NONLINEAR PHENOMENA OF SMALL-SCALE SOUND IN A GAS WITH EXPONENTIAL STRATIFICATION

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Abstract: The nonlinear dynamics of perturbations, quickly varying in space, with comparatively large characteristic wavenumbers k: $k > 1/H$, is considered. H is the scale of density and pressure reduction in unperturbed gas, as the coordinate increases (H is the so-called height of the uniform equilibrium gas). Coupling nonlinear equations which govern the sound and the entropy mode in a weakly nonlinear flow are derived. They describe the dynamics of the gas in the leading order, with an accuracy up to the terms $(kH)^{-1}$. In the field of the dominative sound mode, other induced modes contain parts which propagate approximately with their own linear speeds and the speed of the dominative mode. The scheme of successive approximations of nonlinear links between perturbations in the progressive mode is established. The numerical calculations for some kinds of impulses confirm the theory.

Keywords: Sound propagation, Non-uniform media, Nonlinear acoustics

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1. Introduction

An external force which acts on a fluid, makes its thermodynamic parameters vary in space and time. The simplest example is a constant mass force which causes the pressure and density of an isothermal fluid vary exponentially [1]. The characteristic scale of spatial inhomogeneity yields in dispersion, *i.e.*, in the dependence of the phase speed of any harmonic signal on its wavenumber. The perturbations typically occupy the bounded space and contain a wide range of wavenumbers. This suggests a description of the dynamics of a waveform as a whole by means of some integro-differential equations [2]. For this purpose, linear modes (modes of an infinitely small magnitude) taking place in non-uniform media should be determined as links of specific perturbations. As usual, dispersion essentially complicates the definition of modes and establishment of dynamic equations which govern every mode. The equations governing every mode include the first order partial derivatives with respect to time. In view of that, this set of equations is more preferable for analytical or numerical solution than equations describing the total field perturbations, which include the fifth-order partial derivatives in a three-dimensional flow. The modes in one-dimensional flow in stratified gas and precise linear and weakly nonlinear dynamic equations for every mode have been derived in [3]. The possibility of distinguishing modes and an analytical prediction of the waveform dynamics is not only an advance in the theory but it is also of importance in meteorology and atmosphere-ocean dynamics applications [4]. It may be resolved with the help of linear operators which distinguish modes uniquely in the linear flow, which have been derived by the author for a waveform with an arbitrary spectrum [3, 5].

Modes of a linear flow propagate independently. Nonlinearity is essential in the inviscid flow of a fluid with exponential stratification because the magnitude of the velocity grows due to the decreasing background density. Nonlinear interactions of modes are described by coupling equations which readily follow from the projecting. The author has established that induced modes in a homogeneous fluid contain parts which propagate with the speed of the dominative mode and with their own speed [6]. The parts of the induced modes which follow the dominative sound, contribute to the corrected nonlinear relations of perturbations in the progressive mode. The procedure may be repeated in order to obtain relations within any accuracy as series in powers of the Mach number. This gives an idea to apply the procedure of "successive approximations" in a weakly dispersive flow. In this study, we develop the idea, establishing a way to obtain more accurate relations for sound perturbations in a weakly dispersive flow. The examplary dominative mode is sound progressive upwards. The nonlinear equations are derived and solved in the leading order, and their numerical evaluations for some kinds of initial disturbances are discussed.

2. Conservation equations and dispersion relations in one dimensional flow

We start from the conservation equations of momentum, energy and mass in a one-dimensional gas flow without any mechanical and thermal losses which is affected by the external constant mass force, g [1, 4]:

$$\begin{gathered}\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial z}-g\\ \rho\left(\frac{\partial E}{\partial t}+v\frac{\partial E}{\partial z}\right)+p\frac{\partial v}{\partial z}=0\\ \frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial z}=0\end{gathered}\tag{1}$$

where ρ, p, E, v denote the fluid density, pressure, internal energy and particle velocity, g is directed oppositely to axis OZ, it may be readily associated with the constant gravity acceleration, and z equals the distance from a boundary (which may correspond to the Earth surface), and t denotes time. The unperturbed pressure and density are functions on the coordinate: $\rho_0 = \rho_{00}\exp(-z/H)$, $p_0 = p_{00}\exp(-z/H) = \rho_{00}gH\exp(-z/H)$, where $\rho_{00} = \rho_0(0)$, $p_{00} = p_0(0)$ denote the values of density and pressure at $z=0$, and $H=(C_p-C_v)T_0/g$, where T_0 is the unperturbed temperature of a gas. The thermodynamic relation for the internal energy of an ideal gas completes the system (1):

$$E = \frac{p}{\rho(\gamma-1)} \tag{2}$$

with $\gamma = C_p/C_v$ being the specific heat ratio. The well-known substitution

$$R=\rho'\cdot\exp(z/2H), \qquad P=p'\cdot\exp(z/2H), \qquad V=v\cdot\exp(-z/2H) \tag{3}$$

where ρ', p' denote perturbations, makes it possible to eliminate exponential factors in linear equations and to apply the Fourier-analysis in the studies of infinitely-small signal flows [1, 4].

The weakly nonlinear dynamics of a gas is described by the following system of equations:

$$\begin{aligned}
&\frac{\partial V}{\partial t}+\frac{1}{\rho_{00}}\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)P+\frac{gR}{\rho_{00}}=\varphi_1\\
&\frac{\partial P}{\partial t}+\gamma gH\rho_{00}\left(\frac{\partial V}{\partial z}+\frac{1}{\gamma H}(\gamma/2-1)V\right)=\varphi_2\\
&\frac{\partial R}{\partial t}+\rho_{00}\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)V=\varphi_3\\
&\varphi_1=-\exp(z/2H)\left(V\left(\frac{\partial}{\partial z}+\frac{1}{2H}\right)V-\frac{R}{\rho_{00}^2}\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)P-\frac{g}{\rho_{00}^2}R^2\right)\\
&\varphi_2=-\exp(z/2H)\left(V\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)P+\gamma P\left(\frac{\partial}{\partial z}+\frac{1}{2H}\right)V\right)\\
&\varphi_3=-\exp(z/2H)\left(R\frac{\partial V}{\partial z}+V\frac{\partial R}{\partial z}\right)
\end{aligned} \tag{4}$$

It has been derived in [5] and accounts for only quadratic nonlinear terms among all other nonlinear ones in Equations (1) which are represented by the right-hand parts of equations φ_1, φ_2, φ_3. The motions of infinitely small magnitudes satisfy the linear system (4) with a zero right-hand side. Any of the quantities R, P, V may be represented by the Fourier integrals as follows:

$$\Psi=\int_{-\infty}^{\infty}\Psi_k\cdot\exp(i\omega t-ikz)dk+cc \tag{5}$$

The dispersion relations of the linear flow are well-known, they are the roots of the dispersion equation resulting from the linearized version of Equation (4), after substitution of Equations (5) in the system (4) and letting φ_1, φ_2, φ_3 be zero. We remind that:

$$\omega_{1,\mathrm{ac}} = c_0\sqrt{k^2+\frac{1}{4H^2}}, \qquad \omega_{2,\mathrm{ac}} = -c_0\sqrt{k^2+\frac{1}{4H^2}}, \qquad \omega_{\mathrm{ent}} = 0 \tag{6}$$

where

$$c_0 = \sqrt{\gamma\frac{p_{00}}{\rho_{00}}} \tag{7}$$

is the speed of sound of an infinitely-small scale as compared to H. The two first roots correspond to the acoustic modes, progressive in different directions of axis OZ, and the last one determines the entropy mode.

3. Specific perturbations in a linear flow

The dispersion relations (6) determine linear links of the Fourier transforms P_k, R_k, V_k which specify every mode of a linear flow. The correspondent links for acoustic and entropy modes in the (z,t) space are integro-differential, they have been established by the author in [6]. These links are valid for perturbations of any scale, *i.e.*, short and large as compared to H. We remind the links connecting acoustic pressure, excess density and velocity in the first acoustic mode:

$$\begin{aligned} P_{1,\mathrm{ac}}(z,t) &= \frac{\rho_{00}}{\pi c_0}\int_{-\infty}^{\infty} dz' F(z-z')\left(\frac{2-\gamma}{2}g - c_0^2\frac{\partial}{\partial z'}\right)V_{1,\mathrm{ac}}(z',t) \\ R_{1,\mathrm{ac}}(z,t) &= \frac{\rho_{00}}{\pi c_0}\left(-\frac{\partial}{\partial z}+\frac{1}{2H}\right)\int_{-\infty}^{\infty} dz' F(z-z')V_{1,\mathrm{ac}}(z',t) \end{aligned} \tag{8}$$

where $F(z)$ reflects the dispersive features of a stratified gas,

$$F(z) = \frac{2}{\pi}\left(I_0(z/2H) - L_0(z/2H)\right) = \int_0^{\infty} dk\frac{\sin(kz)}{\sqrt{k^2+1/4H^2}} \tag{9}$$

with I_0, L_0 denoting the modified Bessel function of zero order, and the Struve function, respectively. These links are valid at any time. Velocity is continued at the whole axis OZ in an odd way. This condition may not reflect the physical conditions of the flow at the boundary $z=0$. This requires additional analysis.

We concentrate on the quickly varying perturbations. This allows expanding the links of specific perturbations in the Taylor series in the vicinity of $1/kH = 0$ valid for $kH > 1$. The links for acoustic modes, propagating upwards and downwards (indexed by 1 and 2, in accordance with the numbering of dispersion relations), take the forms which have been firstly established in [5]:

$$\begin{aligned} P_{1,\mathrm{ac}} &= c_0\rho_{00}\left(1+\tfrac{\gamma-2}{2\gamma H}\int^z dz\right)V_{1,\mathrm{ac}}, \qquad & R_{1,\mathrm{ac}} &= \tfrac{\rho_{00}}{c_0}\left(1-\tfrac{1}{2H}\int^z dz\right)V_{1,\mathrm{ac}} \\ P_{2,\mathrm{ac}} &= -c_0\rho_{00}\left(1+\tfrac{\gamma-2}{2\gamma H}\int^z dz\right)V_{2,\mathrm{ac}}, \qquad & R_{2,\mathrm{ac}} &= -\tfrac{\rho_{00}}{c_0}\left(1-\tfrac{1}{2H}\int^z dz\right)V_{2,\mathrm{ac}} \end{aligned} \tag{10}$$

The lower limit of integration is ∞, where all perturbations vanish. The entropy mode is stationary, its specific velocity equals zero, and perturbations in pressure and density are related by the equalities

$$P_{\text{ent}} = -\frac{c_0^2}{\gamma H}\int^{z} R_{\text{ent}}\,dz \tag{11}$$

where R_{ent} is any smooth function of z. In contrast to the motion over the uniform background, the excess pressure of the stationary mode does not equal zero. Making use of the relations (10)–(11), the matrix operators may be readily derived, which project the vector of total perturbations,

$$\psi = \begin{pmatrix} V(z,t) & P(z,t) & R(z,t) \end{pmatrix}^T \tag{12}$$

into each of the specific modes. The details may be found in [5]. The rows which project ψ onto specific excess densities, follow from the algebraic relations

$$\Pi_{1,\text{ac}}\psi = R_{1,\text{ac}}, \quad \Pi_{2,\text{ac}}\psi = R_{2,\text{ac}}, \quad \Pi_{\text{ent}}\psi = R_{\text{ent}} \tag{13}$$

They take the leading-order forms

$$\begin{aligned}
\Pi_{1,\text{ac}} &= \Big(\tfrac{\rho_{00}}{2c_0}\Big(1 - \tfrac{1}{2H}\int^z dz\Big) \quad \tfrac{1}{2c_0^2} - \tfrac{1}{2c_0^2 H}\int^z dz \quad \tfrac{1}{2\gamma H}\int^z dz \Big)\\
\Pi_{2,\text{ac}} &= \Big(-\tfrac{\rho_{00}}{2c_0}\Big(1 - \tfrac{1}{2H}\int^z dz\Big) \quad \tfrac{1}{2c_0^2} - \tfrac{1}{2c_0^2 H}\int^z dz \quad \tfrac{1}{2\gamma H}\int dz \Big)\\
\Pi_{\text{ent}} &= \Big(0 \quad \tfrac{1}{c_0^2}\Big(-1 + \tfrac{1}{H}\int^z dz\Big) \quad 1 - \tfrac{1}{\gamma H}\int^z dz \Big)
\end{aligned} \tag{14}$$

The projecting row $\Pi_{1,\text{ac}}$, applying to the system (4), reduces all the terms of the second acoustic and entropy modes in the linear left side and yields a quadratic nonlinear "source" in its right-hand side. Analogously, $\Pi_{2,\text{ac}}$ and Π_{ent} reduce all foreign terms in the linear part of the dynamic equations.

4. Nonlinear dynamics and correction in relations of specific perturbations

We will consider the upwards progressive mode as dominative, *i.e.*, the initial perturbations correspond mostly to the first acoustic mode. This means that the perturbations which specify this mode are much larger initially than those of the other modes. The initial disturbances are represented by the zero-order vector

$$\begin{aligned}
\psi^0 &= \big(V_{1,\text{ac}}^0(z,t) \quad P_{1,\text{ac}}^0(z,t) \quad R_{1,\text{ac}}^0(z,t) \big)^T =\\
&\Big(\tfrac{c_0}{\rho_{00}}\Big(1 + \tfrac{1}{2H}\int^z dz\Big) \quad c_0^2\Big(1 + \tfrac{\gamma-1}{\gamma H}\int^z dz\Big) \quad 1 \Big)^T R_{1,\text{ac}}^0(z,t)
\end{aligned} \tag{15}$$

This allows considering only the terms which belong to the first acoustic mode among all nonlinear terms. Applying of $\Pi_{2,\text{ac}}$, Π_{ent} at the system (4), one arrives at dynamic equations describing specific excess densities:

$$\frac{\partial R_{2,\text{ac}}^1}{\partial t} - c_0\frac{\partial R_{2,\text{ac}}^1}{\partial z} = \Lambda_2, \qquad \frac{\partial R_{\text{ent}}^1}{\partial t} = \Lambda_3 \tag{16}$$

where

$$\Lambda_2 = -\frac{(\gamma+1)c_0}{2\rho_{00}} e^{z/2H} R^0_{1,\text{ac}} \frac{\partial R^0_{1,\text{ac}}}{\partial z} - \\ \frac{c_0}{H\rho_{00}} e^{z/2H} \left(\frac{\gamma-1}{4} (R^0_{1,\text{ac}})^2 + \frac{\gamma-1}{2} \frac{\partial R^0_{1,\text{ac}}}{\partial z} \int^z R^0_{1,\text{ac}} dz \right)$$
$$\Lambda_3 = \frac{(\gamma-1)c_0}{\rho_{00}} e^{z/2H} R^0_{1,\text{ac}} \frac{\partial R^0_{1,\text{ac}}}{\partial z} + \\ \frac{c_0(\gamma-1)}{2H\rho_{00}} e^{z/2H} \left(\left(R^0_{1,\text{ac}}\right)^2 + 2 \frac{\partial R^0_{1,\text{ac}}}{\partial z} \int^z R^0_{1,\text{ac}} dz \right) \tag{17}$$

These equations determine the dynamics of the first-order perturbations of the foreign modes induced in the field of the dominative sound, $R^1_{2,\text{ac}}$ and R^1_{ent} (they are of the order squared Mach number, Ma^2). During integration, we bear in mind that

$$H^{-1} \int^z e^{z'/2H} \frac{\partial \left(R^0_{1,\text{ac}}(z',t)\right)^2}{\partial z'} dz' = e^{z/2H} \left(R^0_{1,\text{ac}}(z,t)\right)^2 \tag{18}$$

in the frames of the accepted accuracy for quickly varying perturbations. The zero-order excess density, $R^0_{1,\text{ac}}$, satisfies the zero-order equation

$$\frac{\partial R^0_{1,\text{ac}}}{\partial t} + c_0 \frac{\partial R^0_{1,\text{ac}}}{\partial z} = 0 \tag{19}$$

and depends exclusively on $z - c_0 t$. Equations (16) may be readily integrated along the correspondent characteristics:

$$R^1_{2,\text{ac}} = \int_0^t \Lambda_2(z + c_0(t-\tau), \tau) d\tau, \quad R^1_{\text{ent}} = \int_0^t \Lambda_3(z,\tau) d\tau \tag{20}$$

Integration along the characteristics is possible because sound does not experience dispersive distortions up to terms of order $(kH)^{-1}$. The result of integration in a gas without stratification consists evidently of two parts, one propagating with the linear speed of the first acoustic mode, and the second which has the speed of the induced mode. Both parts do not change their shapes in the course of propagation. This has been analyzed in details in [6]. In the case of weak dispersion, the integrands are not so simple and the conclusions concerning the case of a uniform equilibrium gas are not valid any longer. It is expected that the result of integration still contains mainly parts of different speeds which are affected by dispersion and are additionally distorted due to the factor $\exp(z/2H)$ in the integrants. If so, the corrected nonlinear links for the perturbations inside the first acoustic mode may be evaluated by adding the terms in $R^1_{2,\text{ac}}, R^1_{\text{ent}}$ which propagate in the positive direction of axis OZ, to the first-order disturbance, $R^0_{1,\text{ac}}$. Analogously, the correspondent parts in the perturbations in pressure and velocity may correct the nonlinear links of perturbations in the first progressive mode.

Other specific perturbations of velocity and pressure are governed by the equations

$$\begin{aligned} &\frac{\partial V^1_{2,\mathrm{ac}}}{\partial t} - c_0 \frac{\partial V^1_{2,\mathrm{ac}}}{\partial z} = \Theta_2, \qquad \frac{\partial V^1_{\mathrm{ent}}}{\partial t} = 0, \\ &\frac{\partial P^1_{2,\mathrm{ac}}}{\partial t} - c_0 \frac{\partial P^1_{2,\mathrm{ac}}}{\partial z} = \Phi_2, \qquad \frac{\partial P^1_{\mathrm{ent}}}{\partial t} = \Phi_3 \end{aligned} \tag{21}$$

$$\begin{aligned} \Theta_2 = &-\frac{(\gamma+1)}{2\rho_0^2} e^{z/2H} R^0_{1,\mathrm{ac}} \frac{\partial R^0_{1,\mathrm{ac}}}{\partial z} - \\ &\frac{1}{8H\rho_0^2} e^{z/2H} \left((3+3\gamma)\left(R^0_{1,\mathrm{ac}}\right)^2 + 4\gamma \frac{\partial R^0_{1,\mathrm{ac}}}{\partial z} \int R^0_{1,\mathrm{ac}} dz \right) \\ \Phi_2 = &-\frac{(\gamma+1)c_0^3}{2\rho_{00}} e^{z/2H} R^0_{1,\mathrm{ac}} \frac{\partial R^0_{1,\mathrm{ac}}}{\partial z} - \\ &\frac{c_0^3}{8\gamma H \rho_{00}} e^{z/2H} \left(\left(4\gamma^2 - \gamma - 1\right)\left(R^0_{1,\mathrm{ac}}\right)^2 + 4(\gamma-1)\gamma \frac{\partial R^0_{1,\mathrm{ac}}}{\partial z} \int^z R^0_{1,\mathrm{ac}} dz \right) \\ \Phi_3 = &-\frac{c_0^3}{2\gamma H \rho_{00}} e^{z/2H} (\gamma-1)\left(R^0_{1,\mathrm{ac}}\right)^2 \end{aligned} \tag{22}$$

The zero-order velocity and excess pressure, $V^0_{1,\mathrm{ac}}$ and $P^0_{1,\mathrm{ac}}$, satisfy the zero-order equations

$$\frac{\partial V^0_{1,\mathrm{ac}}}{\partial t} + c_0 \frac{\partial V^0_{1,\mathrm{ac}}}{\partial z} = 0, \qquad \frac{\partial P^0_{1,\mathrm{ac}}}{\partial t} + c_0 \frac{\partial P^0_{1,\mathrm{ac}}}{\partial z} = 0 \tag{23}$$

and the foreign quantities, induced in their field, take the forms

$$\begin{aligned} &V^1_{2,\mathrm{ac}} = \int_0^t \Theta_2\left(z + c_0(t-\tau), \tau\right) d\tau, \qquad V^1_{\mathrm{ent}} = 0 \\ &P^1_{2,\mathrm{ac}} = \int_0^t \Phi_2\left(z + c_0(t-\tau), \tau\right) d\tau, \qquad P^1_{\mathrm{ent}} = \int_0^t \Phi_3(z,\tau) d\tau \end{aligned} \tag{24}$$

The solutions (20)–(24) satisfy the zero initial conditions at $t = 0$.

The result of integration of the leading-order term in Λ_2 along the characteristics takes the form:

$$\begin{aligned} R^1_{2,\mathrm{ac}} &= -\frac{(\gamma+1)c_0}{4\rho_{00}} \int_0^t e^{\xi/2H} \frac{\partial \left(R^0_{1,\mathrm{ac}}\right)^2}{\partial \xi}\Bigg|_{\xi = z + c_0(t-\tau),\tau} d\tau \\ &= \frac{(\gamma+1)}{8\rho_{00}} e^{(z+c_0 t)/2H} \int_0^t e^{-c_0\tau/2H} \frac{\partial \left(R^0_{1,\mathrm{ac}}\right)^2}{\partial \tau}\Bigg|_{\xi = z + c_0(t-\tau),\tau} d\tau \\ &= \frac{(\gamma+1)}{8\rho_{00}} \Bigg(e^{z/2H} \left(R^0_{1,\mathrm{ac}}(z,t)\right)^2 - e^{(z+c_0 t)/2H} \left(R^0_{1,\mathrm{ac}}(z + c_0 t, 0)\right)^2 - \\ &\qquad \frac{c_0}{2H} \int_0^t \left(R^0_{1,\mathrm{ac}}\right)^2 e^{-c_0\tau/2H} d\tau \Bigg) \end{aligned} \tag{25}$$

We have taken into account the fact that $R^0_{1,\mathrm{ac}}$ is a function of $z-c_0t$ in accordance with Equation (19). Hence, the leading-order value of $R^1_{2,\mathrm{ac}}$ equals

$$\frac{(\gamma+1)}{8\rho_{00}}\left(e^{z/2H}\left(R^0_{1,\mathrm{ac}}(z,t)\right)^2-e^{(z+c_0t)/2H}\left(R^0_{1,\mathrm{ac}}(z+c_0t,0)\right)^2\right) \tag{26}$$

and the part which follows the first mode equals evidently

$$\frac{(\gamma+1)}{8\rho_{00}}e^{z/2H}\left(R^0_{1,\mathrm{ac}}(z,t)\right)^2 \tag{27}$$

the second term propagates with the speed $-c_0$, and the last term in Equation (25) exhibits weak dispersion, it is proportional to H^{-1}. The leading-order value of R^1_{ent} equals

$$R^1_{\mathrm{ent}}=-\frac{(\gamma-1)}{2\rho_{00}}e^{z/2H}\left(\left(R^0_{1,\mathrm{ac}}(z,t)\right)^2-\left(R^0_{1,\mathrm{ac}}(z,0)\right)^2\right) \tag{28}$$

Taking a sum of the parts which follow the first acoustic mode, we arrive at the corrected links for the progressive mode. In the leading order, the corrected vector of perturbations in the progressive mode takes the form:

$$\begin{aligned}\psi_1=&\begin{pmatrix}\dfrac{c_0}{\rho_{00}} & c_0^2 & 1\end{pmatrix}^T R^0_{1,\mathrm{ac}}+\begin{pmatrix}-\dfrac{c_0}{\rho_{00}} & c_0^2 & 1\end{pmatrix}^T\frac{\gamma+1}{8\rho_{00}}\,e^{z/2H}\left(R^0_{1,\mathrm{ac}}\right)^2-\\&(0\quad 0\quad 1)^T\,\frac{\gamma-1}{2\rho_{00}}\,e^{z/2H}\left(R^0_{1,\mathrm{ac}}\right)^2=\left(\overline{V}_{1,\mathrm{ac}}\quad \overline{P}_{1,\mathrm{ac}}\quad \overline{R}_{1,\mathrm{ac}}\right)^T\end{aligned} \tag{29}$$

Collecting parts of the secondary modes which propagate in accordance with the dominative mode, we arrive at the leading-order equalities

$$\begin{aligned}\overline{P}_{1,\mathrm{ac}}&=\rho_{00}c_0\overline{V}_{1,\mathrm{ac}}+\frac{\gamma+1}{4}\rho_{00}e^{z/2H}\overline{V}^2_{1,\mathrm{ac}}\\ \overline{R}_{1,\mathrm{ac}}&=\frac{\rho_{00}}{c_0}\overline{V}_{1,\mathrm{ac}}-\frac{\gamma-3}{4c_0^2}\rho_{00}e^{z/2H}\overline{V}^2_{1,\mathrm{ac}}\end{aligned} \tag{30}$$

Equations (29)–(30) are the leading order links, they include only terms proportional to H^0. They follow from Equations (16)–(24). The total progressive velocity and excess pressure are readily expressed in the terms of the total progressive in the positive direction of axis OZ excess density. The correspondence of the links to these in the Riemann wave in a homogeneous equilibrium gas is analyzed in the subsequent subsection.

4.1. The Riemann wave

It seems useful to remind at this point the Riemann acoustic wave propagating over the background ideal gas of constant density and pressure in the absence of attenuation [7, 8]. It is an exact solution of a system of hydrodynamic equations which is strictly progressive. The precise links of the rightwards propagating (in the positive direction of axis OZ) Riemann wave are as follows:

$$P_R = \frac{\rho_0 c_0^2}{\gamma}\left(1+\frac{\gamma-1}{2}\frac{V_R}{c_0}\right)^{\frac{2\gamma}{\gamma-1}} - \frac{\rho_0 c_0^2}{\gamma} \qquad R_R = \rho_0\left(1+\frac{\gamma-1}{2}\frac{V_R}{c_0}\right)^{\frac{2}{\gamma-1}} - \rho_0 \tag{31}$$

The Earnshaw equation is the exact dynamic equation governing velocity in the nonlinear Riemann wave:

$$\frac{\partial V_R}{\partial t} + c_0\frac{\partial V_R}{\partial z} + \frac{\gamma+1}{2}V_R\frac{\partial V_R}{\partial z} = 0 \tag{32}$$

It corresponds to the equation describing the first-order velocity in the sound wave:

$$\frac{\partial V_{1,\mathrm{ac}}^1}{\partial t} + c_0\frac{\partial V_{1,\mathrm{ac}}^1}{\partial z} = -\frac{(\gamma+1)}{2}e^{z/2H}V_{1,\mathrm{ac}}^1\frac{\partial V_{1,\mathrm{ac}}^1}{\partial z} - \frac{1}{8H}e^{z/2H}\left((\gamma+1)\left(V_{1,\mathrm{ac}}^1\right)^2 + 2(\gamma-1)\frac{\partial V_{1,\mathrm{ac}}^1}{\partial z}\int V_{1,\mathrm{ac}}^1 dz\right) \tag{33}$$

when $H\to\infty$. Equation (33) readily follows from the projecting. The Riemann wave is an exclusive example of exact complete separating of acoustic quantities in the nonlinear flow. The leading-order relations of perturbations in the Riemann wave

$$P_R = \rho_{00}c_0V_R + \frac{\gamma+1}{4}\rho_{00}V_R^2, \qquad R_R = \frac{\rho_{00}}{c_0}V_R - \frac{\gamma-3}{4c_0^2}\rho_{00}V_R^2 \tag{34}$$

coincide with Equations (30) when $H\to\infty$.

Thus, we specify a way to establish new nonlinear links for progressive perturbations. The zero-order values denote linear perturbations proportional to Ma, and the first order values denote perturbations proportional to Ma^2, *i.e.*, these values which are generated by the quadratic nonlinear sources (Ma denotes the Mach number). The procedure may be repeated in order to obtain quantities of order Ma^3 and higher with any desired accuracy. In view of small nonlinearity (that is valid at heights where the magnitude of $\frac{R_{1,\mathrm{ac}}^1}{\rho_{00}}e^{z/2H}$ is considerably smaller than the unit), the solution may be sought as a function of $(z-c_0t, \eta t)$ [8, 9], where η is a small parameter responsible for nonlinear distortions. In this case, the acoustic forces of the second-order modes may be integrated along the characteristics with an account for weak dependence of $R_{1,\mathrm{ac}}^1$ on the second variable [2, 6].

5. Progressing waves caused by some initial perturbations

In this section, we discuss numerical evaluations of the secondary modes in accordance with Equations (16)–(17) for some initial conditions which correspond to the dominative upwards propagating sound. Note, that for this mode to be dominative, the disturbances in velocity and pressure should correspond in the leading order to the linear links. This is the first line in Equations (10). Otherwise, an initial perturbation contains contributions of other modes which may be of the same magnitude or lager. This may be readily examined by projection [5].

5.1. Nonlinear effects of impulses

The perturbation of density in the first and second examples takes the forms

$$R_{1,\mathrm{ac}}^{0}(z,0)=\mathrm{Ma}\rho_{00}\sqrt{2e\mu}\left(\frac{z-z_0}{H}\right)\exp\left(-\mu\left(\frac{z-z_0}{H}\right)^2\right) \tag{35}$$

and

$$R_{1,\mathrm{ac}}^{0}(z,0)=\mathrm{Ma}\rho_{00}\exp\left(-\mu\left(\frac{z-z_0}{H}\right)^2\right)\left(2\mu\left(\frac{z-z_0}{H}\right)^2-1\right) \tag{36}$$

Figure 1 shows the initial perturbations of density in both impulses.

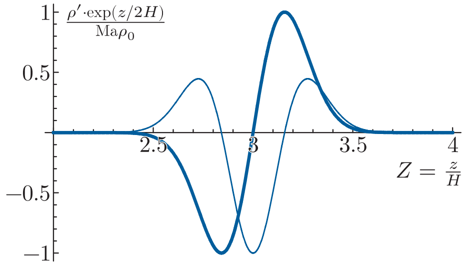


Figure 1. Initial excess dimensional density $R_{1,\mathrm{ac}}^{0}/(\mathrm{Ma}\rho_{00})$ at $t=0$ as function of dimensionless distance $Z=z/H$; the bold line corresponds to Equation (35), and the normal line corresponds to Equation (36) when $\mu=20$ and $z_0=3H$

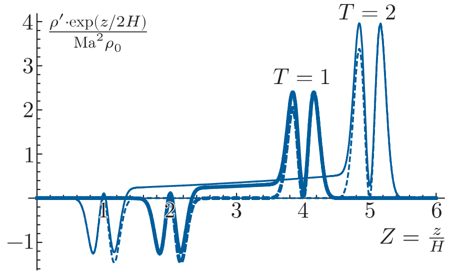
Excess densities of the secondary modes generated in the field of the first intense mode, are shown in Figure 2. In the series of calculations, $z_0=3H$, $\mu=20$ and $\gamma=1.4$. The dotted lines are the leading-order expressions, Equations (26)–(28), and the normal lines represent the results of the numerical simulations of Equations (20), (17).

The initial excess density of the first mode in the following example,

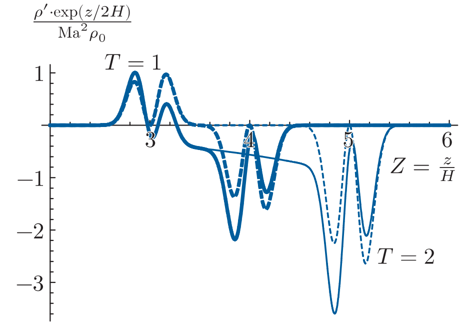
$$R_{1,\mathrm{ac}}^{0}(z,0)=-\mathrm{Ma}\rho_{00}\exp\left(-\left(\frac{z-z_0}{H}\right)^2\right)\cdot \left(\frac{2(z-z_0)}{\mu H}\cos\left(\mu\frac{z-z_0}{H}\right)+\sin\left(\mu\frac{z-z_0}{H}\right)\right) \tag{37}$$

is plotted in Figure 3. The dynamics of the secondary second and third modes are shown in Figure 4.

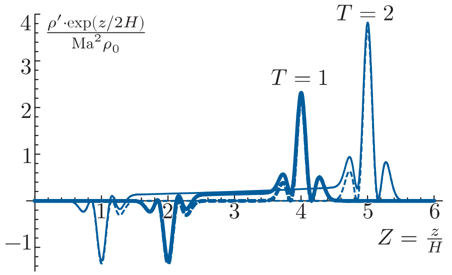
Figures 2 and 4 reveal that the induced modes consist of two parts, one approximately propagating with the high-frequency speed of a dominative mode, and the second with the speed of an induced mode, c_0. The parts which follow the dominative mode, evidently increase their magnitude, and these which propagate with the speeds of the secondary modes, do not. This agrees with the approximate formulas (26)–(28). The numerical simulations account for the weak dispersion which also is reflected by the term proportional to H^{-1} in these formulas.



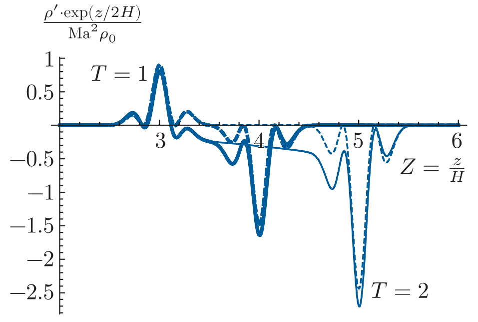
(a) $R^1_{2,\mathrm{ac}}$



(b) R^1_{ent}



(c) $R^1_{2,\mathrm{ac}}$



(d) R^1_{ent}

Figure 2. Induced excess dimensionless densities $R^1_{2,\mathrm{ac}}/\left(\mathrm{Ma}^2\rho_{00}\right)$ (a), (c) and $R^1_{\mathrm{ent}}/\left(\mathrm{Ma}^2\rho_{00}\right)$ (b), (d) as functions of dimensionless distance $Z=z/H$; the bold lines relate to time $T=tc_0/H=1$, and the normal lines to time $T=2$ from the beginning of evolution; in the series of evaluations, $R^0_{1,\mathrm{ac}}$ is determined by Equation (35) (a), (c) or by Equation (36) (b), (d); the dotted lines relate to the leading-order quantities (Equations (26)–(28))

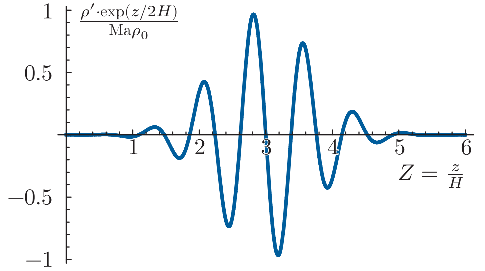


Figure 3. Initial dimensionless excess density $R^0_{1,\mathrm{ac}}/(\mathrm{Ma}\rho_{00})$ at $t=0$ in accordance with Equation (37) as a function of dimensionless distance $Z=z/H$, $\mu=8$, $z_0=3H$

6. Conclusions

In this study, we consider the case of weak dispersion which corresponds to the quickly varying perturbations with large characteristic wavenumber k: $kH \gg 1$. In the previous investigations of the author, precise and simplified operators were employed in the studies of the dynamics of short impulses. The usage of approximate projecting is in a good agreement with the exact formulas

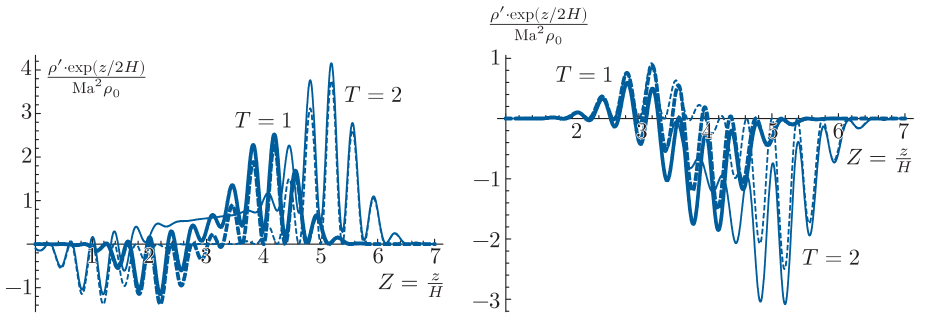


Figure 4. Induced excess dimensionless densities $R^1_{2,\mathrm{ac}}$ (a) and R^1_{ent} (b) as functions of dimensionless distance $Z=z/H$; the bold lines relate to time $T=tc_0/H=1$, and the normal lines to time $T=2$ from the beginning of evolution; in the series of evaluations, $R^0_{1,\mathrm{ac}}$ is determined by Equation (37) with $\mu=8$, $z_0=3H$; the dotted lines relate to the leading-order quantities (Equations (26)–(28))

even for fairly extended waveforms: $1/k\sim H$ [4]. The expansion in series of the acoustic dispersion relation in powers of $1/kH$ (up to terms proportional to $(kH)^0$ and $(kH)^{-1}$) are simply $\omega_{1,2}=c_0k$ in the leading order and do not reveal dispersion of sound. In spite of this, both linear relations of specific perturbations in acoustic and entropy modes, and the acoustic force which induces the secondary modes, include terms responsible for the dispersion. In the leading order, short acoustic perturbations are functions exclusively on $z-c_0t$ or $z+c_0t$ for the progressive in the positive direction of axis OZ or in the negative direction of axis OZ acoustic waves, respectively.

The numerical calculations of an excess density in the induced modes caused by an acoustic source which includes terms of order $(kH)^{-1}$, somewhat differ from the leading-order predictions, but on the whole, parts of different speeds are recognizable. These parts change their shape due to the dispersion but have approximately linear speeds of the dominative and induced modes. The parts of induced modes which follow the dominative sound, being added to the dominative mode, form new links in a progressive mode. These links may be established with any accuracy. The main result of this study is the proposal of a procedure which allows establishing more precise analytical links of perturbations in a progressive weakly dispersive wave. This accuracy concerns order in powers of the Mach number, but cannot exceed the accuracy up to $(kH)^{-1}$. This study contains results which are valid with an accuracy up to terms proportional to Ma^2 and $(kH)^{-1}$.

As for the perturbation in density, ρ', which specifies the second mode, its leading-order value consists of two parts, in accordance with Equation (26). One part follows the dominative mode without changing its shape, and the second propagates with the speed $-c_0$ and linearly increases its magnitude as time increases. As for the entropy mode, its components of different speeds do not vary their shape. This is because the formulas include factor $\exp(z/2H)$ in the integrands which assumed to be constant in the integration of short impulses. The results may be readily applied in studies of a fluid different from an ideal gas (by involving the correspondent equations of state) in the presence of a constant mass force other than the force of gravity [10].

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