Joanna Raczek

NP-COMPLETENESS OF WEAKLY CONVEX AND CONVEX DOMINATING SET DECISION PROBLEMS

Abstract. The convex domination number and the weakly convex domination number are new domination parameters. In this paper we show that the decision problems of convex and weakly convex dominating sets are NP-complete for bipartite and split graphs. Using a modified version of Warshall algorithm we can verify in polynomial time whether a given subset of vertices of a graph is convex or weakly convex.

Keywords: dominating set, NP-completeness, distance, convex set.

Mathematics Subject Classification: 05C69, 05C85.

1. INTRODUCTION

Let $G = (V, E)$ be a graph with $|V| = n$ and let H be a subgraph of G. We say that H is an *induced subgraph* of G if every two vertices $u, v \in V$ are adjacent in H if and only if they are adjacent in G. The induced subgraph H with $V' = V(H)$ is called the *subgraph induced by* V' and is denoted by $\langle V' \rangle$.

A *split graph* is a graph G whose vertex set can be partitioned into two sets V' and V'', where $\langle V' \rangle$ is a complete graph and $\langle V'' \rangle$ has no edge. A graph G is *chordal* if every cycle of G of length greater than three has a chord, that is, an edge between two nonconsecutive vertices of the cycle.

A set $V' \subset V$ is a *dominating set* in G if every vertex $u \in V - V'$ is adjacent to a vertex $v \in V'$.

For unexplained terms and symbols see [1].

Convex and weakly convex dominating subsets of vertices of a graph G were first introduced by Jerzy Topp, Gdansk University of Technology, 2002.

A set $V' \subset V$ is a *weakly convex dominating set* in G if it is dominating and if $d_G(u, v) = d_{\langle V' \rangle}(u, v)$ for every two vertices $u, v \in V'$, where $d_G(u, v)$ denotes distance between u and v in G.

A set $V' \subset V$ is a *convex dominating set* in G if it is a weakly convex dominating set and if all shortest paths between any two vertices of V' are the same in $\langle V' \rangle$ as in G. Equivalently, a set $V' \subset V$ is a *convex dominating set* in G if it is a weakly convex dominating set and if the number of shortest paths between any two vertices $u, v \in V'$ is the same in $\langle V' \rangle$ as in G.

In this paper we consider decision problems of convex and weakly convex dominating sets as follows:

WEAKLY CONVEX DOMINATING SET
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a weakly convex dominating set of size $\leq k$?

CONVEX DOMINATING SET
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a convex dominating set of size $\leq k$?

The class NP is the class of problems which can be verified in polynomial time. A verification algorithm takes as input an instance of the problem of WEAKLY CONVEX DOMINATING SET (also CONVEX DOMINATING SET) and a candidate solution to the problem, called a *certificate*. Then it verifies in polynomial time whether the certificate is a proper solution to the given problem instance.

Lemma 1. *Decision problem of WEAKLY CONVEX DOMINATING SET for arbitrary graphs is in NP-class of decisions problems.*

Proof. Let G be a given graph and $V' \subset V(G)$ be a certificate. To verify whether V' is a weakly convex set in G, let us use the Warshall algorithm [2, 3] (Algorithm 1) which finds the distance $d_G(v_i, v_j)$ between every two vertices $v_i, v_j \in V$.

The input to this algorithm is the matrix $\mathbf{W}_0$, where $\mathbf{W}_0[i,j] = 1$ if $v_i v_j \in E(G)$ and $\mathbf{W}_0[i,j] = \infty$ otherwise. The output of the algorithm is the matrix $\mathbf{W}^*$, where $\mathbf{W}^*[i,j] = d_G(v_i, v_j)$ for $v_i, v_j \in V(G)$. It has been known, that time complexity of the Warshall algorithm is $O(n^3)$.

Algorithm 1. Warshall Algorithm

```
begin
  W := W_0;
  for k := 1 to n do
      for i := 1 to n do
          for j := 1 to n do
              if W[i,j] > W[i,k] + W[k,j]
                 then
                          W[i,j] := W[i,k] + W[k,j];
              fi
          od
      od
```

```
   od
   W* := W;
end.
```

Certainly in polynomial time we can verify whether V' is a dominating subset of vertices in G. If $|V'| > k$, then the verification procedure stops as the candidate is not a proper solution to the problem.

If G is disconnected, then verification procedure stops, as such a graph cannot have a weakly convex dominating set. Therefore we assume that G is connected and so the procedure goes on.

For a given set $V' \subset V$ let H be the spanning subgraph of G obtained from G by removing all the edges incident to at least one vertex of $V - V'$.

Observe that the set V' is weakly convex if $d_G(v_i, v_j) = d_H(v_i, v_j)$ for every two vertices $v_i, v_j \in V'$.

As time complexity of Warshall algorithm is $O(n^3)$, time complexity of the above verification procedure is also polynomial. □

Lemma 2. *Decision problem of CONVEX DOMINATING SET for arbitrary graphs is in NP-class of decisions problems.*

Proof. Let G be a given graph and $V' \subset V$ be a certificate. To verify whether V' is a convex set in G, we modify the Warshall algorithm [2, 3]. Let $c_G(v_i, v_j)$ be the number of shortest paths between vertices $v_i, v_j \in V(G)$.

The input to the modified algorithm is the matrix $\mathbf{W}_0$ constructed as above. Our algorithm finds the distance between every two vertices in G, that is $\mathbf{W}^*[i,j] = d_G(v_i, v_j)$ and the numbers of shortest paths, that is $\mathbf{B}^*[i,j] = c_G(v_i, v_j)$.

Algorithm 2. Modified version of Warshall Algorithm

```
begin
   W := W0;
   for i := 1 to n do
      for j := 1 to n do
         B[i,j] := 0;
      od
   od
   for k := 1 to n do
      for i := 1 to n do
         for j := 1 to n do
            if W[i,j] > W[i,k] + W[k,j]
               then
                  W[i,j] := W[i,k] + W[k,j];
                  B[i,j] := 1;
            fi
         else
```

```
17                  if W[i,j] = W[i,k] + W[k,j]
18                    then
19                          B[i,j] := B[i,j] + 1;
20                  fi
21              od
22          od
23      od
24      W* := W;
25      B* := B;
26  end.
```

The matrix **B** keeps numbers of shortest paths between vertices. In the beginning the matrix **B** is a zero matrix (line 5 of the Algorithm 2). If the algorithm finds a shorter path between vertices v_i and v_j than known so far, it sets value 1 to $\mathbf{B}[i,j]$ (line 14). If the path is one of the shortest paths not known so far, the algorithm increases $\mathbf{B}[i,j]$ by 1 (line 19). The time complexity of that version of Warshall algorithm is again $O(n^3)$.

If $|V'| > k$, then the verification procedure stops as the candidate is not a proper solution to the problem.

It is well-known that it is possible to verify in polynomial time whether V' is a dominating set in G.

If G is disconnected, then verification procedure stops as such a graph has no convex dominating set. For this reason we assume that G is connected.

For a given set $V' \subset V$, let H be the spanning subgraph of G obtained from G by removing the edges incident to at least one vertex of $V - V'$.

Observe that the set V' is a convex set in G if for every two vertices from V' is $d_G(v_i, v_j) = d_H(v_i, v_j)$ and $c_G(v_i, v_j) = c_H(v_i, v_j)$.

Certainly time complexity of presented procedure is $O(n^3)$, which is polynomial. $\square$

2. *NP*-COMPLETENESS OF WEAKLY CONVEX DOMINATING SET PROBLEM

In this section we show that WEAKLY CONVEX DOMINATING SET problem on split and bipartite graphs is NP-complete by giving a polynomial reduction of the EXACT COVER BY 3-SETS problem to the WEAKLY CONVEX DOMINATING SET problem.

WEAKLY CONVEX DOMINATING SET
INSTANCE: A connected split graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a weakly convex dominating set of size $\leq k$?

Theorem 1. *WEAKLY CONVEX DOMINATING SET problem is NP-complete for split graphs.*

Proof. We have already seen in Lemma 1 that WEAKLY CONVEX DOMINATING SET problem for arbitrary graphs is in NP-class of decision problems.

To prove the statement we use a polynomial reduction from *exact cover by 3-sets*, which is known to be NP-complete problem.

EXACT COVER BY 3-SETS
INSTANCE: A finite set $X = \{x_1, x_2, \ldots, x_{3q}\}$ of cardinality $3q$, $q \in \mathbb{N}$, and a set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of 3-element subsets of X.
QUESTION: Does $\mathcal{C}$ contain an exact cover for X?

For a given instance of EXACT COVER BY 3-SETS problem we construct a split graph $G = (V, E)$ where

$$\begin{array}{rcl} V_C & = & \{C_1, C_2, \ldots, C_m\}, \\ V_X & = & \{x_1, x_2, \ldots, x_{3q}\}, \\ V & = & V_C \cup V_X, \\ E & = & \{(C_i, C_j)\colon 1 \le i, j \le m,\ i \ne j\} \\ & & \cup\{(x_i, C_j)\colon x_i \in C_j,\ 1 \le i \le 3q,\ 1 \le j \le m\}. \end{array}$$

Certainly G is a split graph in which $\langle V_C \rangle$ is a complete graph and $\langle V_X \rangle$ has no edge (see Fig. 1).

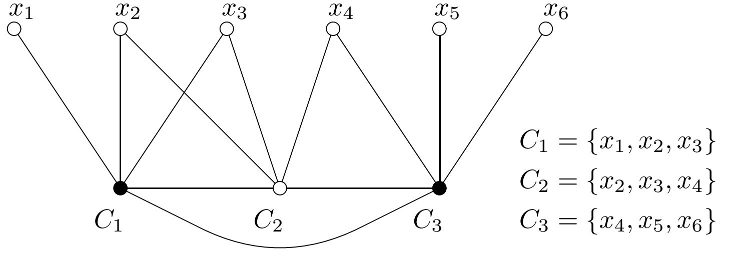

Fig. 1. Split graph

Assume that $\{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$ is an exact cover for X. Then $D = \{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$ is a weakly convex dominating set of size $k = q$ of graph G. Then every vertex x_i is dominated by exactly one vertex of D. As $\langle V_C \rangle$ is complete, D is a weakly convex set.

Conversely, assume that D is a weakly convex dominating set of cardinality at most q. We shall show that $\mathcal{C}$ contains an exact cover for X. Note that each vertex x_i must be in the dominating set or be dominated by vertices of D. Thus $|D| \ge q$ as each vertex C_j dominates exactly 3 vertices belonging to V_X. Consequently $|D| = q$ and every vertex x_i is dominated by exactly one vertex of $D = \{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$. Therefore the family $\{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$ is an exact cover for X.

It is obvious that the used transformation is polynomial as G has $3q+m$ vertices and $3m + \binom{m}{2}$ edges. □

Since every split graph is a chordal graph, we have the following corollary.

Corollary 1. *WEAKLY CONVEX DOMINATING SET problem is NP-complete for chordal graphs.*

WEAKLY CONVEX DOMINATING SET
INSTANCE: A connected bipartite graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a weakly convex dominating set of size $\leq k$?

Theorem 2. *WEAKLY CONVEX DOMINATING SET problem is NP-complete for bipartite graphs.*

Proof. By Lemma 1, WEAKLY CONVEX DOMINATING SET problem for arbitrary graphs is in NP-class of decision problems.

To prove the statement we use a polynomial reduction from EXACT COVER BY 3-SETS problem which is known to be NP-complete.

For a given instance of EXACT COVER BY 3-SETS problem we construct a bipartite graph $G = (V, E)$ where

$$\begin{aligned}
V_C &= \{C_1, C_2, \ldots, C_m\}, \\
V_X &= \{x_1, x_2, \ldots, x_{3q}\}, \\
V &= V_C \cup V_X \cup \{y\}, \\
E &= \{(y, C_j)\colon 1 \leq j \leq m\} \\
&\quad \cup \{(x_i, C_j)\colon x_i \in C_j,\ 1 \leq i \leq 3q,\ 1 \leq j \leq m\}.
\end{aligned}$$

As $V = V_C \cup (V_X \cup \{y\})$, obtained graph G is bipartite (see Fig. 2).

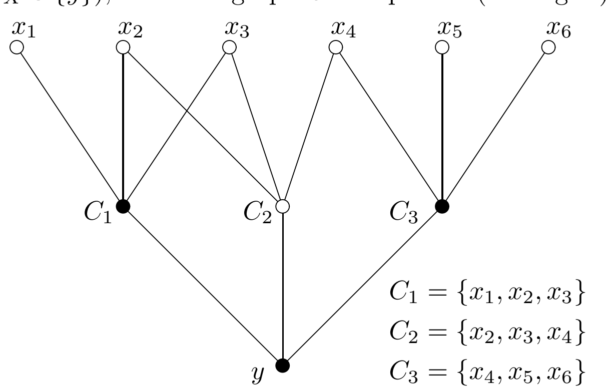

Fig. 2. Bipartite graph

Assume first that the family $\mathcal{C}$ contains an exact cover $\{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$ for X. Then $D = \{C_{j_1}, C_{j_2}, \ldots, C_{j_q}, y\}$ is a weakly convex dominating set of size $k = q + 1$. Every vertex x_i is dominated by exactly one vertex belonging to D and each vertex which belongs to V_C is dominated by y. It is obvious that $\langle D \rangle$ is a connected subgraph of G and every shortest path between any two vertices from $V_C \cap D$ is of length 2. Thus, D is a weakly convex dominating set of G.

Assume now that G has a weakly convex dominating set of size at most $k = q+1$. We shall show that $\mathcal{C}$ contains an exact cover for X. Let us observe that y is in every connected dominating set of G of size at most k. Moreover, each vertex x_i must be in weakly convex dominating set or be dominated by vertices from D. As each vertex C_j dominates exactly 3 vertices belonging to V_X, $|D| \geq k$. Consequently $|D| = q+1 = k$ and every vertex x_i is dominated by exactly one vertex of $D - \{y\} = \{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$. Therefore the family $\{C_{j_1}, C_{j_2}, \ldots, C_{j_q}\}$ is an exact cover for X.

It is obvious that the used transformation is polynomial as G has $3q + m + 1$ vertices and $4m$ edges. □

3. *NP*-COMPLETENESS OF THE CONVEX DOMINATING SET PROBLEM

In this section we show that CONVEX DOMINATING SET problem on split and bipartite graphs is *NP*-complete by giving a polynomial reduction of the EXACT COVER BY 3-SETS problem to the CONVEX DOMINATING SET problem.

CONVEX DOMINATING SET
INSTANCE: A connected chordal graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a convex dominating set of size $\leq k$?

CONVEX DOMINATING SET
INSTANCE: A connected bipartite graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a convex dominating set of size $\leq k$?

By Lemma 2 the CONVEX DOMINATING SET problem for arbitrary graphs is in *NP*-class of decision problems. The proofs of the following theorems are similar to those of Theorem 1 and Theorem 2.

Theorem 3. *CONVEX DOMINATING SET problem is NP-complete for split graphs.*

Corollary 2. *CONVEX DOMINATING SET problem is NP-complete for chordal graphs.*

Theorem 4. *CONVEX DOMINATING SET problem is NP-complete for bipartite graphs.*

REFERENCES

[1] Haynes T. W., Hedetniemi S. T., Slater P. J.: *Fundamentals of domination in graphs.* New York, Basel, Hong Kong, Marcel Dekker Inc. 1998.

[2] Ross K. A., Wright C. R. B.: *Matematyka dyskretna.* Warszawa, Wydawnictwo Naukowe PWN 1996.

[3] *Handbook of Discrete and Combinatorial Mathematics.* Editor-in-Chief: Rosen K. H., London, Washington, D.C. CRC Press Boca Raton 2000.

Joanna Raczek
gardenia@pg.gda.pl

Gdańsk University of Technology
Department of Mathematics
ul. Narutowicza 11/12, 80-952 Gdańsk, Poland

Received: November 4, 2003.