J. Math. Soc. Japan
Vol. 65, No. 1 (2013) pp. 37–50
doi: 10.2969/jmsj/06510037

Non-isolating 2-bondage in graphs

By Marcin Krzywkowski

(Received Dec. 14, 2010)
(Revised Apr. 30, 2011)

Abstract. A 2-dominating set of a graph $G = (V, E)$ is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. The non-isolating 2-bondage number of G, denoted by $b_2'(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G-E') \geq 1$ and $\gamma_2(G-E') > \gamma_2(G)$. If for every $E' \subseteq E$, either $\gamma_2(G-E') = \gamma_2(G)$ or $\delta(G-E') = 0$, then we define $b_2'(G) = 0$, and we say that G is a γ_2-non-isolatingly strongly stable graph. First we discuss the basic properties of non-isolating 2-bondage in graphs. We find the non-isolating 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. Finally, we characterize all γ_2-non-isolatingly strongly stable trees.

1. Introduction.

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong if it is adjacent to at least two leaves. Let $\delta(G)$ mean the minimum degree among all vertices of G. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a tree H if there is a neighbor of v, say x, such that the tree resulting from T by removing the edge vx, and which contains the vertex x, is a tree H. Let $K_{p,q}$ denote a complete bipartite graph the partite sets of which have cardinalities p and q. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph that can be obtained from a star by joining a positive number of vertices to one of the leaves.

2010 *Mathematics Subject Classification.* Primary 05C05, 05C69.

Key Words and Phrases. 2-domination, bondage, non-isolating 2-bondage, graph, tree.

The research was supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set, abbreviated 2DS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [**3**], and further studied for example in [**1**], [**13**]. For a comprehensive survey of domination in graphs, see [**7**], [**8**].

The bondage number $b(G)$ of a graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma(G - E') > \gamma(G)$. If for every $E' \subseteq E$ we have $\gamma(G - E') = \gamma(G)$, then we define $b(G) = 0$, and we say that G is a γ-strongly stable graph. Bondage in graphs was introduced in [**4**], and further studied for example in [**2**], [**5**], [**6**], [**9**], [**10**], [**11**], [**12**] and [**14**].

We define the non-isolating 2-bondage number of a graph G, denoted by $b_2'(G)$, to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_2(G - E') > \gamma_2(G)$. Thus $b_2'(G)$ is the minimum number of edges of G that have to be removed in order to obtain a graph with no isolated vertices, and with the 2-domination number greater than that of G. If for every $E' \subseteq E$, either $\gamma_2(G - E') = \gamma_2(G)$ or $\delta(G - E') = 0$, then we define $b_2'(G) = 0$, and we say that G is a γ_2-non-isolatingly strongly stable graph.

First we discuss the basic properties of non-isolating 2-bondage in graphs. We find the non-isolating 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. Finally, we characterize all γ_2-non-isolatingly strongly stable trees.

2. Results.

We begin with the following observations.

Observation 1: *Every leaf of a graph G is in every $\gamma_2(G)$-set.*

Observation 2: *If $H \subseteq G$ and $V(H) = V(G)$, then $\gamma_2(H) \geq \gamma_2(G)$.*

Observation 3: *For every positive integer n we have $\gamma_2(K_n) = \min\{2, n\}$.*

Observation 4: *If n is a positive integer, then $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.*

Observation 5: *For every integer $n \geq 3$ we have $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$.*

Observation 6: *For every integer $n \geq 4$ we have*

$$\gamma_2(W_n) = \begin{cases} 2 & \text{if } n = 4, 5; \\ \lfloor (n+1)/3 \rfloor + 1 & \text{if } n \geq 6. \end{cases}$$

Observation 7: *Let p and q be positive integers such that $p \leq q$. Then*

$$\gamma_2(K_{p,q}) = \begin{cases} \max\{q, 2\} & \text{if } p = 1; \\ \min\{p, 4\} & \text{if } p \geq 2. \end{cases}$$

Since the definition of the non-isolating 2-bondage does not allow isolated vertices in the searched subgraphs of a given graph, in this paper, we do not consider removing edges that produces an isolated vertex.

First we find the non-isolating 2-bondage numbers of complete graphs.

Remark 8. For every positive integer n we have

$$b_2'(K_n) = \begin{cases} 0 & \text{if } n = 1, 2, 3; \\ \lfloor 2n/3 \rfloor & \text{otherwise.} \end{cases}$$

Proof. Of course, $b_2'(K_1) = 0$, $b_2'(K_2) = 0$, and $b_2'(K_3) = 0$. Now assume that $n \geq 4$. Let $E(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let G be a graph with at least two vertices. Let us observe that $\gamma_2(G) = 2$ if and only if G has two vertices which are both adjacent to every vertex other than they. Let $E' \subseteq E(K_n)$. Let us observe that $\gamma_2(K_n - E') > 2$ if and only if at most one vertex of K_n is not incident to any edge of E', and every edge of E' is adjacent to some other edge of E'. We want to choose a smallest set $E' \subseteq E(K_n)$ satisfying the condition above while $\delta(K_n - E') \geq 1$. Let us observe that the most efficient way of choosing edges of K_n is to choose for example edges v_1v_2, v_2v_3, v_4v_5, v_5v_6, and so on. In this way no vertex becomes isolated. Let k be a positive integer.

If $n = 3k$, then we remove $2k$ edges. Thus $b_2'(K_{3k}) = 2k = \lfloor 2n/3 \rfloor$. If $n = 3k+1$, then we also remove $2k$ edges as one vertex can remain universal. We have $b_2'(K_{3k+1}) = 2k = \lfloor 2k + 2/3 \rfloor = \lfloor 2(3k+1)/3 \rfloor = \lfloor 2n/3 \rfloor$. Now assume that $n = 3k+2$. If we remove the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \ldots, v_{2k-2}v_{2k-1}, v_{2k-1}v_{2k}$, then the vertices v_{3k+1} and v_{3k+2} remain universal. Therefore $b_2'(K_{3k+2}) > 2k$. Let us observe that removing the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \ldots, v_{2k-2}v_{2k-1}$, $v_{2k-1}v_{2k}, v_{2k}v_{2k+1}$ increases the 2-domination number. This implies that $b_2'(K_{3k+2}) = 2k + 1 = \lfloor 2k + 4/3 \rfloor = \lfloor 2(3k+2)/3 \rfloor = \lfloor 2n/3 \rfloor$. $\square$

Now we calculate the non-isolating 2-bondage numbers of paths.

Remark 9. If n is a positive integer, then

$$b_2'(P_n) = \begin{cases} 0 & \text{for } n = 1, 2, 3; \\ 1 & \text{for } n \geq 4. \end{cases}$$

Now we investigate the non-isolating 2-bondage in cycles.

Remark 10. For every integer $n \geq 3$ we have

$$b_2'(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n \text{ is even;} \\ 2 & \text{otherwise.} \end{cases}$$

Now we calculate the non-isolating 2-bondage numbers of wheels.

Remark 11. For every integer $n \geq 4$ we have

$$b_2'(W_n) = \begin{cases} 1 & \text{if } n = 5; \\ 2 & \text{if } n \neq 3k + 2; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let $E(W_n) = \{v_1v_2, v_1v_3, \dots, v_1v_n, v_2v_3, v_3v_4, \dots, v_{n-1}v_n, v_nv_2\}$. Since $W_4 = K_4$, by Remark 8 we get $b_2'(W_4) = b_2'(K_4) = \lfloor 8/3 \rfloor = 2$. By Observation 6 we have $\gamma_2(W_5) = 2$. Let us observe that $\gamma_2(W_5 - v_2v_3) = 3 > 2 = \gamma_2(W_5)$. Thus $b_2'(W_5) = 1$. Now let us assume that $n \geq 6$. If we remove an edge incident with v_1, say v_1v_2, then we get $\gamma_2(W_n - v_1v_2) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$-set that contains the vertices v_1 and v_2; such set is also a 2DS of the graph $W_n - v_1v_2$. If we remove an edge non-incident with v_1, say v_2v_3, then we get $\gamma_2(W_n - v_2v_3) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$-set that does not contain the vertices v_2 and v_3; such set is also a 2DS of the graph $W_n - v_2v_3$. This implies that $b_2'(W_n) = 0$ or $b_2'(W_n) \geq 2$. First assume that $n = 3k$ or $n = 3k + 1$. Let us remove the edges $v_{n-1}v_n$ and v_nv_2. We find a relation between the numbers $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_{n-1})$. Let D be any $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$-set. By Observation 1 we have $v_n \in D$. Let us observe that $D \setminus \{v_n\}$ is a 2DS of the graph W_{n-1}. Therefore $\gamma_2(W_{n-1}) \leq \gamma_2(W_n - v_{n-1}v_n - v_nv_2) - 1$. Using Observation 6 we get $\gamma_2(W_n - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_{n-1}) + 1 = \lfloor n/3 \rfloor + 2 = \lfloor (n+1)/3 \rfloor + 2 > \lfloor (n+1)/3 \rfloor + 1 = \gamma_2(W_n)$. Therefore $b_2'(W_n) = 2$ if $n = 3k$ or $n = 3k + 1$. Now assume that $n = 3k + 2$. It is not difficult to verify that now removing any two edges does not increase the 2-domination number. This implies that $b_2'(W_n) = 0$ or $b_2'(W_n) \geq 3$. Let us remove the edges $v_{n-2}v_{n-1}$, $v_{n-1}v_n$, and v_nv_2. We find a relation between the numbers $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_{n-2})$. Let D be any $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$-set. By Observation 1 we have $v_{n-1}, v_n \in D$. Let us observe that $D \setminus \{v_{n-1}, v_n\}$ is a 2DS of the graph W_{n-2}. Therefore $\gamma_2(W_{n-2}) \leq \gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) - 2$. Now we get $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_{n-2}) + 2 = \lfloor (n-1)/3 \rfloor + 3 = \lfloor (n+2)/3 \rfloor + 2 > \lfloor (n+1)/3 \rfloor + 1 = \gamma_2(W_n)$. Therefore $b_2'(W_n) = 3$ if $n = 3k + 2$. □

Now we investigate the non-isolating 2-bondage in complete bipartite graphs.

Remark 12. Let p and q be positive integers such that $p \leq q$. Then

$$b_2'(K_{p,q}) = \begin{cases} 3 & \text{if } p = q = 3; \\ 5 & \text{if } p = q = 4; \\ p-1 & \text{otherwise.} \end{cases}$$

Proof. Let $E(K_{p,q}) = \{a_ib_j \colon 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. If $p = 1$, then obviously $b_2'(K_{p,q}) = 0 = p - 1$ as removing an edge gives us an isolated vertex. Now assume that $p = 2$. By Observation 7 we have $\gamma_2(K_{2,q}) = 2$. Let us observe that $\gamma_2(K_{2,q} - a_1b_1) = 3$ as the vertex b_1 has to belong to every 2DS of the graph $K_{2,q} - a_1b_1$. Thus $b_2'(K_{2,q}) = 1 = p - 1$.

Now let us assume that $p = 3$. By Observation 7 we have $\gamma_2(K_{3,q}) = 3$. Let us observe that removing one edge does not increase the 2-domination number. This implies that $b_2'(K_{3,q}) = 0$ or $b_2'(K_{3,q}) \geq 2$. If $q = 3$, then it is easy to verify that removing any two edges does not increase the 2-domination number. This implies that $b_2'(K_{3,3}) = 0$ or $b_2'(K_{3,q}) \geq 3$. Let us observe that $\gamma_2(K_{3,3} - a_1b_1 - a_1b_2 - a_2b_1) = 4 > 3 = \gamma_2(K_{3,3})$. Therefore $b_2'(K_{3,3}) = 3$. Now assume that $q \geq 4$. We have $\gamma_2(K_{3,q} - a_1b_1 - a_2b_1) = 4$ as the vertex b_1 has to belong to every 2DS of the graph $K_{3,q} - a_1b_1 - a_2b_1$. Thus $b_2'(K_{3,q}) = 2$ if $q \geq 4$.

Now assume that $p \geq 4$. By Observation 7 we have $\gamma_2(K_{p,q}) = 4$. If $q = 4$, then it is not difficult to verify that removing any four edges does not increase the 2-domination number. This implies that $b_2'(K_{4,4}) = 0$ or $b_2'(K_{4,4}) \geq 5$. We have $\gamma_2(K_{4,4} - a_1b_1 - a_1b_2 - a_1b_3 - a_2b_1 - a_3b_1) = 5$ as the vertices a_1 and b_1 have to belong to every 2DS of the graph $K_{4,4} - a_1b_1 - a_1b_2 - a_1b_3 - a_2b_1 - a_3b_1$. Thus $b_2'(K_{4,4}) = 5$. Now assume that $q \geq 5$. Let us observe that removing any $p - 2$ edges does not increase the 2-domination number. This implies that $b_2'(K_{p,q}) = 0$ or $b_2'(K_{p,q}) \geq p - 1$. We have $\gamma_2(K_{p,q} - a_1b_1 - a_2b_1 - \cdots - a_{p-1}b_1) = 5$ as the vertex b_1 has to belong to every 2DS of the graph $K_{p,q} - a_1b_1 - a_2b_1 - \cdots - a_{p-1}b_1$. Therefore $b_2'(K_{p,q}) = p - 1$ if $p \geq 4$ and $q \geq 5$. □

A paired dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. The paired domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired dominating set of G. The paired bondage number, denoted by $b_p(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_p(G - E') > \gamma_p(G)$. If for every $E' \subseteq E$, either $\gamma_p(G - E') = \gamma_p(G)$ or $\delta(G - E') = 0$, then we define $b_p(G) = 0$, and we say that G is a γ_p-strongly stable graph. Raczek [**11**] observed that if $H \subseteq G$, then $b_p(H) \leq b_p(G)$. Let us observe that no inequality of such type is true for the non-isolating 2-bondage. Consider the complete bipartite graphs $K_{3,3}$, $K_{3,5}$, and $K_{4,5}$. Of course, $K_{3,3} \subseteq K_{3,5} \subseteq K_{4,5}$. Using Remark 12 we get $b'_2(K_{3,3}) = 3 > 2 = b'_2(K_{3,5}) < 3 = b'_2(K_{4,5})$.

The authors of [**4**] proved that the bondage number of any tree is either one or two. Let us observe that for every non-negative integer there exists a tree having such non-isolating 2-bondage number. For positive integers k consider trees T_k of the form presented in Figure 1. It is not difficult to verify that $b'_2(T_k) = k - 1$.

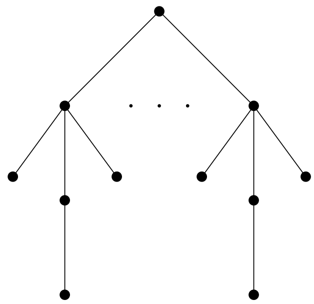

Figure 1. A tree T_k having $5k + 1$ vertices.

Hartnell and Rall [**5**] characterized all trees with bondage number equaling two. We characterize all trees with the non-isolating 2-bondage number equaling zero, that is, all γ_2-non-isolatingly strongly stable trees.

We have the following property of γ_2-non-isolatingly strongly stable trees.

Lemma 13. *Let T be a tree with $b'_2(T) = 0$, and let x be a vertex of T which is neither a leaf nor a support vertex. Then $\gamma_2(T) = \gamma_2(T - x) + 1$.*

Proof. The neighbors of x we denote by $y_1, y_2, \ldots, y_k$. Let T_i mean the component of $T - x$ which contains the vertex y_i. Let $E_0 = \{xy_i : 3 \leq i \leq k\}$, $E_1 = E_0 \cup \{xy_2\}$, and $E_2 = E_0 \cup \{xy_1\}$. Since $b'_2(T) = 0$, we have $\gamma_2(T) = \gamma_2(T - E_0) = \gamma_2(T - E_1) = \gamma_2(T - E_2)$. By T'_i we denote the component of $T - E_i$ which contains the vertex x. For $i = 1, 2$, let D'_i be any $\gamma_2(T'_i)$-set. By Observation 1 we have $x \in D'_i$. It is easy to observe that $D'_1 \cup D'_2$ is a 2DS of the tree T'_0. Thus

$\gamma_2(T_0') \leq \gamma_2(T_1')+\gamma_2(T_2')-1$. Now let D_1 be any $\gamma_2(T_1)$-set. Of course, $D_1 \cup \{x\}$ is a 2DS of the tree T_1'. Thus $\gamma_2(T_1') \leq \gamma_2(T_1)+1$. Suppose that $\gamma_2(T_1') < \gamma_2(T_1)+1$. Now we get $\gamma_2(T) = \gamma_2(T-E_0) = \gamma_2(T_0')+\gamma_2(T_3)+\gamma_2(T_4)+\dots+\gamma_2(T_k) \leq \gamma_2(T_1')+\gamma_2(T_2')-1+\gamma_2(T_3)+\gamma_2(T_4)+\dots+\gamma_2(T_k) < \gamma_2(T_1)+\gamma_2(T_2')+\gamma_2(T_3)+\gamma_2(T_4)+\dots+\gamma_2(T_k) = \gamma_2(T-E_2) = \gamma_2(T)$, a contradiction. Therefore $\gamma_2(T_1') = \gamma_2(T_1)+1$. Now we get $\gamma_2(T) = \gamma_2(T-E_1) = \gamma_2(T_1')+\gamma_2(T_2)+\gamma_2(T_3)+\dots+\gamma_2(T_k) = \gamma_2(T_1)+\gamma_2(T_2)+\dots+\gamma_2(T_k)+1 = \gamma_2(T-x)+1$. □

We have the following sufficient condition for that a subtree of a γ_2-non-isolatingly strongly stable tree is also γ_2-non-isolatingly strongly stable.

LEMMA 14. *Let T be a γ_2-non-isolatingly strongly stable tree. Assume that $T' \neq K_1$ is a subtree of T such that $T-T'$ has no isolated vertices. Then $b_2'(T')=0$.*

PROOF. Let E_1 mean the minimum subset of the set of edges of T such that T' is a component of $T-E_1$. Now let E' be a subset of the set of edges of T' such that $\delta(T'-E') \geq 1$. The assumption $b_2'(T)=0$ implies that $\gamma_2(T-E_1-E') = \gamma_2(T)$. We have $T-E_1-E' = T'-E' \cup (T-T')$, and consequently, $\gamma_2(T-E_1-E') = \gamma_2(T'-E')+\gamma_2(T-T')$. Now we get $\gamma_2(T'-E') = \gamma_2(T-E_1-E')-\gamma_2(T-T') = \gamma_2(T)-\gamma_2(T)+\gamma_2(T') = \gamma_2(T')$. This implies that $b_2'(T')=0$. □

Now we prove that attaching a path P_3 by joining it through the support vertex increases the 2-domination number of any graph by two.

LEMMA 15. *Let G be a graph, and let H obtained from G by attaching a path P_3 by joining the support vertex to any vertex of G. Then $\gamma_2(H) = \gamma_2(G)+2$.*

PROOF. Let $v_1v_2v_3$ mean the attached path. Let D' be any $\gamma_2(G)$-set. It is easy to see that $D' \cup \{v_1, v_3\}$ is a 2DS of the graph H. Thus $\gamma_2(H) \leq \gamma_2(G)+2$. Now let us observe that there exists a $\gamma_2(H)$-set that does not contain the vertex v_2. Let D be such a set. By Observation 1 we have $v_1, v_3 \in D$. Observe that $D \setminus \{v_1, v_3\}$ is a 2DS of the graph G. Therefore $\gamma_2(G) \leq \gamma_2(H)-2$. This implies that $\gamma_2(H) = \gamma_2(G)+2$. □

Now we need to define trees G_1 and G_2, see Figure 2. The tree G_1 is a star $K_{1,3}$ and the tree G_2 is a double star with five vertices.

For the purpose of characterizing all γ_2-non-isolatingly strongly stable trees, that is, all trees T such that for every $E' \subseteq E$, either $\gamma_2(T-E') = \gamma_2(T)$ or $\delta(T-E')=0$, we introduce a family $\mathcal{T}$ of trees $T=T_k$ that can be obtained as follows. Let $T_1 \in \{P_1, P_2, P_3\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

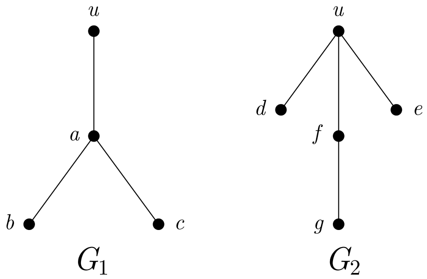

Figure 2. The trees G_1 and G_2.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to a strong support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_3 by joining the support vertex to a leaf of $T_k \neq P_3$ the neighbor of which has degree at most two.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining the support vertex to a vertex of T_k which is not a leaf.
- Operation $\mathcal{O}_4$: Let x mean a vertex of T_k adjacent to a tree G_1 through the vertex u. Remove that tree G_1 and attach a tree G_2 by joining the vertex u to the vertex x.
- Operation $\mathcal{O}_5$: Attach a path P_3 by joining the support vertex to a leaf of T_k the neighbor of which is adjacent to at least three leaves.

Now we characterize all γ_2-non-isolatingly strongly stable trees.

Theorem 16. *Let T be a tree. Then $b_2'(T) = 0$ if and only if $T \in \mathcal{T}$.*

Proof. Let T be a tree of the family $\mathcal{T}$. We use the induction on the number k of operations performed to construct the tree T. If $T = P_1$, then obviously $b_2'(T) = 0$. If $T = P_2$, then also $b_2'(T) = 0$ as removing the edge gives us isolated vertices. Similarly, $b_2'(P_3) = 0$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let x mean the attached vertex, and let y mean its neighbor. Let D be any $\gamma_2(T)$-set. By Observation 1 we have $x \in D$. Let us observe that $D \setminus \{x\}$ is a 2DS of the tree T' as the vertex y has at least two neighbors in $D \setminus \{x\}$. Therefore $\gamma_2(T') \leq \gamma_2(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. Since x is a leaf of T, we have $xy \notin E'$. The assumption $b_2'(T') = 0$ implies that $\gamma_2(T' - E') = \gamma_2(T')$. Let D' be any $\gamma_2(T' - E')$-set. Of course, $D' \cup \{x\}$ is a 2DS of $T - E'$. Thus $\gamma_2(T - E') \leq \gamma_2(T' - E') + 1$. Now we get $\gamma_2(T - E') \leq$

$\gamma_2(T'-E')+1=\gamma_2(T')+1\le\gamma_2(T)$. On the other hand, by Observation 2 we have $\gamma_2(T-E')\ge\gamma_2(T)$. This implies that $\gamma_2(T-E')=\gamma_2(T)$, and consequently, $b_2'(T)=0$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. The leaf to which is attached P_3 we denote by x. Let y mean the neighbor of x. The attached path we denote by $v_1v_2v_3$. If $d_T(y)=1$, then T' is a path P_2. It is not difficult to verify that $b_2'(T)=0$. Now assume that $d_T(y)=2$. The neighbor of y other than x we denote by z. Since $T'\ne P_3$, we have $d_{T'}(z)\ge 2$. Let t mean a neighbor of z other than y. By Lemma 15 we have $\gamma_2(T)=\gamma_2(T')+2$. Let E' be a subset of the set of edges of T such that $\delta(T-E')\ge 1$. Since v_1 and v_3 are leaves of T, we have $v_1v_2, v_2v_3\notin E'$. If $xv_2\in E'$, then $\gamma_2(T-E')=\gamma_2(P_3\cup T'-(E'\setminus\{xv_2\}))=\gamma_2(T'-(E'\setminus\{xv_2\}))+\gamma_2(P_3)=\gamma_2(T')+2=\gamma_2(T)$. Now assume that $xv_2\notin E'$. If $xy\notin E'$, then using Lemma 15 we get $\gamma_2(T-E')=\gamma_2(T'-E')+2=\gamma_2(T')+2=\gamma_2(T)$. Now assume that $xy\in E'$. By T'_y we denote the component of $T'-(E'\setminus\{xy\})$ which contains the vertex y. Let us observe that $T'_y\ne P_3$. Suppose that $T'_y=P_3$. Let $E''=E'\setminus\{xy,zt\}$ and $E'''=E''\cup\{yz\}$. Since $b_2'(T')=0$, we have $\gamma_2(T'-E'')=\gamma_2(T')$ and $\gamma_2(T'-E''')=\gamma_2(T')$. This implies that $\gamma_2(T'-E'')=\gamma_2(T'-E''')$. Let D''' be any $\gamma_2(T'-E''')$-set. By Observation 1 we have $x,y,z\in D'''$. Let us observe that $D'''\setminus\{y\}$ is a 2DS of $T'-E''$. Consequently, $\gamma_2(T'-E'')\le\gamma_2(T'-E''')-1$, a contradiction. Therefore $T'_y\ne P_3$. Since $b_2'(T')=0$, we have $\gamma_2(T'-(E'\setminus\{xy\})-yz)=\gamma_2(T')$. Let D' be any $\gamma_2(T'-(E'\setminus\{xy\})-yz)$-set. By Observation 1 we have $x,y\in D'$. Let us observe that $D'\cup\{v_1,v_3\}$ is a 2DS of $T-E'$. Thus $\gamma_2(T-E')\le\gamma_2(T'-(E'\setminus\{xy\})-yz)+2$. We get $\gamma_2(T-E')\le\gamma_2(T'-(E'\setminus\{xy\})-yz)+2=\gamma_2(T')+2=\gamma_2(T)$. Now we conclude that $\gamma_2(T-E')=\gamma_2(T)$. This implies that $b_2'(T)=0$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. The vertex to which is attached P_3 we denote by x. Let $v_1v_2v_3$ mean the attached path. Let E' be a subset of the set of edges of T such that $\delta(T-E')\ge 1$. If $xv_2\in E'$, then similarly as when considering the previous operation we get $\gamma_2(T-E')\le\gamma_2(T)$. Now assume that $xv_2\notin E'$. If the component of $T-E'$ which contains the vertex x is not a star $K_{1,3}$, then similarly as when considering the previous operation we get $\gamma_2(T-E')\le\gamma_2(T)$. Now assume that the component of $T-E'$ which contains the vertex x is a star $K_{1,3}$. Let us observe that $b_2'(T'-x)=0$. Suppose that $b_2'(T'-x)>0$. This implies that there is a component of $T'-x$, say T_i, such that $b_2'(T_i)>0$. Since x is not a leaf of T', the graph $T'-T_i$ has no isolated vertices. By Lemma 14 we have $b_2'(T_i)=0$, a contradiction. Therefore $b_2'(T'-x)=0$. This implies that $\gamma_2(T'-x-(E'\cap E(T'-x)))=\gamma_2(T'-x)$. Let D' be any $\gamma_2(T'-x-(E'\cap E(T'-x)))$-set. It is easy to observe that $D'\cup\{x,v_1,v_3\}$ is a 2DS of $T-E'$. Thus $\gamma_2(T-E')\le\gamma_2(T'-x-(E'\cap E(T'-x)))+3$. Using Lemmas 13 and 15 we get $\gamma_2(T-E')\le\gamma_2(T'-x-(E'\cap E(T'-x)))+3=\gamma_2(T'-x)+3=$

$\gamma_2(T')+2=\gamma_2(T)$. Now we conclude that $\gamma_2(T-E')=\gamma_2(T)$, and consequently, $b_2'(T)=0$.

Now assume that T is obtained from T' by operation $\mathcal{O}_4$. Let us observe that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. By Observation 1 we have $d,e,g\in D$. The set D is minimal, thus $f\notin D$. Let us observe that $D\cup\{b,c\}\setminus\{d,e,g\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T')\leq\gamma_2(T)-1$. Now let E' be a subset of the set of edges of T such that $\delta(T-E')\geq 1$. Since d, e, and g are leaves of T, we have $ud,ue,fg\notin E'$. First assume that $ux\in E'$. The assumption $b_2'(T')=0$ implies that $\gamma_2(T'-(E'\cap E(T')))=\gamma_2(T')$. We have $\gamma_2(G_1)=3$ and $\gamma_2(G_2)=4$. Now we get $\gamma_2(T-E')=\gamma_2(T'-(E'\cap E(T')))-\gamma_2(G_1)+\gamma_2(G_2)=\gamma_2(T')-3+4=\gamma_2(T')+1\leq\gamma_2(T)$. Now assume that $ux\notin E'$. First assume that x is a leaf of $T'-(E'\cap E(T'))$. Since $b_2'(T')=0$, we have $\gamma_2(T'-(E'\cap E(T')))=\gamma_2(T')$. Let us observe that there exists a $\gamma_2(T'-(E'\cap E(T')))$-set that contains the vertex u. Let D' be such a set. By Observation 1 we have $b,c,x\in D'$. The set D' is minimal, thus $a\notin D'$. Let us observe that $D'\setminus\{u,b,c\}\cup\{d,e,f,g\}$ is a 2DS of $T-E'$. Thus $\gamma_2(T-E')\leq\gamma_2(T'-(E'\cap E(T')))+1$. Now we get $\gamma_2(T-E')\leq\gamma_2(T'-(E'\cap E(T')))+1=\gamma_2(T')+1\leq\gamma_2(T)$. Now assume that x is not a leaf of $T'-(E'\cap E(T'))$. Since $b_2'(T')=0$, we have $\gamma_2(T'-(E'\cap E(T'))-ux)=\gamma_2(T')$. Let D' be any $\gamma_2(T'-(E'\cap E(T'))-ux)$-set. By Observation 1 we have $b,c,u\in D'$. The set D' is minimal, thus $a\notin D'$. Let us observe that now also $D'\setminus\{u,b,c\}\cup\{d,e,f,g\}$ is a 2DS of $T-E'$. Thus $\gamma_2(T-E')\leq\gamma_2(T'-(E'\cap E(T'))-ux)+1$. Now we get $\gamma_2(T-E')\leq\gamma_2(T'-(E'\cap E(T'))-ux)+1=\gamma_2(T')+1\leq\gamma_2(T)$. We conclude that $\gamma_2(T-E')=\gamma_2(T)$, and consequently, $b_2'(T)=0$.

Now assume that T is obtained from T' by operation $\mathcal{O}_5$. The leaf to which is attached P_3 we denote by x. Let y mean the neighbor of x. The attached path we denote by $v_1v_2v_3$. Let E' be a subset of the set of edges of T such that $\delta(T-E')\geq 1$. If $xv_2\in E'$, then similarly as when considering the operation $\mathcal{O}_2$ we get $\gamma_2(T-E')\leq\gamma_2(T)$. Now assume that $xv_2\notin E'$. If the component of $T-E'$ which contains the vertex x is not a star $K_{1,3}$, then similarly as when considering the operation $\mathcal{O}_2$ we get $\gamma_2(T-E')\leq\gamma_2(T)$. Now assume that the component of $T-E'$ which contains the vertex x is a star $K_{1,3}$. Since $b_2'(T')=0$, we have $\gamma_2(T'-(E'\setminus\{xy\}))=\gamma_2(T')$. Let D' be any $\gamma_2(T'-(E'\setminus\{xy\}))$-set. By Observation 1 we have $x\in D'$. Let us observe that $D'\cup\{v_1,v_3\}$ is a 2DS of $T-E'$ as the vertex y is adjacent to at least two leaves in $T-E'$. Thus $\gamma_2(T-E')\leq\gamma_2(T'-(E'\setminus\{xy\}))+2$. Using Lemma 15 we get $\gamma_2(T-E')\leq\gamma_2(T'-(E'\setminus\{xy\}))+2=\gamma_2(T')+2=\gamma_2(T)$. Now we conclude that $\gamma_2(T-E')=\gamma_2(T)$. Consequently, $b_2'(T)=0$.

Now assume that T is a γ_2-non-isolatingly strongly stable tree. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T)=0$, then $T=P_1\in\mathcal{T}$. If $\mathrm{diam}(T)=1$, then $T=P_2\in\mathcal{T}$. If

$\mathrm{diam}(T) = 2$, then T is a star. If $T = P_3$, then $T \in \mathcal{T}$. If T is a star different from P_3, then it can be obtained from P_3 by a proper number of operations $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now let us assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. Let a and b mean the support vertices of T. Without loss of generality we assume that $d_T(a) \leq d_T(b)$. If $T = P_4$, then by Remark 9 we have $b_2'(T) = 1 \neq 0$. Now assume that T is a double star different from P_4. First assume that $d_T(a) = 1$. If $d_T(b) = 2$, then the tree T can be obtained from P_2 by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. If $d_T(b) \geq 3$, then the tree T can be obtained from P_2 by first, operation $\mathcal{O}_2$, and then a proper number of operations $\mathcal{O}_1$ performed on the strong support vertex. Thus $T \in \mathcal{T}$. Now assume that $d_T(a) \geq 2$. The tree T can be obtained from P_3 by first, operation $\mathcal{O}_3$ performed on the support vertex, and then possibly proper numbers of operations $\mathcal{O}_1$ performed on the support vertices. Thus $T \in \mathcal{T}$.

Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is adjacent to at least three leaves. Let y mean a leaf adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_2(T')$-set. Of course, $D' \cup \{y\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now let E' be a subset of the set of edges of T' such that $\delta(T' - E') \geq 1$. Since $b_2'(T) = 0$, we have $\gamma_2(T - E') = \gamma_2(T)$. Let D be any $\gamma_2(T - E')$-set. By Observation 1 we have $y \in D$. Let us observe that $D \setminus \{y\}$ is a 2DS of $T' - E'$ as the vertex y is adjacent to at least two leaves in $T' - E'$. Therefore $\gamma_2(T' - E') \leq \gamma_2(T - E') - 1$. Now we get $\gamma_2(T' - E') \leq \gamma_2(T - E') - 1 = \gamma_2(T) - 1 \leq \gamma_2(T')$. On the other hand, by Observation 2 we have $\gamma_2(T' - E') \geq \gamma_2(T')$. This implies that $\gamma_2(T' - E') = \gamma_2(T')$, and consequently, $b_2'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is adjacent to at most two leaves.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(v) = 2$. Assume that among the descendants of u there is a support vertex, say x, different from v. It suffices to consider only the possibilities when x is adjacent to one or two leaves. First assume that x is adjacent to two leaves. Let $T' = T - T_x$. Lemma 14 implies that $b_2'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that x is adjacent to exactly one leaf. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 1$.

We have $T-uv=T'\cup P_2$. Now we get $\gamma_2(T-uv)=\gamma_2(T'\cup P_2)=\gamma_2(T')+\gamma_2(P_2)=\gamma_2(T')+2\geq\gamma_2(T)+1>\gamma_2(T)$. Therefore $b'_2(T)=1$, a contradiction.

Now assume that every descendant of u excluding v is a leaf. First assume that u is adjacent to two leaves, say x and y. Let T' be a tree obtained from $T-T_u$ by attaching a tree G_1 by joining the vertex u to the vertex w. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex u. Let D' be such a set. By Observation 1 we have $b,c\in D'$. The set D' is minimal, thus $a\notin D'$. Let us observe that $D'\setminus\{b,c\}\cup\{t,x,y\}$ is a 2DS of the tree T. Thus $\gamma_2(T)\leq\gamma_2(T')+1$. Now let E' be a subset of the set of edges of T' such that $\delta(T'-E')\geq 1$. Since b and c are leaves of T', we have $ab,ac\notin E'$. The assumption $b'_2(T)=0$ implies that $\gamma_2(T-(E'\cap E(T)))=\gamma_2(T)$. Let us observe that there exists a $\gamma_2(T-(E'\cap E(T)))$-set that contains the vertex u. Let D be such a set. By Observation 1 we have $t,x,y\in D$. The set D is minimal, thus $v\notin D$. Let us observe that $D\cup\{b,c\}\setminus\{t,x,y\}$ is a 2DS of $T'-E'$. Therefore $\gamma_2(T'-E')\leq\gamma_2(T-(E'\cap E(T)))-1$. Now we get $\gamma_2(T'-E')\leq\gamma_2(T-(E'\cap E(T)))-1=\gamma_2(T)-1\leq\gamma_2(T')$. We conclude that $\gamma_2(T'-E')=\gamma_2(T')$, and consequently, $b'_2(T')=0$. By the inductive hypothesis we have $T'\in\mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T\in\mathcal{T}$.

Now assume that u is adjacent to exactly one leaf, say x. Let $E'=\{wu,uv\}$ and $T'=T-T_u$. Let D' be any $\gamma_2(T')$-set. It is easy to observe that $D'\cup\{u,t,x\}$ is a 2DS of the tree T. Thus $\gamma_2(T)\leq\gamma_2(T')+3$. We have $T-E'=T'\cup P_2\cup P_2$. Now we get $\gamma_2(T-E')=\gamma_2(T'\cup P_2\cup P_2)=\gamma_2(T')+2\gamma_2(P_2)=\gamma_2(T')+4\geq\gamma_2(T)+1>\gamma_2(T)$. This implies that $b'_2(T)\in\{1,2\}$, a contradiction.

Now assume that $d_T(u)=2$. Let $T'=T-T_v$. Let D' be any $\gamma_2(T')$-set. By Observation 1 we have $u\in D'$. It is easy to see that $D'\cup\{t\}$ is a 2DS of the tree T. Thus $\gamma_2(T)\leq\gamma_2(T')+1$. We have $T-uv=T'\cup P_2$. Now we get $\gamma_2(T-uv)=\gamma_2(T'\cup P_2)=\gamma_2(T')+\gamma_2(P_2)=\gamma_2(T')+2\geq\gamma_2(T)+1>\gamma_2(T)$. Therefore $b'_2(T)=1$, a contradiction.

Now assume that $d_T(v)=3$. The leaf adjacent to v and different from t we denote by a. Assume that $d_T(u)\geq 3$. Let $T'=T-T_v$. Lemma 14 implies that $b'_2(T')=0$. By the inductive hypothesis we have $T'\in\mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T\in\mathcal{T}$.

Now assume that $d_T(u)=2$. First assume that w is adjacent to two leaves. Let $T'=T-T_v$. Lemma 14 implies that $b'_2(T')=0$. By the inductive hypothesis we have $T'\in\mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_5$. Thus $T\in\mathcal{T}$.

Now assume that w is adjacent to exactly one leaf, say x. Let G' be a graph obtained from T by removing all edges incident to w excluding wx. Let D' be any $\gamma_2(G')$-set. By Observation 1 we have $u,w,x\in D'$. Let us observe that $D'\setminus\{w\}$ is a 2DS of the tree T. Thus $\gamma_2(T)\leq\gamma_2(G')-1$. Therefore $b'_2(T)>0$, a contradiction.

Now assume that there is a descendant of w, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is isomorphic to T_u. The descendant of k we denote by l, and the leaves adjacent to l we denote by m and p. Let G' be a graph obtained from T by removing all edges incident to w excluding wu. Let us observe that there exists a $\gamma_2(G')$-set that contains the vertex u. Let D' be such a set. By Observation 1 we have $w, k \in D'$. Let us observe that $D' \setminus \{w\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(G') - 1$. This implies that $b_2'(T) > 0$, a contradiction.

Now assume that there is a descendant of w, say k, such that the distance of w to the most distant vertex of T_k is two. It suffices to consider only the possibilities when k is adjacent to one or two leaves. First assume that k is adjacent to two leaves. Let $T' = T - T_k$. Lemma 14 implies that $b_2'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that k is adjacent to exactly one leaf. Let $T' = T - T_k$. Similarly as when T_k is isomorphic to T_u we conclude that $b_2'(T) > 0$, a contradiction.

Now assume that $d_T(w) = 2$. Let $T' = T - T_v$. Lemma 14 implies that $b_2'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. $\square$

Acknowledgments. Thanks are due to the anonymous referee for comments that helped to make the paper correct.

References

[1] M. Blidia, M. Chellali and O. Favaron, Independence and 2-domination in trees, Australas. J. Combin., **33** (2005), 317–327.

[2] G. S. Domke and R. C. Laskar, The bondage and reinforcement numbers of γ_f for some graphs, Discrete Math., **167/168** (1997), 249–259.

[3] J. F. Fink and M. S. Jacobson, n-domination in graphs, In: Graph Theory with Applications to Algorithms and Computer Science (Kalamazoo, Mich., 1984), Wiley-Intersci. Publ., Wiley, New York, 1985, pp. 282–300.

[4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph, Discrete Math., **86** (1990), 47–57.

[5] B. L. Hartnell and D. F. Rall, A characterization of trees in which no edge is essential to the domination number, Ars Combin., **33** (1992), 65–76.

[6] B. L. Hartnell and D. F. Rall, Bounds on the bondage number of a graph, Discrete Math., **128** (1994), 173–177.

[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Monogr. Textbooks Pure Apple. Math., **208**, Marcel Dekker, New York, 1998.

[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds.), Domination in Graphs: Advanced Topics, Monogr. Textbooks Pure Apple. Math., **209**, Marcel Dekker, New York, 1998.

[9] L. Kang and J. Yuan, Bondage number of planar graphs, Discrete Math., **222** (2000),

191–198.

[10] H. Liu and L. Sun, The bondage and connectivity of a graph, Discrete Math., **263** (2003), 289–293.

[11] J. Raczek, Paired bondage in trees, Discrete Math., **308** (2008), 5570–5575.

[12] U. Teschner, New results about the bondage number of a graph, Discrete Math., **171** (1997), 249–259.

[13] L. Volkmann, A Nordhaus-Gaddum-type result for the 2-domination number, J. Combin. Math. Combin. Comput., **64** (2008), 227–235.

[14] Y. P. Wu and Q. Fan, The bondage number of four domination parameters for trees, J. Math. (Wuhan), **24** (2004), 267–270.

Marcin Krzywkowski

Faculty of Electronics
Telecommunications and Informatics
Gdańsk University of Technology
Narutowicza 11/12
80–233 Gdańsk, Poland
E-mail: marcin.krzywkowski@gmail.com