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Nonlinear boundary value problems for second order differential equations with causal operators

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Abstract

In this paper we deal with second order differential equations with causal operators. To obtain sufficient conditions for existence of solutions we use a monotone iterative method. We investigate both differential equations and differential inequalities. An example illustrates the results obtained.

Keywords: Nonlinear boundary conditions; Second order differential equations; Causal operators; Differential inequalities with positive linear operators; Monotone iterative method; Extremal solutions; Unique solution

1. Introduction

Let $J = [0, T]$, $E = C(J, \mathbb{R})$ and $Q \in C(E, E)$. We shall say that Q is a causal operator, or nonanticipative, if the following property holds: for each couple of elements of E such that $u(s) = v(s)$ for $0 \leqslant s \leqslant t$, there results $(Qu)(s) = (Qv)(s)$ for $0 \leqslant s \leqslant t$ with $t < T$ arbitrary, for details see [1].

Note that $(Q_1x)(t) = \int_0^t W(t, s, x(s))\,ds$, $t \in [0, c)$ and $(Q_2x)(t) = h(t, x(t))$, $t \in [0, c)$ are examples of causal operators. Indeed, W and h are continuous functions with values in $\mathbb{R}^p$. In the literature operator Q_1 is known under the name "Volterra operator" and Q_2 is known as "Niemytskii operator."

In this paper, we investigate nonlinear four-point boundary value problems for second order differential equations with a causal operator Q of the form

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$$\begin{cases} x''(t) = (Qx)(t), & t \in J = [0, T], \\ 0 = g_1\big(x(0), x(\delta)\big), & 0 < \delta < T, \\ 0 = g_2\big(x(T), x(\gamma)\big), & 0 < \gamma < T, \end{cases} \tag{1}$$

where $g_1, g_2 \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Functional equations with causal operators are discussed in book [1], see also the references therein. To obtain approximate solutions of nonlinear differential problems we can apply the monotone iterative technique. This technique is well known and we have a lot of applications of this method to differential equations both initial and boundary conditions, see for example [3–6,8–12]. Recently, this method is also applied to first order differential equations with causal operators, see [2] (periodic conditions) and also [7] (nonlinear boundary conditions). This paper extends the application of this method to nonlinear four-point boundary problems for second order differential equations with causal operators. In Section 2, we discuss differential inequalities with positive linear operators to obtain a comparison result. This result is useful to prove the existence of solutions of problems of type (1). In Section 3, we formulate sufficient conditions which guarantee that problem (1) has extremal solutions. A one-sided Lipschitz condition (with corresponding linear operators) is imposed on the causal operator Q. The problem when (1) has the unique solution is also investigated. At the end of this section, an example is added to illustrate theoretical results. In Section 4, we discuss the situation when problem (1) has quasi-solutions and then also the unique solution.

2. Differential inequalities

To apply the monotone iterative method to problems of type (1) we need a fundamental result on differential inequalities.

Lemma 1. *Assume that*:

H_1: *$M \in C(J, [0, \infty))$, $M(t) > 0$, $t \in (0, T)$, $M(0) \geqslant 0$, $M(T) \geqslant 0$,*
H_2: *$\mathcal{L} \in C(E, E)$ is a positive linear causal operator i.e. $(\mathcal{L}m)(t) \geqslant 0$, $t \in J$ provided that $m(t) \geqslant 0$ on J,*
H_3: *$\rho \equiv \int_0^T (\int_s^T [M(t) + (\mathcal{L}\mathbf{1})(t)]\,dt)\,ds \leqslant 1$, where $\mathbf{1}(t) = 1$, $t \in J$.*

Let $p \in C^2(J, \mathbb{R})$ and

$$\begin{cases} p''(t) \geqslant M(t)p(t) + (\mathcal{L}p)(t), & t \in J, \\ p(0) \leqslant 0, \quad p(T) \leqslant 0. \end{cases}$$

Then $p(t) \leqslant 0$ on J.

Proof. Suppose that the inequality $p(t) \leqslant 0$, $t \in J$ is not true. It means that there exists t_0 such that

$$p(t_0) = \max_{t \in J} p(t) = d > 0.$$

Note that $p''(t_0) \leqslant 0$ and $p'(t_0) = 0$, $t_0 \in (0, T)$.

Case 1. Assume that $p(t) \geqslant 0$, $t \in [0, t_0]$. Then

$$0 \geqslant p''(t_0) \geqslant M(t_0)d > 0.$$

It is a contradiction.

Case 2. There exists $t_1 \in [0, t_0)$ such that $p(t_1) < 0$. Then there exists $\xi \in [0, t_0)$ such that

$$p(\xi) = \min_{t \in [0, t_0]} p(t) < 0.$$

It yields

$$p''(t) \geqslant p(\xi)\big[M(t) + (\mathcal{L}\mathbf{1})(t)\big], \quad t \in [0, t_0].$$

Integrating the above inequality from s to t_0 we get

$$-p'(s) = p'(t_0) - p'(s) \geqslant p(\xi) \int_s^{t_0} \big[M(t) + (\mathcal{L}\mathbf{1})(t)\big]\, dt.$$

Next, we integrate the above inequality from ξ to t_0 to obtain

$$p(\xi) > -p(t_0) + p(\xi) \geqslant p(\xi) \int_\xi^{t_0} \left(\int_s^{t_0} \big[M(t) + (\mathcal{L}\mathbf{1})(t)\big]\, dt \right) ds.$$

Dividing by $p(\xi)$, we finally get

$$1 < \int_0^T \left(\int_s^T \big[M(t) + (\mathcal{L}\mathbf{1})(t)\big]\, dt \right) ds = \rho \leqslant 1$$

since $p(\xi) < 0$. It is a contradiction. This proves the lemma. □

Remark 1. Let the operator $\mathcal{L}$ be defined by

$$(\mathcal{L}p)(t) = \sum_{i=1}^{r} L_i(t) p\big(\alpha_i(t)\big),$$

where $L_i \in C(J, \mathbb{R}_+)$, $\alpha_i \in C(J, J)$, $\alpha_i(t) \leqslant t$, $i = 1, 2, \ldots, r$. Then

$$\rho = \int_0^T \left(\int_s^T \left[M(t) + \sum_{i=1}^{r} L_i(t) \right] dt \right) ds.$$

If $M(t) = L_0 > 0$, $L_i(t) = L_i > 0$, $t \in J$, $i = 1, 2, \ldots, r$. Then

$$\rho = \frac{1}{2} T^2 \sum_{i=0}^{r} L_i.$$

3. Extremal solutions. Unique solution

A function $y_0 \in C^2(J, \mathbb{R})$ is said to be a lower solution of (1) if

$$y_0''(t) \geqslant (Qy_0)(t), \quad t \in J, \qquad g_1\big(y_0(0), y_0(\delta)\big) \leqslant 0, \qquad g_2\big(y_0(T), y_0(\gamma)\big) \leqslant 0.$$

A function $z_0 \in C^2(J, \mathbb{R})$ is said to be an upper solution of problem (1) if the above inequalities are reversed.

To show that problem (1) has a solution we construct two sequences which elements are solutions of corresponding linear problems. Existence of solutions of such problems is discussed in the next theorem.

Theorem 1. *Let assumptions* H_1–H_3 *be satisfied. In addition we assume that*

H_4: $Q \in C(E, E)$ *is a causal operator,* $g_i \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$,
H_5: $y_0, z_0 \in C^2(J, \mathbb{R})$ *are lower and upper solutions of problem* (1)*, respectively, and* $y_0(t) \leqslant z_0(t)$, $t \in J$,
H_6: $m \in C^2(J, \mathbb{R})$ *and* $y_0(t) \leqslant m(t) \leqslant z_0(t)$, $t \in J$,
H_7: *the following condition*

$$(Qu)(t) - (Q\bar{u})(t) \geqslant -M(t)\big[\bar{u}(t) - u(t)\big] - \big(\mathcal{L}(\bar{u} - u)\big)(t) \tag{2}$$

holds for $y_0(t) \leqslant u(t) \leqslant \bar{u}(t) \leqslant z_0(t)$,
H_8: g_i, $i = 1, 2$ *are nonincreasing with respect to the second variable and there exist positive constants* a, b *such that*

$$g_1(\bar{u}, v) - g_1(u, v) \leqslant a(\bar{u} - u),$$
$$g_2(\bar{u}_1, v_1) - g_2(u_1, v_1) \leqslant b(\bar{u}_1 - u_1)$$

for $y_0(0) \leqslant u \leqslant \bar{u} \leqslant z_0(0)$, $y_0(T) \leqslant u_1 \leqslant \bar{u}_1 \leqslant z_0(T)$, $y_0(\delta) \leqslant v \leqslant z_0(\delta)$, $y_0(\gamma) \leqslant v_1 \leqslant z_0(\gamma)$.

Let $y \in C^2(J, \mathbb{R})$ *and*

$$\begin{cases} y''(t) = M(t)y(t) + (\mathcal{L}y)(t) + \sigma(t), & t \in J, \\ y(0) = k_1 \in \mathbb{R}, \qquad y(T) = k_2 \in \mathbb{R}, \end{cases} \tag{3}$$

where

$$\sigma(t) = (Qm)(t) - M(t)m(t) - (\mathcal{L}m)(t),$$
$$k_1 = -\frac{1}{a} g_1\big(m(0), m(\delta)\big) + m(0), \qquad k_2 = -\frac{1}{b} g_2\big(m(T), m(\gamma)\big) + m(T).$$

Then problem (3) *has a unique solution* $y \in C^2(J, \mathbb{R})$ *and* $y \in [y_0, z_0]_*$, *where* $[y_0, z_0]_* = \{w \in C^2(J, \mathbb{R}):\ y_0(t) \leqslant w(t) \leqslant z_0(t),\ t \in J\}$.

Proof. Note that problem (3) has at most one solution. To see it let us assume that it has two distinct solutions $z, w \in C^2(J, \mathbb{R})$. Put $p = z - w$. Then $p(0) = p(T) = 0$ and $p''(t) = M(t)p(t) + (\mathcal{L}p)(t)$ on J. In view of assumption H_3 and Lemma 1, we have $p \leqslant 0$, so $z(t) \leqslant w(t)$, $t \in J$. Now putting $p = w - z$, we have $w(t) \leqslant z(t)$, $t \in J$, by Lemma 1. Hence $w(t) = z(t)$, $t \in J$.

It shows that problem (3) has at most one solution. Denote this solution by y. We need to show that $y \in [y_0, z_0]_*$. Put $p = y_0 - y$. Then, in view of assumptions H_5, H_6, H_8, we have

$$\begin{aligned} p(0) &= y_0(0) + \frac{1}{a}\big[g_1\big(m(0), m(\delta)\big) - g_1\big(y_0(0), y_0(\delta)\big) + g_1\big(y_0(0), y_0(\delta)\big)\big] - m(0) \\ &\leqslant y_0(0) + \frac{1}{a}\big[g_1\big(m(0), y_0(\delta)\big) - g_1\big(y_0(0), y_0(\delta)\big)\big] - m(0) \leqslant 0, \\ p(T) &= y_0(T) + \frac{1}{b}\big[g_2\big(m(T), m(\gamma)\big) - g_2\big(y_0(T), y_0(\gamma)\big) + g_2\big(y_0(T), y_0(\gamma)\big)\big] - m(T) \\ &\leqslant y_0(T) + \frac{1}{b}\big[g_2\big(m(T), y_0(\gamma)\big) - g_2\big(y_0(T), y_0(\gamma)\big)\big] - m(T) \leqslant 0. \end{aligned}$$

Moreover,

$$\begin{aligned} p''(t) &\geqslant (Qy_0)(t) - (Qm)(t) - M(t)\big[y(t) - m(t)\big] - \big(\mathcal{L}(y-m)\big)(t) \\ &\geqslant -M(t)\big[m(t) - y_0(t)\big] - \big(\mathcal{L}(m - y_0)\big)(t) - M(t)\big[y(t) - m(t)\big] - \big(\mathcal{L}(y-m)\big)(t) \\ &= M(t)p(t) + (\mathcal{L}p)(t), \end{aligned}$$

by assumption H_7. This result and Lemma 1 show that $y_0(t) \leqslant m(t)$, $t \in J$. Similarly, we can show that $m(t) \leqslant z_0(t)$, $t \in J$. It means that if problem (3) has a solution then it belongs to $[y_0, z_0]_*$.

Now we need to show that problem (3) has a solution. To do it we write problem (3) in the following way

$$y(t) = \int_0^T G(t,s)\big[M(s)y(s) + (\mathcal{L}y)(s) + \sigma(s)\big]\,ds + \frac{k_2 - k_1}{T}t + k_1, \quad t \in J, \tag{4}$$

where the Green function G is defined by

$$G(t,s) = -\frac{1}{T}\begin{cases} (T-t)s & \text{if } 0 \leqslant s \leqslant t \leqslant T, \\ (T-s)t & \text{if } 0 \leqslant t \leqslant s \leqslant T. \end{cases}$$

Denote by A the operator defined by the right-hand side of (4). Note that E is a Banach space with the norm $\|y\| = \max_{t\in J}\|y(t)\|$. We employ Schauder's fixed point theorem to show that operator A has a fixed point. Let $y \in E$. Note that $M(t)y(t) + (\mathcal{L}y)(t) + \sigma(t)$ is bounded in J, so operator $A : E \to E$ is continuous and bounded. In fact A is a compact map. Let

$$\big|M(t)y(t) + (\mathcal{L}y)(t) + \sigma(t)\big| \leqslant K, \quad K > 0.$$

Take $t_1, t_2 \in J$, $t_1 < t_2$ such that $|t_1 - t_2| < \frac{T\epsilon}{4KT^2 + |k_2 - k_1|}$ for $\epsilon > 0$. Then we have

$$\begin{aligned} &\big|Ay(t_1) - Ay(t_2)\big| \\ &\quad = \left|\int_0^T \big[G(t_1,s) - G(t_2,s)\big]\big\{M(s)y(s) + (\mathcal{L}y)(s) + \sigma(s)\big\}\,ds\right| + k|t_1 - t_2| \\ &\quad = \frac{1}{T}\left|(t_1 - t_2)\int_0^{t_1} s\big\{M(s)y(s) + (\mathcal{L}y)(s) + \sigma(s)\big\}\,ds\right. \\ &\qquad - t_1\int_{t_1}^{t_2}(T-s)\big\{M(s)y(s) + (\mathcal{L}y)(s) + \sigma(s)\big\}\,ds \\ &\qquad + (T - t_2)\int_{t_1}^{t_1} s\big\{M(s)y(s) + (\mathcal{L}y)(s) + \sigma(s)\big\}\,ds \\ &\qquad \left. + (t_2 - t_1)\int_{t_2}^{T}(T-s)\big\{M(s)y(s) + (\mathcal{L}y)(s) + \sigma(s)\big\}\,ds\right| + k|t_1 - t_2| \\ &\quad \leqslant (4KT + k)|t_1 - t_2| < \epsilon, \end{aligned}$$

where $k = \frac{|k_2 - k_1|}{T}$. Consequently $A : E \to E$ is compact. Schauder's fixed point theorem guarantees that A has a fixed point in E. In view of (4), we have $y(0) = k_1$, $y(T) = k_2$, and y'' exists and $y'' \in E$. Moreover, $y \in C^2(J, \mathbb{R})$ and $y''(t) = M(t)y(t) + (\mathcal{L}y)(t) + \sigma(t)$, so y is a solution of problem (3). It shows that y is the unique solution of (3). This ends the proof. □

Theorem 2. *Let assumptions from* H_1–H_5 *and* H_7, H_8 *be satisfied. Then problem* (1) *has extremal solutions in the sector* $[y_0, z_0]_*$.

Proof. Let us define two sequences $\{y_n, z_n\}$ by relations:

$$\begin{cases} y_n''(t) = (Qy_{n-1})(t) + M(t)\big[y_n(t) - y_{n-1}(t)\big] + \big(\mathcal{L}(y_n - y_{n-1})\big)(t), & t \in J, \\ y_n(0) = -\dfrac{1}{a} g_1\big(y_{n-1}(0), y_{n-1}(\delta)\big) + y_{n-1}(0), \\ y_n(T) = -\dfrac{1}{b} g_2\big(y_{n-1}(T), y_{n-1}(\gamma)\big) + y_{n-1}(T), \end{cases}$$

$$\begin{cases} z_n''(t) = (Qz_{n-1})(t) + M(t)\big[z_n(t) - z_{n-1}(t)\big] + \big(\mathcal{L}(z_n - z_{n-1})\big)(t), & t \in J, \\ z_n(0) = -\dfrac{1}{a} g_1\big(z_{n-1}(0), z_{n-1}(\delta)\big) + z_{n-1}(0), \\ z_n(T) = -\dfrac{1}{b} g_2\big(z_{n-1}(T), z_{n-1}(\gamma)\big) + z_{n-1}(T) \end{cases}$$

for $n = 1, 2, \ldots$. Note that y_1, z_1 are well defined, by Theorem 1.

First of all we want to show that

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J. \tag{5}$$

Put $p = y_0 - y_1$. Then

$$p(0) = y_0(0) + \frac{1}{a} g_1\big(y_0(0), y_0(\delta)\big) - y_0(0) \leqslant 0,$$
$$p(T) = y_0(T) + \frac{1}{b} g_2\big(y_0(T), y_0(\gamma)\big) - y_0(T) \leqslant 0,$$

by assumption H_5. Moreover,

$$\begin{aligned} p''(t) &\geqslant (Qy_0)(t) - (Qy_0)(t) - M(t)\big[y_1(t) - y_0(t)\big] - \big(\mathcal{L}(y_1 - y_0)\big)(t) \\ &= M(t)p(t) + (\mathcal{L}p)(t), \end{aligned}$$

by assumption H_5. This result and Lemma 1 show that $y_0(t) \leqslant y_1(t)$, $t \in J$. Similarly, we can show that $z_1(t) \leqslant z_0(t)$, $t \in J$. Now let $p = y_1 - z_1$. Then

$$\begin{aligned} p(0) &= \frac{1}{a}\big[g_1\big(z_0(0), z_0(\delta)\big) - g_1\big(y_0(0), y_0(\delta)\big)\big] + y_0(0) - z_0(0) \\ &\leqslant \frac{1}{a}\big[g_1\big(z_0(0), y_0(\delta)\big) - g_1\big(y_0(0), y_0(\delta)\big)\big] + y_0(0) - z_0(0) \\ &\leqslant z_0(0) - y_0(0) + y_0(0) - z_0(0) = 0, \\ p(T) &= y_0(T) + \frac{1}{b}\big[g_2\big(z_0(T), z_0(\gamma)\big) - g_2\big(y_0(T), y_0(\gamma)\big)\big] - z_0(T) \leqslant 0, \end{aligned}$$

by assumption H_8. Moreover,

$$\begin{aligned}
p''(t) &= (Qy_0)(t) - (Qz_0)(t) + M(t)\big[y_1(t) - y_0(t) - z_1(t) + z_0(t)\big] \\
&\quad + \big(\mathcal{L}(y_1 - y_0 - z_1 + z_0)\big)(t) \\
&\geqslant -M(t)\big[z_0(t) - y_0(t)\big] - \big(\mathcal{L}(z_0 - y_0)\big)(t) \\
&\quad + M(t)\big[y_1(t) - y_0(t) - z_1(t) + z_0(t)\big] + \big(\mathcal{L}(y_1 - y_0 - z_1 + z_0)\big)(t) \\
&= M(t)p(t) + (\mathcal{L}p)(t),
\end{aligned}$$

by assumption H_7. In view of Lemma 1, $y_1(t) \leqslant z_1(t)$, $t \in J$. It proves (5).

Now we show that y_1 is a lower solution of problem (1). Note that

$$\begin{aligned}
0 &= y_1(0) + \frac{1}{a}\big[g_1\big(y_0(0), y_0(\delta)\big) - g_1\big(y_1(0), y_1(\delta)\big) + g_1\big(y_1(0), y_1(\delta)\big)\big] - y_0(0) \\
&\geqslant y_1(0) + \frac{1}{a}\big[g_1\big(y_0(0), y_1(\delta)\big) - g_1\big(y_1(0), y_1(\delta)\big) + g_1\big(y_1(0), y_1(\delta)\big)\big] - y_0(0) \\
&\geqslant y_1(0) - y_0(0) - y_1(0) + y_0(0) + \frac{1}{a}g_1\big(y_1(0), y_1(\delta)\big) = \frac{1}{a}g_1\big(y_1(0), y_1(\delta)\big), \\
0 &= y_1(T) + \frac{1}{b}\big[g_2\big(y_0(T), y_0(\gamma)\big) - g_2\big(y_1(T), y_1(\gamma)\big) + g_2\big(y_1(T), y_1(\gamma)\big)\big] - y_0(T) \\
&\geqslant \frac{1}{b}g_2\big(y_1(T), y_1(\gamma)\big),
\end{aligned}$$

by assumption H_8. Moreover,

$$\begin{aligned}
y_1''(t) &= (Qy_0)(t) - (Qy_1)(t) + (Qy_1)(t) + M(t)\big[y_1(t) - y_0(t)\big] + \big(\mathcal{L}(y_1 - y_0)\big)(t) \\
&\geqslant -M(t)\big[y_1(t) - y_0(t)\big] - \big(\mathcal{L}(y_1 - y_0)\big)(t) + (Qy_1)(t) + M(t)\big[y_1(t) - y_0(t)\big] \\
&\quad + \big(\mathcal{L}(y_1 - y_0)\big)(t) = (Qy_1)(t),
\end{aligned}$$

by assumption H_7. It proves that y_1 is a lower solution of (1). Similarly, we can prove that z_1 is an upper solution of problem (1).

By mathematical induction we can show that

$$y_0(t) \leqslant \cdots \leqslant y_{n-1}(t) \leqslant y_n(t) \leqslant z_n(t) \leqslant z_{n-1}(t) \leqslant \cdots \leqslant z_0(t), \quad t \in J$$

for $n = 1, 2, \ldots$.

It implies that $\{y_n\}, \{z_n\}$ are uniformly bounded. We can show that they are equicontinuous on J. The Arzeli–Ascoli theorem guarantees the existence of subsequences $\{y_{n_k}\}, \{z_{n_k}\}$ and functions $\bar{y}, \bar{z} \in C(J, \mathbb{R})$ with y_{n_k}, z_{n_k} converging uniformly on J to $\bar{y}$ and $\bar{z}$, respectively, if $n_k \to \infty$. However, since the sequences $\{y_n\}, \{z_n\}$ are monotonic, we conclude that the whole sequences $\{y_n\}, \{z_n\}$ converge uniformly on J to $\bar{y}$ and $\bar{z}$, respectively, if $n \to \infty$. Indeed, $\bar{y}, \bar{z}$ are solutions of problem (1).

We need to show now that $(\bar{y}, \bar{z})$ are extremal solutions of problem (1) in the segment $[y_0, z_0]_*$. To prove it, we assume that $\tilde{y}$ is another solution of problem (1), and $y_{n-1}(t) \leqslant \tilde{y}(t) \leqslant z_{n-1}(t)$, $t \in J$ for some positive integer n. Put $p(t) = y_n(t) - \tilde{y}(t)$, $q(t) = \tilde{y}(t) - z_n(t)$, $t \in J$. Hence

$$\begin{aligned}
p(0) &= y_{n-1}(0) - \tilde{y}(0) + \frac{1}{a}\big[g_1\big(\tilde{y}(0), \tilde{y}(\delta)\big) - g_1\big(y_{n-1}(0), y_{n-1}(\delta)\big)\big] \\
&\leqslant y_{n-1}(0) - \tilde{y}(0) + \frac{1}{a}\big[g_1\big(\tilde{y}(0), y_{n-1}(\delta)\big) - g_1\big(y_{n-1}(0), y_{n-1}(\delta)\big)\big] \leqslant 0, \\
p(T) &= y_{n-1}(T) - \tilde{y}(T) + \frac{1}{b}\big[g_2\big(\tilde{y}(T), \tilde{y}(\gamma)\big) - g_2\big(y_{n-1}(T), y_{n-1}(\gamma)\big)\big]
\end{aligned}$$

$$\leqslant y_{n-1}(T) - \tilde{y}(T) + \frac{1}{b}\big[g_2\big(\tilde{y}(T), y_{n-1}(\gamma)\big) - g_2\big(y_{n-1}(T), y_{n-1}(\gamma)\big)\big] \leqslant 0.$$

This and assumption H_7 yield

$$\begin{aligned} p''(t) &= (Qy_{n-1})(t) + M(t)\big[y_n(t) - y_{n-1}(t)\big] + \big(\mathcal{L}(y_n - y_{n-1})\big)(t) - (Q\tilde{y})(t) \\ &\geqslant -M(t)\big[\tilde{y}(t) - y_{n-1}(t)\big] - \big(\mathcal{L}(\tilde{y} - y_{n-1})\big)(t) + M(t)\big[y_n(t) - y_{n-1}(t)\big] \\ &\quad + \big(\mathcal{L}(y_n - y_{n-1})\big)(t) = M(t)p(t) + (\mathcal{L}p)(t). \end{aligned}$$

By a similar way we can show that

$$q(0) \leqslant 0, \qquad q(T) \leqslant 0 \quad \text{and} \quad q''(t) \geqslant M(t)q(t) + (\mathcal{L}q)(t).$$

By Lemma 1, $y_n(t) \leqslant \tilde{y}(t) \leqslant z_n(t)$, $t \in J$. If $n \to \infty$, it yields $y_0(t) \leqslant \bar{y}(t) \leqslant \tilde{y}(t) \leqslant \bar{z}(t) \leqslant z_0(t)$, $t \in J$. It proves that $\bar{y}, \bar{z}$ are extremal solutions of problem (1) in the segment $[y_0, z_0]_*$. This ends the proof. □

Now we investigate the case when problem (1) has the unique solution but first we need the following

Lemma 2. *Assume that*

H_9: $\delta, \gamma \in (0, T)$ *and* $0 \leqslant k\gamma < T$, $l \geqslant 0$,
H_{10}: (i) $l \in (0, 1)$, $k = 1$ *or* (ii) $l = 0$, $k \geqslant 0$.

Let $p \in C^2(J, \mathbb{R})$, $B \in C(J \times E, \mathbb{R})$ *and*

$$\begin{cases} p''(t) \geqslant B(t, p), & t \in J, \\ p(0) \leqslant lp(\delta), & p(T) \leqslant kp(\gamma). \end{cases} \tag{6}$$

Then function p *satisfies the following inequality*

$$\begin{aligned} p(t) \leqslant \frac{1}{\Delta}\Bigg\{&(T - k\gamma)l\int_0^\delta\left(\int_0^s B(\tau, p)\,d\tau\right)ds \\ &+ l\delta\left[-\int_0^T\left(\int_0^s B(\tau, p)\,d\tau\right)ds + k\int_0^\gamma\left(\int_0^s B(\tau, p)\,d\tau\right)ds\right] \\ &+ t(1-l)\left[-\int_0^T\left(\int_0^s B(\tau, p)\,d\tau\right)ds + k\int_0^\gamma\left(\int_0^s B(\tau, p)\,d\tau\right)ds\right] \\ &- t(1-k)l\int_0^\delta\left(\int_0^s B(\tau, p)\,d\tau\right)ds\Bigg\} + \int_0^t\left(\int_0^s B(\tau, p)\,d\tau\right)ds, \end{aligned}$$

where $\Delta = (1 - l)(T - k\gamma) + l\delta(1 - k)$.

Proof. We replace problem (6) by

$$\begin{cases} p''(t) = B(t, p) + A, & t \in J, \\ p(0) = lp(\delta) + a, & p(T) = kp(\gamma) + b \end{cases}$$

with $A \geqslant 0$, $a \leqslant 0$, $b \leqslant 0$. Integrating it two times on $[0, T]$, we have

$$p(t) = p(0) + p'(0)t + \frac{1}{2}At^2 + D(t), \quad t \in J, \tag{7}$$

where $D(t) = \int_0^t (\int_0^s B(\tau, p)\, d\tau)\, ds$.

Using the boundary conditions, we have the system

$$\begin{cases} (1-l)p(0) - l\delta p'(0) = a + l\left(\frac{1}{2}A\delta^2 + D(\delta)\right), \\ (1-k)p(0) + (T - k\gamma)p'(0) = b - \frac{1}{2}AT^2 - D(T) + k\left(\frac{1}{2}A\gamma^2 + D(\gamma)\right), \end{cases} \tag{8}$$

for finding $p(0)$ and $p'(0)$. Solving system (8) and substituting the solutions in formula (7) we obtain

$$p(t) = aa_1(t) + bb_1(t) + \frac{1}{2}A\big[c_1 + d_1(t)\big] + h_1(t),$$

where

$$\begin{aligned} a_1(t) &= \frac{1}{\Delta}\big[T - k\gamma + t(k-1)\big], \qquad b_1(t) = \frac{1}{\Delta}\big[l\delta + t(1-l)\big], \\ c_1 &= \frac{1}{\Delta}l\delta\big[\delta(T - k\gamma) - T^2 + k\gamma^2\big], \\ d_1(t) &= \frac{1}{\Delta}t\big[(1-l)\big(-T^2 + k\gamma^2\big) - (1-k)l\delta^2\big] + t^2, \\ h_1(t) &= \frac{1}{\Delta}\big[(T - k\gamma)lD(\delta) + l\delta\big(-D(T) + kD(\gamma)\big) + t(1-l)\big(-D(T) + kD(\gamma)\big) \\ &\quad - t(1-k)lD(\delta)\big] + D(t). \end{aligned}$$

Assume that $l \in (0, 1)$, $k = 1$. Then $\Delta > 0$, $a_1(t) > 0$, $b_1(t) > 0$ and

$$\begin{aligned} c_1 &\leqslant \frac{1}{\Delta}l\delta\big[T(T-\gamma) - T^2 + \gamma^2\big] = \frac{1}{\Delta}l\delta\gamma(\gamma - T) < 0, \\ d_1(t) &= \frac{1}{\Delta}t(1-l)\big(-T^2 + \gamma^2\big) + t^2 \leqslant \frac{1}{\Delta}t(1-l)\big(-T^2 + T\gamma\big) + t^2 = t(t-T) \leqslant 0. \end{aligned}$$

This and (7) give $p(t) \leqslant h_1(t)$ because $A \geqslant 0$, $a \leqslant 0$, $b \leqslant 0$. The case (ii) can be discussed in the same way as above. It ends the proof. □

Theorem 3. *Assume that all assumptions of Theorem* 2 *and* H_9, H_{10} *are satisfied. In addition, we assume that*

H_{11}: *there exist a function* $L \in C(J, \mathbb{R}_+)$ *and a positive linear operator* $\mathcal{L}_1 \in C(E, E)$ *such that*

$$(Q\bar{u})(t) - (Qu)(t) \geqslant -L_1(t)\big[\bar{u}(t) - u(t)\big] - \big(\mathcal{L}_1(\bar{u} - u)\big)(t)$$

for $y_0(t) \leqslant u(t) \leqslant \bar{u}(t) \leqslant z_0(t)$,

H_{12}: *there exist constants* $0 < M_1 \leqslant a$, $0 < M_2 \leqslant b$, $N_i \geqslant 0$, $i = 1, 2$, *such that*

$$\begin{aligned} g_1(\bar{u}, \bar{v}) - g_1(u, v) &\geqslant M_1(\bar{u} - u) - N_1(\bar{v} - v), \\ g_2(\bar{u}_1, \bar{v}_1) - g_2(u_1, v_1) &\geqslant M_2(\bar{u}_1 - u_1) - N_2(\bar{v}_1 - v_1) \end{aligned}$$

for $y_0(0) \leqslant u \leqslant \bar{u} \leqslant z_0(0)$, $y_0(\delta) \leqslant v \leqslant \bar{v} \leqslant z_0(\delta)$, $y_0(T) \leqslant u_1 \leqslant \bar{u}_1 \leqslant z_0(T)$, $y_0(\gamma) \leqslant v_1 \leqslant \bar{v}_1 \leqslant z_0(\gamma)$

and

$$\frac{1}{\Delta}\big[l\delta + T(1-l)\big]\int_0^T\left(\int_0^s\big[L_1(\tau) + (\mathcal{L}_1\mathbf{1})(\tau)\big]\,d\tau\right)ds < 1 \tag{9}$$

with $l = \frac{N_1}{M_1}$, $k = \frac{N_2}{M_2}$, $\Delta = (1-l)(T-k\gamma) + l\delta(1-k)$.

Then problem (1) *has, in the sector* $[y_0, z_0]_*$, *the unique solution.*

Proof. Theorem 2 says that problem (1) has, in the sector $[y_0, z_0]_*$, extremal solutions $\bar{y}, \bar{z}$ and $y_0(t) \leqslant \bar{y}(t) \leqslant \bar{z}(t) \leqslant z_0(t)$, $t \in J$. We want to show that $\bar{z} = \bar{y}$. Put $p = \bar{z} - \bar{y}$, so $p(t) \geqslant 0$, $t \in J$. In view of assumption H_{12}, we get

$$\begin{aligned}
0 &= g_1\big(\bar{z}(0), \bar{z}(\delta)\big) - g_1\big(\bar{y}(0), \bar{y}(\delta)\big) \geqslant M_1\big[\bar{z}(0) - \bar{y}(0)\big] - N_1\big[\bar{z}(\delta) - \bar{y}(\delta)\big] \\
&= M_1 p(0) - N_1 p(\delta), \\
0 &= g_2\big(\bar{z}(T), \bar{z}(\gamma)\big) - g_2\big(\bar{y}(T), \bar{y}(\gamma)\big) \geqslant M_2\big[\bar{z}(T) - \bar{y}(T)\big] - N_2\big[\bar{z}(\gamma) - \bar{y}(\gamma)\big] \\
&= M_2 p(T) - N_2 p(\gamma),
\end{aligned}$$

so

$$p(0) \leqslant lp(\delta), \qquad p(T) \leqslant kp(\gamma).$$

Moreover, in view of assumption H_{11}, we see that

$$p''(t) = (Q\bar{z})(t) - (Q\bar{y})(t) \geqslant -L_1(t)p(t) - (\mathcal{L}_1 p)(t) \equiv B(t, p).$$

It is obvious that $B(t, p) \leqslant 0$, $t \in J$. From Lemma 2, we obtain

$$p(t) \leqslant \frac{1}{\Delta}\left\{-l\delta\int_0^T\left(\int_0^s B(\tau, p)\,d\tau\right)ds - t(1-l)\int_0^T\left(\int_0^s B(\tau, p)\,d\tau\right)ds\right\}. \tag{10}$$

Suppose that $\max_{t\in J} p(t) = p(t_1) = d > 0$. Then $-B(t, p) \leqslant d[L_1(t) + (\mathcal{L}_1\mathbf{1})(t)]$. From (10), we have now

$$\begin{aligned}
d &\leqslant \frac{d}{\Delta}\big[l\delta + t_1(1-l)\big]\int_0^T\left(\int_0^s\big[L_1(\tau) + (\mathcal{L}_1\mathbf{1})(\tau)\big]\,d\tau\right)ds \\
&\leqslant \frac{d}{\Delta}\big[l\delta + T(1-l)\big]\int_0^T\left(\int_0^s\big[L_1(\tau) + (\mathcal{L}_1\mathbf{1})(\tau)\big]\,d\tau\right)ds,
\end{aligned}$$

so

$$d\left\{1 - \frac{1}{\Delta}\big[l\delta + T(1-l)\big]\int_0^T\left(\int_0^s\big[L_1(\tau) + (\mathcal{L}_1\mathbf{1})(\tau)\big]\,d\tau\right)ds\right\} \leqslant 0.$$

Hence $d \leqslant 0$, by condition (9), so $p(t) = 0$, $t \in J$. It proves that problem (1) has the unique solution. It ends the proof. □

Example. Consider the problem

$$\begin{cases} x''(t) = (Qx)(t), \quad t \in [0, T], \\ 0 = x(0) - \frac{1}{3}x^2\left(\frac{1}{2}T\right) \equiv g_1\left(x(0), x\left(\frac{1}{2}T\right)\right), \\ 0 = x(T) - x\left(\frac{1}{3}T\right) \equiv g_2\left(x(T), x\left(\frac{1}{3}T\right)\right), \end{cases} \tag{11}$$

where

$$(Qx)(t) = -a_1 \cos x(t) - a_2(t)x(t) - b_1(t) \sin x\left(\frac{1}{4}t\right) + b_2(t)x\left(\frac{1}{3}t\right) - k_1.$$

Note that $\delta = \frac{1}{2}T$, $\gamma = \frac{1}{3}T$. Assume that $a_1 > 0$, $a_2, b_1, b_2 \in C(J, [0, \infty))$, $k_1 \geqslant 0$ and

$$-a_1 \cos 1 - a_2(t) - b_1(t) \sin 1 + b_2(t) - k_1 \geqslant 0, \tag{12}$$

$$\frac{a_1 T^2}{2} + \int_0^T \int_s^T \left[b_1(\tau) + b_2(\tau)\right] d\tau\, ds \leqslant 1, \tag{13}$$

$$\frac{a_1 T^2}{2} + \int_0^T \int_0^s a_2(\tau)\, d\tau\, ds < \frac{1}{3}. \tag{14}$$

Take $y_0(t) = 0$, $z_0(t) = 1$, $t \in J$. Then

$$(Qy_0)(t) = -a_1 - k_1 < 0 = y_0''(t),$$
$$(Qz_0)(t) = -a_1 \cos 1 - a_2(t) - b_1(t) \sin 1 + b_2(t) - k_1 \geqslant 0 = z_0''(t),$$

by (12). Moreover,

$$g_1\left(y_0(0), y_0\left(\frac{1}{2}T\right)\right) = g_1(0, 0) = 0, \qquad g_2\left(y_0(T), y_0\left(\frac{1}{3}T\right)\right) = g_2(0, 0) = 0,$$
$$g_1\left(z_0(0), z_0\left(\frac{1}{2}T\right)\right) = g_1(1, 1) = \frac{2}{3} > 0, \qquad g_2\left(z_0(T), z_0\left(\frac{1}{3}T\right)\right) = g_2(1, 1) = 0.$$

This shows that y_0, z_0 are lower and upper solutions of problem (11), respectively. It is quite easy to see that assumptions H_7, H_8 hold with $a = b = 1$, $M(t) = a_1$ and $(\mathcal{L}u)(t) = b_1(t)u(\frac{1}{4}t) + b_2(t)u(\frac{1}{3}t)$. In view of (13), assumption H_3 holds. It proves that problem (11) has extremal solutions in the sector $[y_0, z_0]_*$, by Theorem 2.

Now we are going to show that all assumptions of Theorem 3 are satisfied. Note that $L_1(t) = a_1 + a_2(t)$, $(\mathcal{L}_1 u)(t) = 0$, $t \in J$, and $M_1 = 1$, $N_1 = \frac{2}{3}$, $M_2 = N_2 = 1$, so $k = 1$, $l = \frac{2}{3} < 1$. Moreover, $\Delta = \frac{2}{9}T$ and assumption (9) holds, by (14). Hence, problem (11) has, in the sector $[y_0, z_0]_*$, the unique solution.

For example, we take $T = 1$, $k_1 = 0$, and

$$a_1 = \frac{1}{8 \cos 1}, \qquad a_2(t) = \frac{1}{8} \sin t, \qquad b_1(t) = \frac{\beta \sin t}{\sin 1}, \qquad b_2(t) = \left(\frac{1}{8} + \beta\right) \sin t + \frac{1}{8}.$$

Then conditions (12)–(14) are satisfied with $\beta \leqslant 1.0143$.

4. Quasi-solutions. Unique solution

This section deals with the problem of existence of quasi-solutions for (1). The case when problem (1) has the unique solution is also investigated.

A pair of functions $y_0, z_0 \in C^2(J,\mathbb{R})$ is called weakly coupled (w.c.) lower and upper solutions of problem (1) if

$$\begin{cases} y_0''(t) \geqslant (Qy_0)(t), \quad t \in J,\\ 0 \geqslant g_1\big(y_0(0), z_0(\delta)\big), \qquad 0 \geqslant g_2\big(y_0(T), z_0(\gamma)\big),\\ z_0''(t) \leqslant (Qz_0)(t), \quad t \in J,\\ 0 \leqslant g_1\big(z_0(0), y_0(\delta)\big), \qquad 0 \leqslant g_2\big(z_0(T), y_0(\gamma)\big). \end{cases}$$

A pair (U,V), $U, V \in C^2(J,\mathbb{R})$ is called a weakly coupled quasi-solution of problem (1) if

$$\begin{cases} U'(t) = (QU)(t), \quad t \in J,\\ 0 = g_1\big(U(0), V(\delta)\big), \qquad 0 = g_2\big(U(T), V(\gamma)\big),\\ V'(t) = (QV)(t), \quad t \in J,\\ 0 = g_1\big(V(0), U(\delta)\big), \qquad 0 = g_2\big(V(T), U(\gamma)\big). \end{cases}$$

A weakly coupled quasi-solution $(\bar U, \bar V)$, $\bar U, \bar V \in C^2(J,\mathbb{R})$ is called the weakly coupled minimal and maximal quasi-solution of problem (1) if for any weakly coupled quasi-solution (U, V) of (1) we have $\bar U(t) \leqslant U(t)$, $V(t) \leqslant \bar V(t)$ on J.

Theorem 4. *Suppose that assumptions* H_1–H_4, H_7 *are satisfied. Let* $y_0, z_0 \in C^2(J,\mathbb{R})$ *be w.c. lower and upper solutions of problem* (1)*, and* $y_0(t) \leqslant z_0(t)$*,* $t \in J$*. In addition, we assume that* g_i*,* $i = 1, 2$ *are nondecreasing with respect to the second variable and there exist positive constants* a, b *such that*

$$g_1(\bar u, v) - g_1(u, v) \leqslant a(\bar u - u),$$
$$g_2(\bar u_1, v_1) - g_2(u_1, v_1) \leqslant b(\bar u_1 - v_1)$$

for $y_0(0) \leqslant u \leqslant \bar u \leqslant z_0(0)$*,* $y_0(T) \leqslant u_1 \leqslant \bar u_1 \leqslant z_0(T)$*,* $y_0(\delta) \leqslant v \leqslant z_0(\delta)$*,* $y_0(\gamma) \leqslant v_1 \leqslant z_0(\gamma)$*.*

Then problem (1) *has, in the sector* $[y_0, z_0]_*$ *the w.c. minimal and maximal quasi-solutions.*

Proof. Let us define two sequences $\{y_n, z_n\}$ by relations:

$$\begin{cases} y_n''(t) = (Qy_{n-1})(t) + M(t)\big[y_n(t) - y_{n-1}(t)\big] + \big(\mathcal{L}(y_n - y_{n-1})\big)(t), \quad t \in J,\\ y_n(0) = -\dfrac{1}{a} g_1\big(y_{n-1}(0), z_{n-1}(\delta)\big) + y_{n-1}(0),\\ y_n(T) = -\dfrac{1}{b} g_2\big(y_{n-1}(T), z_{n-1}(\gamma)\big) + y_{n-1}(T), \end{cases}$$

$$\begin{cases} z_n''(t) = (Qz_{n-1})(t) + M(t)\big[z_n(t) - z_{n-1}(t)\big] + \big(\mathcal{L}(z_n - z_{n-1})\big)(t), \quad t \in J,\\ z_n(0) = -\dfrac{1}{a} g_1\big(z_{n-1}(0), y_{n-1}(\delta)\big) + z_{n-1}(0),\\ z_n(T) = -\dfrac{1}{b} g_2\big(z_{n-1}(T), y_{n-1}(\gamma)\big) + z_{n-1}(T) \end{cases}$$

for $n = 1, 2, \ldots$. The proof is similar to the proof of Theorem 2 and therefore it is omitted. □

Theorem 5. *Assume that all assumptions of Theorem* 4 *and* H_9–H_{11} *are satisfied. In addition, we assume that*:

H'_{12}: *there exist constants* $0 < M_1 \leqslant a$, $0 < M_2 \leqslant b$, $N_i \geqslant 0$, $i = 1, 2$, *such that*

$$g_1(\bar{u}, v) - g_1(u, \bar{v}) \geqslant M_1(\bar{u} - u) - N_1(\bar{v} - v),$$
$$g_2(\bar{u}_1, v_1) - g_2(u_1, \bar{v}_1) \geqslant M_2(\bar{u}_1 - u_1) - N_2(\bar{v}_1 - v_1)$$

for $y_0(0) \leqslant u \leqslant \bar{u} \leqslant z_0(0)$, $y_0(\delta) \leqslant v \leqslant \bar{v} \leqslant z_0(\delta)$, $y_0(T) \leqslant u_1 \leqslant \bar{u}_1 \leqslant z_0(T)$, $y_0(\gamma) \leqslant v_1 \leqslant \bar{v}_1 \leqslant z_0(\gamma)$

and

$$\frac{1}{\Delta}\big[l\delta + T(1-l)\big]\int_0^T\left(\int_0^s \big[L_1(\tau) + (\mathcal{L}_1\mathbf{1})(\tau)\big]\,d\tau\right)ds < 1$$

with $l = \frac{N_1}{M_1}$, $k = \frac{N_2}{M_2}$, $\Delta = (1-l)(T - k\gamma) + l\delta(1-k)$.

Then problem (1) *has, in the sector* $[y_0, z_0]_*$, *the unique solution.*

The proof of Theorem 5 is similar to the proof of Theorem 3 and therefore it is omitted.

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