma 4. *If the step potential function q_G of a graph G has a local maximum at vertex x and $C_G(x) \neq \emptyset$, then for every $u \in C_G(x)$ it holds that $e^{\max}(u) = q_G(u) = q_G(x)$ and q_G attains a local maximum at u.*

Proof. Assume that $e^{\max}(u) < q_G(x)$. Then from Lemma 3 it follows that $u \in D_G(x)$, a contradiction. Therefore, $e^{\max}(u) \geq q_G(x)$ and since q_G has a local maximum at x, it holds that $q_G(x) \geq q_G(u)$. By definition $e^{\max}(u) \leq q_G(u)$. Thus $e^{\max}(u) = q_G(u) = q_G(x)$.

Concerning local maximum at u, on the contrary assume that there exists $v \in N_G(u)$ such that $q_G(v) > q_G(u)$. Clearly, since $q_G(v) > e^{\max}(u)$, v belongs to every set that realizes $q_G(u)$. Let $R(u)$ realize $q_G(u)$ and let $z \notin R(u)$ be a vertex for which $q_G(z) = e^{\max}(u)$. Then, $R(u) \cup \{z\}$ realizes $q_G(u) + 1$, a contradiction. □

Lemma 5. *Every graph G contains a vertex that is critical with respect to q_G.*

Proof. Let x be an arbitrary vertex at which q_G attains a local maximum (note that every graph contains such a vertex). If maximum at x is strict, then by Lemma 3 the vertex x is critical. Now, assume that the maximum in x is not strict and that x is not critical. Then by the definition of a critical vertex $|D_G(x)| < q_G(x) \leq d_G(x)$, and consequently $C_G(x) \neq \emptyset$. Let $u \in C_G(x)$ and let k stand for $q_G(x)$. By Lemma 4, $e^{\max}(u) = q_G(u) = k$ and q_G has a local maximum at u.
If u is critical then the thesis follows. Assume that u is not critical. Hence, $C_G(u) \neq \emptyset$.

Case 1 x is saturated.

Since x is saturated, $x \notin C_G(u)$. Hence, there exists $z \neq x$ such that $z \in C_G(u)$. From Lemma 4 it follows that $e^{\max}(z) = q_G(z) = q_G(u) = k$ and q_G attains a local maximum at z. Now, since $z \in C_G(u)$, z is nonsaturated and there exists $R(z) \subseteq N_G(z) \setminus \{u\}$ that realizes $q_G(z) = k$. Analogously, since $u \in C_G(x)$, u is nonsaturated and there exists $R(u) \subseteq N_G(u) \setminus \{x\}$ that realizes $q_G(u) = k$. Consider $R(u) \cup \{x\}$ and $R(z) \cup \{u\}$, and recalculate q_G for u and z to get $q_G(u) = q_G(z) = k + 1$, which is a contradiction.

Case 2 x is nonsaturated.

Subcase 2.1 $x \in C_G(u)$.

Since $x \in C_G(u)$ and by assumption x is nonsaturated, there exists $R(x) \subseteq N_G(x) \setminus \{u\}$ that realizes $q_G(x) = k$. Analogously, since $u \in C_G(x)$, u is nonsaturated and there exists $R(u) \subseteq N_G(u) \setminus \{x\}$ that realizes $q_G(u) = k$. Consider $R(u) \cup \{x\}$ and $R(x) \cup \{u\}$, and recalculate q_G for u and x to get $q_G(u) = q_G(x) = k + 1$, a contradiction.

Subcase 2.2 $x \notin C_G(u)$.

Since $x \notin C_G(u)$, there exists $z \neq x$ such that $z \in C_G(u)$. Now, in order to complete the proof proceed analogously as in Case 1. □

Corollary 1. *If q_G has a local maximum at x, then x is critical with respect to q_G or there exists $u \in N_G(x)$ that is critical with respect to q_G and nonsaturated.*

Consequently, in view of Theorem 6 and Proposition 2 there follows the main result of this section.

Corollary 2. *If I is an independent set generated by a `GreedyMAX`-type algorithm that in every iteration selects a vertex critical with respect to the step potential function q_G, then*

$$|I| \geq \sum_{v \in V(G)} \frac{1}{q_G(v) + 1}. \tag{11}$$

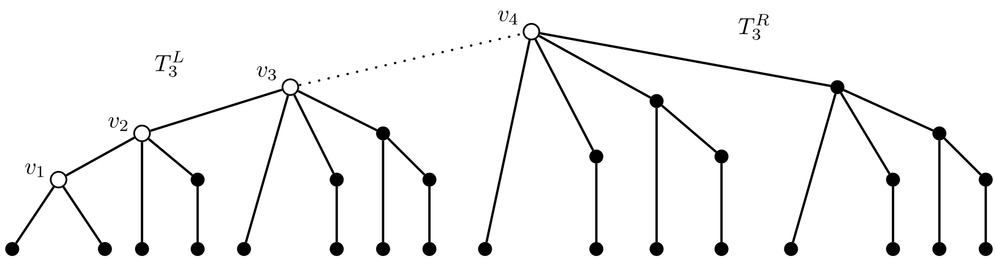


Fig. 2. The trees T_3^L, T_3^R and the resulting tree T_4.

Concerning the worst-case for differences $\mathrm{CW}(G, f_G) - \mathrm{CW}(G, f'_G)$ taken for various functions f_G, f'_G the construction given in [7] reveals that for every integer $\eta > 0$ there exists a connected graph G such that $\mathrm{CW}(G, p_G) - \mathrm{CW}(G, d_G) > \eta$. Before we present a generalization of this result, recall that for the t-step iterative process of calculating the step potential of a graph G, $q_G^{(j)} : V(G) \to \{q^{(j)}(v_1), \ldots, q^{(j)}(v_n)\}$ such that $q_G^{(j)}(v_i) = q^{(j)}(v_i),\ i \in \{1, \ldots, n\}$ is the function obtained after the jth step of the process.

Theorem 7. *For any integers $\eta > 0$, $t \geq 1$ there exists a connected graph G such that after each step $j \in \{1, \ldots, t\}$ of the t-step iterative process of calculating the step potential function q_G it holds that*

$$\mathrm{CW}(G, q_G^{(j)}) - \mathrm{CW}(G, q_G^{(j-1)}) > \eta.$$

Proof. Let us first consider a tree T_ℓ, with the root v_ℓ. In the recursive definition of T_ℓ we distinguish left and right subtrees denoted $T_{\ell-1}^L$ and $T_{\ell-1}^R$, respectively. The appropriate trees can be constructed as follows:

(R) Let T_1^R be isomorphic to a path P_4 and have the root in one of the internal vertices. For $\ell \geq 2$ take two copies of $T_{\ell-1}^R$ and join their roots by an edge. The root of T_ℓ^R is the root of the second copy.

(L) Let T_1^L be isomorphic to a path P_3 and have the root in the middle vertex. For $\ell \geq 2$ take one copy of $T_{\ell-1}^L$ and one copy of $T_{\ell-1}^R$. Then, join roots of both copies by an edge and set the root of $T_{\ell-1}^L$ as the root of T_ℓ^L.

Finally, set $T_\ell = T_\ell^L$. An example in Fig. 2 presents the tree T_4.

Now, consider the iterative process on T_ℓ, $\ell \geq 1$ and observe that the final values of the step potential are calculated in ℓ steps, so that after each step j the functions $q_{T_\ell}^{(j-1)}$ and $q_{T_\ell}^{(j)}$ differ only for a single vertex v_j. For a further reference notice that for $j \in \{1, \ldots, \ell - 1\}$ we have $q_{T_\ell}^{(j-1)}(v_j) = j + 2$, while $q_{T_\ell}^{(j)}(v_j) = j + 1$.

A connected graph G that proves our assertion can be obtained, e.g., by taking a cycle C_m and joining by an edge every vertex of the cycle with the roots of $r \geq \ell + 2$ of mr disjoint copies of T_ℓ. For the graph G the process converges after $\ell + 1$ steps and the following formula holds for all $j \in \{1, \ldots, \ell + 1\}$ (observe that $d_G(v_\ell) = \ell + 2$)

$$\mathrm{CW}(G, q_G^{(j)}) - \mathrm{CW}(G, q_G^{(j-1)}) \geq \begin{cases} mr/(j^2 + 5j + 6), & j \in \{1, \ldots, \ell\} \\ m/(l^2 + 9l + 20), & j = \ell + 1. \end{cases} \tag{12}$$

Note that it involves no loss of generality that in inequality (12) a nonnegative additive term depending on the potentials of the vertices of C_m was omitted for $j = 1$. □

The *performance ratio* $\rho_{\mathtt{A}}$ of an algorithm $\mathtt{A}$ is defined as $\inf_G \mathtt{A}(G)/\alpha(G)$, where $\mathtt{A}(G)$ denotes the size of an independent set generated by $\mathtt{A}$ for a graph G. For the `GreedyMAX` algorithm in its classical setting, Halldórsson and Radhakrishnan [22] used a complete bipartite graph with the removed perfect matching to show that whenever the maximum degree of a graph is bounded by Δ it holds that $\rho_{\mathtt{GreedyMAX}} \leq 2/(\Delta + 1)$, while from the results of Griggs [20] it follows that $\rho_{\mathtt{GreedyMAX}} \geq 1/(\Delta + 1)$. Using the step potential function q_G, as a consequence of Corollary 2, we get the following lower bound which holds for graphs having their maximum step potential bounded by Q.

Theorem 8. *If $\mathtt{A}$ is a `GreedyMAX`-type algorithm that in each step selects a vertex critical with respect to the step potential function q_G, then*

$$\rho_{\mathtt{A}} \geq 1/(Q + 1). \tag{13}$$

Proof. It is enough to observe that $\alpha(G) \leq n$, while by Corollary 2, $\mathtt{A}(G) \geq \mathrm{CW}(G, q_G) \geq n/(Q + 1)$. □

Note that, e.g., for stars $K_{1,k}$ or wheels W_k it holds that $Q(K_{1,k}) = 1$, $\Delta(K_{1,k}) = k$, and $Q(W_k) = 3$, $\Delta(W_k) = k$. In both cases the right-hand side of (13) is constant, while previously known ratios depending on Δ get worse when maximum degree grows. Though for any $\eta > 0$ the existence of graphs for which $\Delta(G) - Q(G) > \eta$ follows from Proposition 4, we would like to mention another interesting class of graphs called *ct*-graphs. We say that G is a *ct-graph* if for every edge $uv \in E(G)$ the value of $|d_G(v) - d_G(u)|$ is constant. The constructions of *ct*-graphs described in [1,32] provide appropriate examples for any $\eta > 0$.

5. Further research

We strongly believe that the following strengthening of Reed's conjecture [39] is true.

Problem 1. Prove that for every graph G it holds that

$$\chi(G) \leq \left\lceil \frac{Q(G) + \omega(G) + 1}{2} \right\rceil. \tag{14}$$

The Chvátal graph (see [13]) is 4-regular, thus its step potential equals 4 and since it is triangle-free and 4-chromatic we can easily see that rounding up in (14) is necessary. There are classes of graphs for which (14) and the original Reed's bound are equal, e.g., for regular graphs $q_G(v) = d_G(v)$ holds for every vertex v. We also know graphs for which the conjectured bound is better than Reed's bound, e.g., stars $K_{1,k}$ and odd wheels W_{2k+1}. In both cases the bound relying on step potential is exact while Reed's bound can be arbitrarily far from the optimum. We conclude with the observation that follows directly from Theorem 4 and the fact that graphs for which $\chi(G) = Q(G) + 1$ are optimally colorable by `Greedy`. Namely, whenever G is such a graph, then (14) holds.

Acknowledgments

We would like to thank the anonymous referees for their valuable comments that permitted us to improve the readability of the paper.

The first author has been partially supported by National Science Centre under contract DEC-2011/02/A/ST6/00201.

References

[1] B.D. Acharya, M.N. Vartak, On the construction of graphs with given constant valence-difference(s) on each of their lines, Wiss. Z. Tech. Hochsch. Ilmenau 23 (6) (1977) 33–60.
[2] V.E. Alekseev, R. Boliac, D.V. Korobitsyn, V.V. Lozin, NP-hard graph problems and boundary classes of graphs, Theoret. Comput. Sci. 389 (2007) 219–236.
[3] N. Alon, J.H. Spencer, The Probabilistic Method, Wiley, New York, 1992.
[4] A. Björklund, T. Husfeldt, M. Koivisto, Set partitioning via inclusion–exclusion, SIAM J. Comput. 39 (2) (2009) 546–563.
[5] A. Blum, New approximation algorithms for graph coloring, J. Assoc. Comput. Mach. 41 (3) (1994) 475–516.
[6] O.V. Borodin, A.V. Kostochka, On an upper bound of a graph's chromatic number, depending on the graph's degree and density, J. Combin. Theory Ser. B 23 (1977) 247–250.
[7] P. Borowiecki, F. Göring, GreedyMAX-type algorithms for the maximum independent set problem, Lect. Notes Comput. Sci. 6543 (2011) 146–156. http://dx.doi.org/10.1007/978-3-642-18381-2_12.
[8] P. Borowiecki, F. Göring, J. Harant, D. Rautenbach, The potential of greed for independence, J. Graph Theory 71 (3) (2012) 245–259. http://dx.doi.org/10.1002/jgt.20644.
[9] P. Borowiecki, E. Sidorowicz, Dynamic coloring of graphs, Fund. Inform. 114 (2) (2012) 105–128. http://dx.doi.org/10.3233/FI-2012-620.
[10] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941) 194–197.
[11] Y. Caro, New Results on the Independence Number, Technical Report, Tel-Aviv University, 1979.
[12] C.A. Christen, S.M. Selkow, Some perfect coloring properties of graphs, J. Combin. Theory Ser. B 27 (1979) 49–59.
[13] V. Chvátal, The smallest triangle-free, 4-chromatic, 4-regular graph, J. Combin. Theory 9 (1970) 93–94.
[14] V. Chvátal, C. McDiarmid, Small transversals in hypergraphs, Combinatorica 12 (1992) 19–26.
[15] P. Erdős, R.J. Wilson, On the chromatic index of almost all graphs, J. Combin. Theory Ser. B 23 (1977) 255–257.
[16] U. Feige, J. Kilian, Zero knowledge and the chromatic number, J. Comput. System Sci. 57 (2) (1998) 187–199.
[17] F.V. Fomin, F. Grandoni, D. Kratsch, Measure and conquer: a simple $O(2^{0.288n})$ independent set algorithm, in: Proc. 17th ACM–SIAM Symposium on Discrete Algorithms, 2006, pp. 18–25.
[18] M.R. Garey, D.S. Johnson, Computers and Intractability. A Guide to the Theory of NP-Completeness, W.H. Freemanan and Company, New York, San Francisco, 1979.
[19] N. Goyal, S. Vishwanathan, NP-completeness of undirected Grundy numbering and related problems, Unpublished Manuscript, Bombay, May 1998, pp. 1–5.
[20] J.R. Griggs, Lower bounds on the independence number in terms of the degrees, J. Combin. Theory Ser. B 34 (1983) 22–39.
[21] A. Gyárfás, Z. Király, J. Lehel, On-line 3-chromatic graphs—II. Critical graphs, Discrete Math. 177 (1997) 99–122.
[22] M.M. Halldórsson, J. Radhakrishnan, Greed is good: approximating independent sets in sparse and bounded–degree graphs, Algorithmica 18 (1997) 145–163.
[23] P.L. Hammer, N.V.R. Mahadev, D. de Werra, The struction of a graph: application to CN-free graphs, Combinatorica 5 (1985) 141–147.
[24] J. Harant, D. Rautenbach, Independence in connected graphs, Discrete Appl. Math. 159 (2011) 79–86.
[25] J. Harant, Z. Ryjaček, I. Schiermeyer, Forbidden subgraphs implying the MIN-algorithm gives a maximum independent set, Discrete Math. 256 (2002) 193–201.
[26] J. Harant, I. Schiermeyer, On the independence number of a graph in terms of order and size, Discrete Math. 232 (2001) 131–138.
[27] J. Håstad, Clique is hard to approximate within $n^{1-\varepsilon}$, Acta Math. 182 (1) (1999) 105–142.
[28] F. Havet, L. Sampaio, On the Grundy number of a graph, Lect. Notes Comput. Sci. 6478 (2010) 170–179.
[29] A. Hertz, Polynomially solvable cases for the maximum stable set problem, Discrete Appl. Math. 60 (1995) 195–210.
[30] A. Hertz, D. de Werra, On the stability number of AH-fre graphs, J. Graph Theory 17 (1993) 53–63.
[31] D.S. Hochbaum, Approximating covering and packing problems: set cover, vertex cover, independent set and related problems, in: D.S. Hochbaum (Ed.), Approximation Algorithms for NP-Hard Problems, PWS Publishing Company, 1995.
[32] S. Jendrol', Z. Ryjaček, 3-polytopes of constant tolerance of edges, Comment. Math. Univ. Carolin. 22 (4) (1981) 843–850.
[33] T.R. Jensen, B. Toft, Graph Coloring Problems, in: Wiley-Intersci. Ser. Discrete Math. Optim., Wiley, New York, 1994.
[34] M. Kubale (Ed.), Graph Colorings, in: Contemporary Mathematics, vol. 352, American Mathematical Society, 2004.
[35] V.V. Lozin, M. Milanič, On the maximum independent set problem in subclasses of planar graphs, J. Graph Algorithms Appl. 14 (2010) 269–286.
[36] A. McRae, January 1994, private communication of S.T. Hedetniemi.
[37] O. Murphy, Lower bounds on the stability number of graphs computed in terms of degrees, Discrete Math. 90 (1991) 207–211.

[38] E. Paschos, Polynomial approximation and graph-coloring, Computing 70 (2003) 41–86.
[39] B.A. Reed, ω, Δ and χ, J. Graph Theory 27 (4) (1989) 177–212.
[40] J.M. Robson, Finding a Maximum independent Set in Time $O(2^{n/4})$, Tech. Rep. 1251-01, LaBRI, Univ. Bordeaux, 2001.
[41] S.M. Selkow, A probabilistic lower bound on the independence number of graphs, Discrete Math. 132 (1994) 363–365.
[42] L. Stacho, New upper bounds for the chromatic number of a graph, J. Graph Theory 36 (2001) 117–120.
[43] G. Szekeres, S.H. Wilf, An inequality for the chromatic number of a graph, J. Combin. Theory 4 (1968) 1–3.
[44] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452.
[45] V.K. Wei, A Lower Bound on the Stability Number of a Simple Graph, Tech. Memorandum No. 81-11217-9, Bell Lab., 1981.
[46] M. Zaker, Grundy chromatic number of the complement of bipartite graphs, Australas. J. Combin. 31 (2005) 325–329.
[47] M. Zaker, New bounds for the chromatic number of graphs, J. Graph Theory 58 (2008) 110–122.
[48] X. Zhu, Graphs with chromatic numbers strictly less than their colouring numbers, Ars Math. Contemp. 4 (2011) 25–27.