NUMERICAL MODELING OF THE NONLINEAR WAVE PROPAGATION IN A BUBBLE LAYER

ANNA BARANOWSKA

Gdansk University of Technology, Department of Mathematical and Numerical Analysis
11/12, G. Narutowicza St., 80-233 Gdansk, Poland
anbar@mif.pg.gda.pl

The aim of the paper is a theoretical analysis of acoustic waves propagation through a bubble layer. The mathematical model of the pressure propagation in bubbly liquid layer is constructed by the linear non-dissipative wave and the Rayleigh-Plesset equations. The acoustic pressure field inside the layer, the reflected and transmitted waves, and suitable power spectral density are studied. Numerical analysis is carried out for different layer thicknesses, different values of physical parameters and generated signals. Some results of numerical investigations are also presented.

INTRODUCTION

The wave generation and propagation inside the layers with different physical properties is a very important problem in practice. The known mathematical models of this problem consist of a system of two differential equations. The first one is the linear non-dissipative wave equation which describes acoustic pressure changes in the bubble layer [3]. The second one is an equation which allows to predict the bubble radius changes, or equivalently, the bubble volume variation. Our mathematical model is based on the Rayleigh-Plesset equation, which allows to analyze radius changes of a bubble.

In the paper we present a mathematical model and the results of numerical investigation of nonlinear waves propagation in a bubbly liquid layer obtained by using own computer programs.

1. MATHEMATICAL MODEL

We assume that plane layer with spherical bubbles of the same size and uniformly distributed is placed between $x = 0$ and $x = L$. The media outside the layer are considered to be linear. The acoustic field is the sum of incident p_i and reflected p_r waves for $x \le 0$. When $x \ge L$, only transmitted wave p_t is propagated.

The mathematical model of the acoustic pressure p propagated inside the layer is built on the basis of linear non-dissipative wave equation [2]:

$$\frac{\partial^2 p}{\partial x^2}(x,t) - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}(x,t) = -\rho_0 \frac{\partial^2 \beta}{\partial t^2}(x,t) \tag{1}$$

where c_0 is sound speed of water, ρ_0 is the density of water at the equilibrium state, β is the local fraction of volume occupied by the gas. Assuming a constant number N of air bubbles per unit volume, the volume fraction is given by

$$\beta(x,t) = \frac{4}{3}\pi N R^3(x,t)$$

where R is the instantaneous radius of the bubbles.

The local bubble radius $R(t)$ is calculated from the Rayleigh - Plesset equation

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 = \frac{1}{\rho_0}\left[p_g\left(\frac{R_0}{R}\right)^{3\gamma} + p_v - p_{stat} - \frac{2\sigma}{R} - P(t) - \rho_0 \delta\omega R\frac{dR}{dt}\right] \tag{2}$$

where p_v is gas and vapor pressure inside a bubble, p_{stat} is ambient static pressure, $P(t)$ is incident signal acoustic pressure, R_0 is the equilibrium bubble radius, ω is angular frequency, γ is polytropic exponent of gas, σ is surface tension coefficient, $p_g = 2\sigma/R_0 + p_{stat} - p_v$, δ is the total damping constant which is the sum of three components $\delta = \delta_{rad} + \delta_{visc} + \delta_{th}$ where $\delta_{rad} = \omega R_0/c_0$ is the acoustic radiation damping constant, $\delta_{visc} = 4\mu/\left(\rho_0 \omega R_0^2\right)$ is the viscous damping constant, δ_{th} is damping constant due to thermal effects and μ is the coefficient of molecular viscosity of seawater.

It is important to notice that the bubble radius R and pressure P in the Rayleigh-Plesset equation (2) are functions of time variable t only. In fact, we consider them as functions of two coordinates: the time coordinate t and the one-dimensional coordinate x respectively. To be precise we put $p(x,t)$ instead of $P(t)$.

The initial conditions for $x \neq 0$ are as follows:

$$\begin{aligned} &p(x,0) = 0 \\ &\frac{\partial p}{\partial t}(x,0) = 0 \\ &R(x,0) = R_0 \\ &\frac{\partial R}{\partial t}(x,0) = 0 \end{aligned} \tag{3}$$

To complete the formulation of our problem, boundary conditions are defined. At the layer boundaries $x = 0$ and $x = L$ the pressure should be continuous, which leads to

$$\begin{aligned} p(0,t) &= p_i(0,t) + p_r(0,t) \\ p(L,t) &= p_t(L,t) \end{aligned} \tag{4}$$

Additionally, taking into account the continuity of velocity we introduce two boundary conditions [1]:

$$\frac{\partial p}{\partial t}(0,t) - c_0 \frac{\partial p}{\partial x}(0,t) = 2\frac{\partial p_i}{\partial t}(0,t)$$
$$\frac{\partial p}{\partial t}(L,t) + c_0 \frac{\partial p}{\partial x}(L,t) = 0 \qquad (5)$$

Next, we consider two situations. Assuming that harmonic signal is generated, we have an incident wave

$$p_i(0,t) = \begin{cases} P_A \sin(\omega t), & 0 \le t \le T_s \\ 0, & t > T_s \end{cases}$$

where T_s is the signal duration. When two different frequency waves are generated then for $x = 0$ we define

$$p_i(0,t) = \begin{cases} P_A \sin(\omega_1 t) + P_A \sin(\omega_2 t), & 0 \le t \le T_s \\ 0, & t > T_s \end{cases}$$

When density and sound speed at the bubble layer and surrounded medium are different then equations (1), (2) and the boundary conditions must be modified. While c_0 and ρ_0 still denote the speed of sound and density in water without bubbles equations (1) and (2) are modified by replacing c_0 and ρ_0 with the sound speed c_L and density ρ_L in the layer respectively. Moreover, the boundary condition at $x = 0$ has the form

$$\frac{\partial p}{\partial t}(0,t) - \frac{c_0\rho_0}{\rho_L}\frac{\partial p}{\partial x}(0,t) = 2\frac{\partial p_i}{\partial t}(0,t) \qquad (6)$$

Similarly, at $x = L$ we find

$$\frac{\partial p}{\partial t}(L,t) + \frac{c_0\rho_0}{\rho_L}\frac{\partial p}{\partial x}(L,t) = 0 \qquad (7)$$

We are looking for the solution of our problem for $x \in [0, L]$ and $t \in [0, T_{\max}]$. To solve the problem nodal points are defined as follows: $x_i = i\Delta x$, $t_n = n\Delta t$, where $\Delta x = L / N_x$, $\Delta t = T_{\max} / N_t$, i=0,1,…,N_x, n=0,1,…,N_t. As a result of numerical calculations we obtain acoustic pressure $p_{i,n} = p(x_i, t_n)$ and bubble radius $R_{i,n} = R(x_i, t_n)$ at nodal points. After calculating $p_{i,m}$ and $R_{i,m}$ for $m \le n$ we can compute $R_{i,n+1}$ using equation (2) and the pressure $p_{i,n+1}$ using equation (1), i.e. we can calculate bubble radius and pressure at time $t = t_{n+1}$ if we know the values of these functions for $t \le t_n$. The finite-difference method was employed to solve equation (1) while equation (2) was solved using the classical Runge-Kutta method.

2. RESULTS OF NUMERICAL INVESTIGATIONS

The first step of our theoretical analysis was to study harmonic wave propagation in bubble layer. We started with examination of correctness of proposed model. Figure 1 presents the incident and reflected waves calculated numerically assuming that the harmonic signal (frequency f = 30 kHz, amplitude P_A = 20 kPa) with the rectangular window is propagated in the layer of thickness $L = 3\lambda$. We put the sound speeds c_0 = 1450 m/s and c_L = 1230 m/s, density ρ_0 = 1000 kg/m^3. Pressure changes calculated for fixed points inside the layer are shown in Figure 2. Calculations were carried out assuming that volume fraction $\beta = 0$. It is equivalent to the situation when only linear effects are considered.

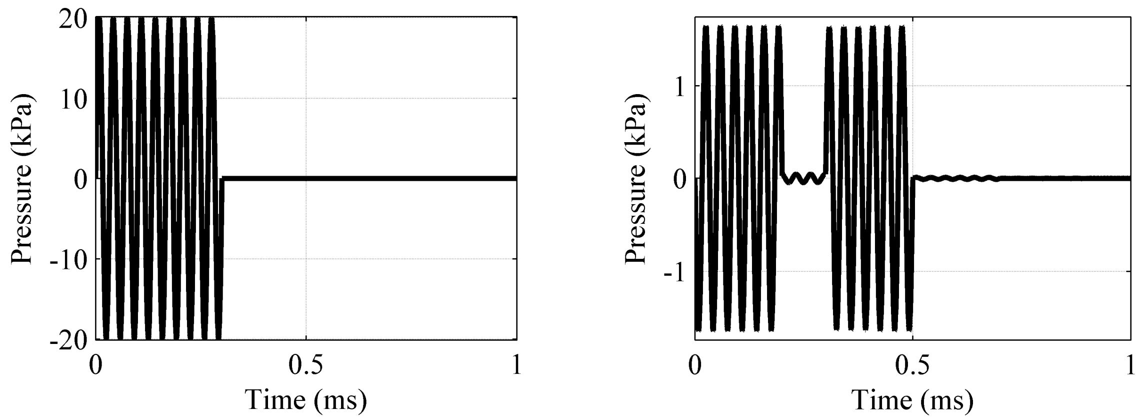

Fig. 1. Incident (left figure) and reflected (right figure) waves: $\beta = 0$

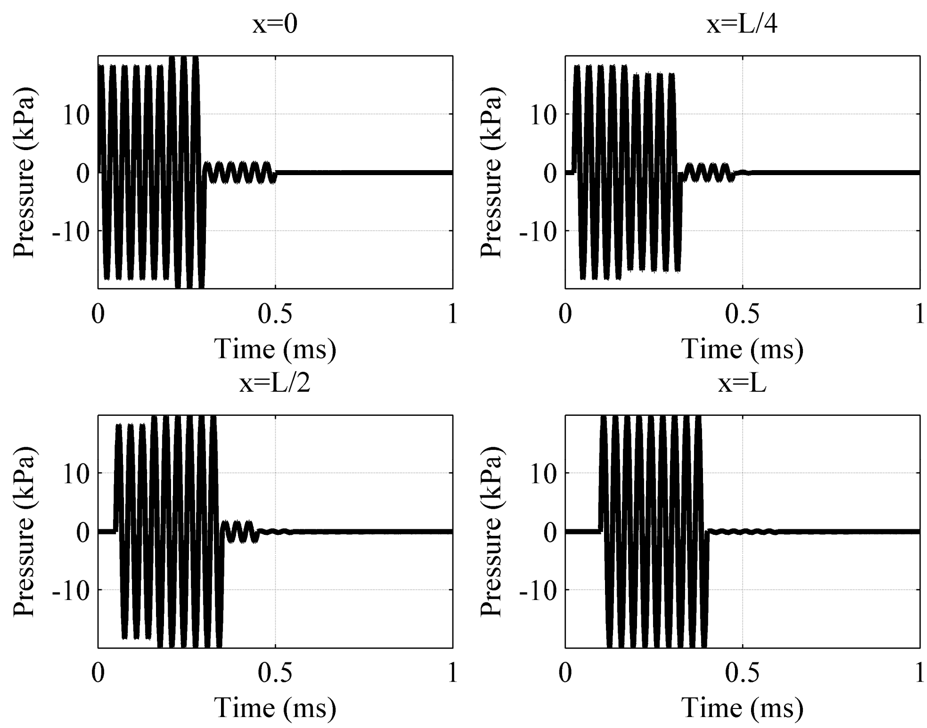

Fig. 2. Pressure as a function of time inside layer at fixed bubble layer points: $\beta = 0$

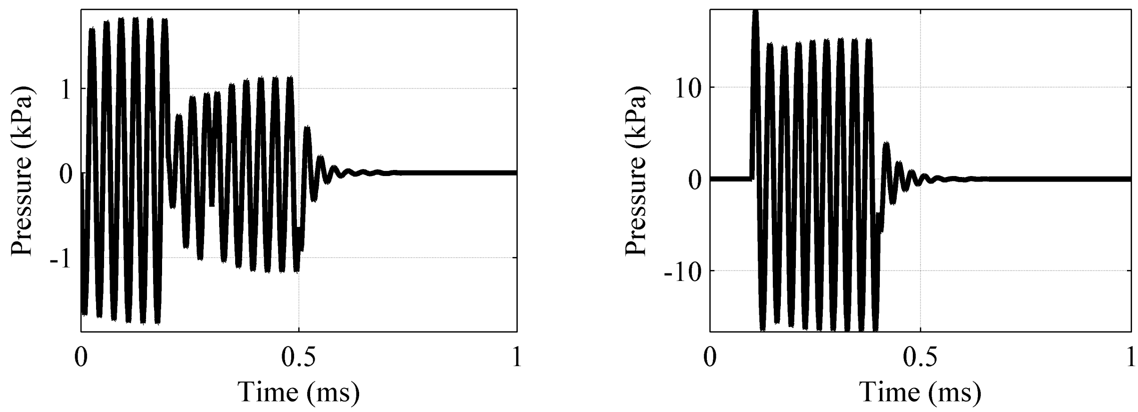

Fig. 3. Reflected (on the left) and transmitted (on the right) waves: $\beta = 10^{-6}$

The reflected wave obtained for volume fraction $\beta = 10^{-6}$ is shown in Figure 3 (on the left). The result obtained for transmitted wave ($x = L$) is given on the right of Figure 3.

The results presented so far were obtained assuming short duration of generated signal. Figure 4 shows a reflected wave and its power spectral density for duration of incident wave T_s = 3 ms (T_{max} = 10 ms). All parameters except for duration and investigated time interval are the same as used earlier.

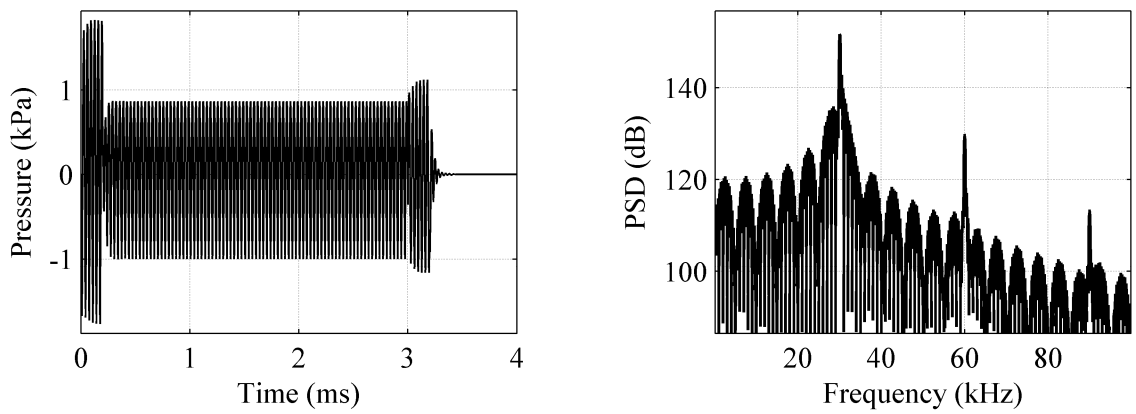

Fig. 4. Reflected wave and power spectral density: $\beta = 10^{-6}$

Different frequency interaction problem is also very important in practice. Examples of numerical calculations related to this problem are presented below. Figure 5 displays reflected wave and power spectral density obtained as a result of calculations assuming that primary wave is the sum of two harmonic waves with frequencies f_1 = 30 kHz and f = 33kHz respectively, and the same amplitudes P_A = 20 kPa. Pressure and power spectral density at fixed bubble layer points ($L = 3\langle \lambda_1, \lambda_2 \rangle$) are shown in Figure 6. The results presented in Figure 6 were achieved for volume fraction $\beta = 10^{-6}$. Figure 7 presents power spectral density of reflected and transmitted waves calculated for $\beta = 10^{-8}$. Similar results obtained for $\beta = 10^{-5}$ are shown in Figure 8. The figure following Figure 8 represents power spectral density of reflected wave when $\beta = 10^{-6}$ and the bubble layer thickness $L = 3\lambda_1$.

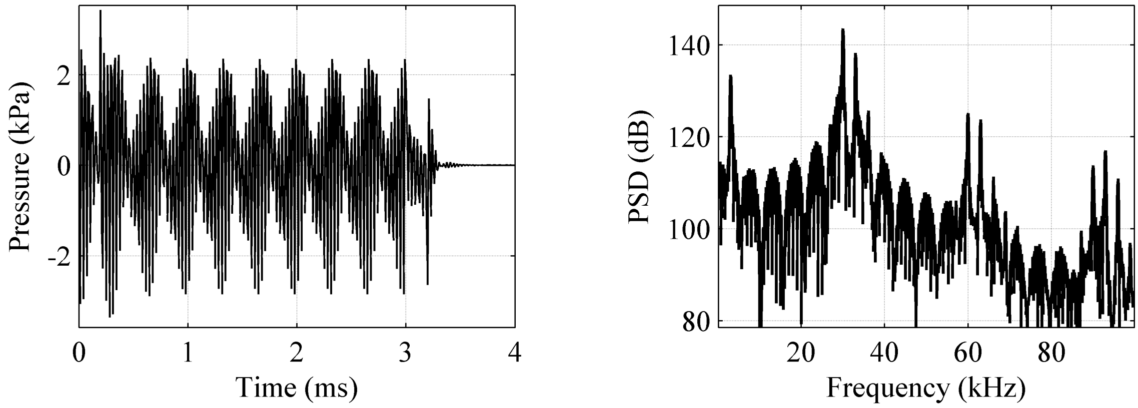

Fig. 5. Reflected wave and power spectral density: $\beta = 10^{-6}$

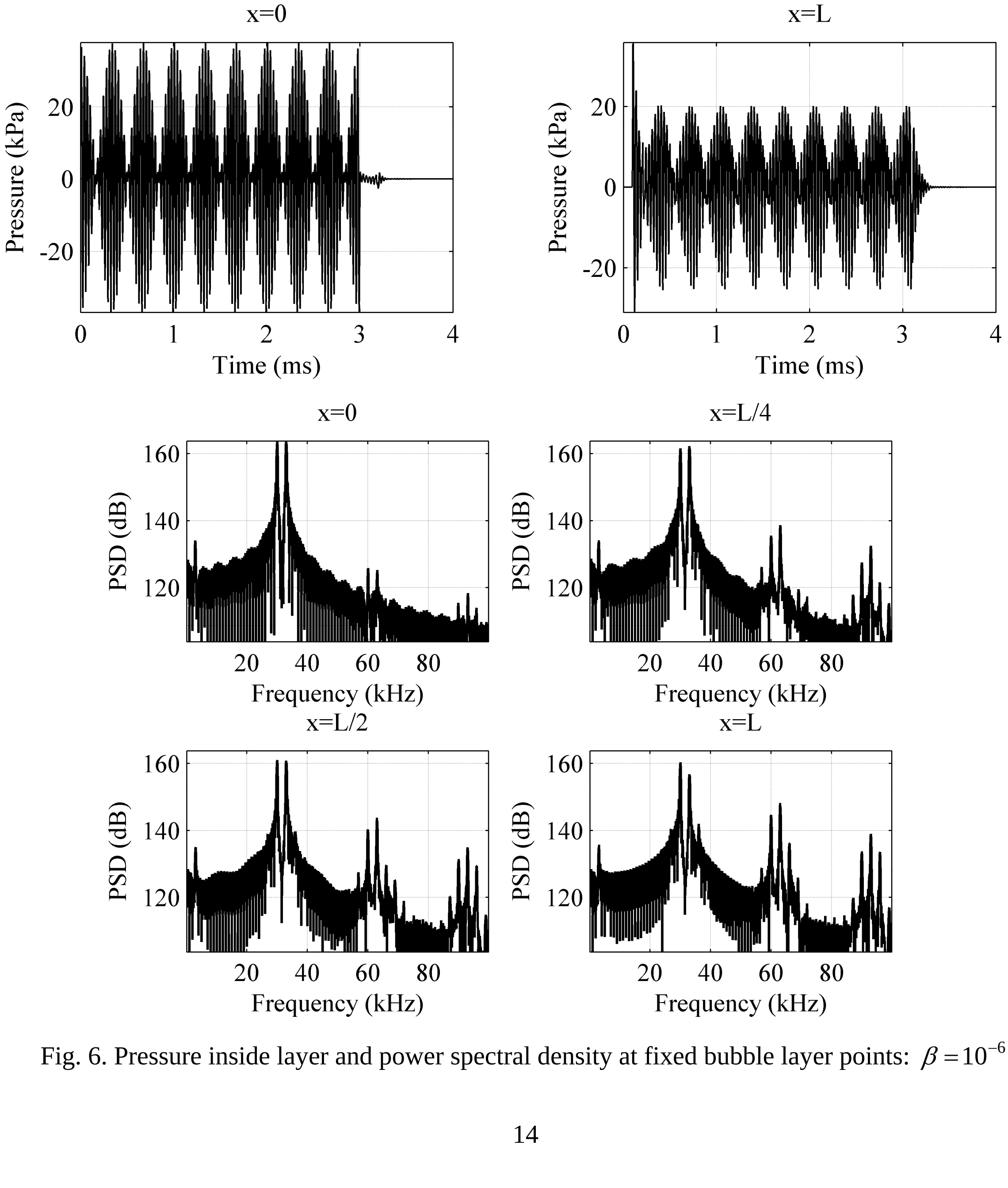

Fig. 6. Pressure inside layer and power spectral density at fixed bubble layer points: $\beta = 10^{-6}$

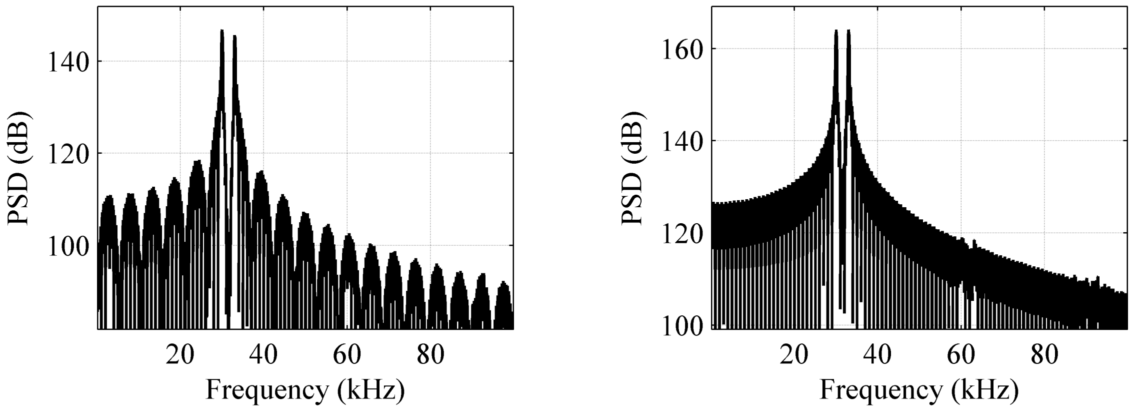

Fig. 7. Power spectral density of reflected (on the left) and transmitted (on the right) waves: $\beta = 10^{-8}$

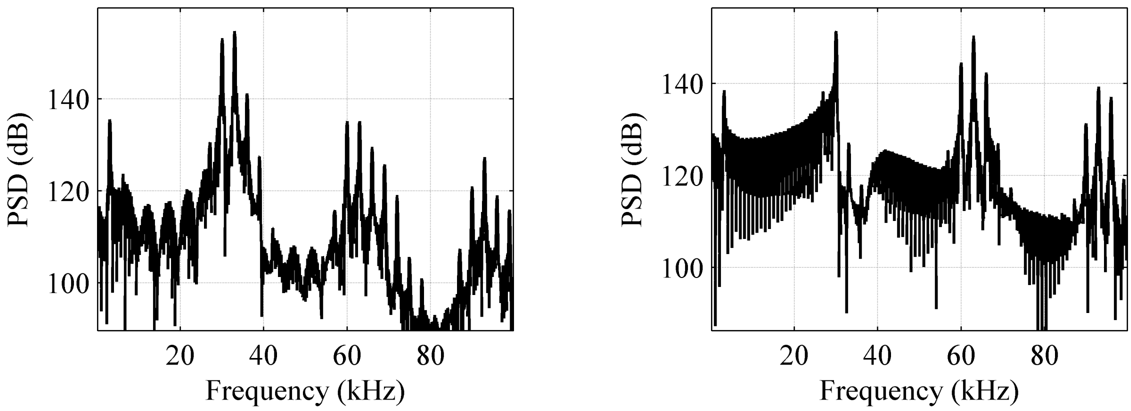

Fig. 8. Power spectral density of reflected (on the left) and transmitted (on the right) waves: $\beta = 10^{-5}$

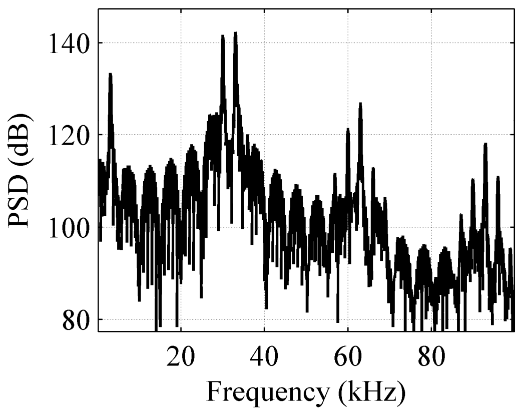

Fig. 9. Power spectral density of reflected wave : $\beta = 10^{-6}$, $L = 3\lambda_1$

A theoretical analysis was carried out for different values of physical parameters. Examples of results obtained for different bubble distributions and layer thicknesses have been presented above. The last figure demonstrates a reflected wave and its power spectral density obtained for water density ρ_0 = 1000 kg/m^3 and density inside the layer ρ_L = 1200 kg/m^3.

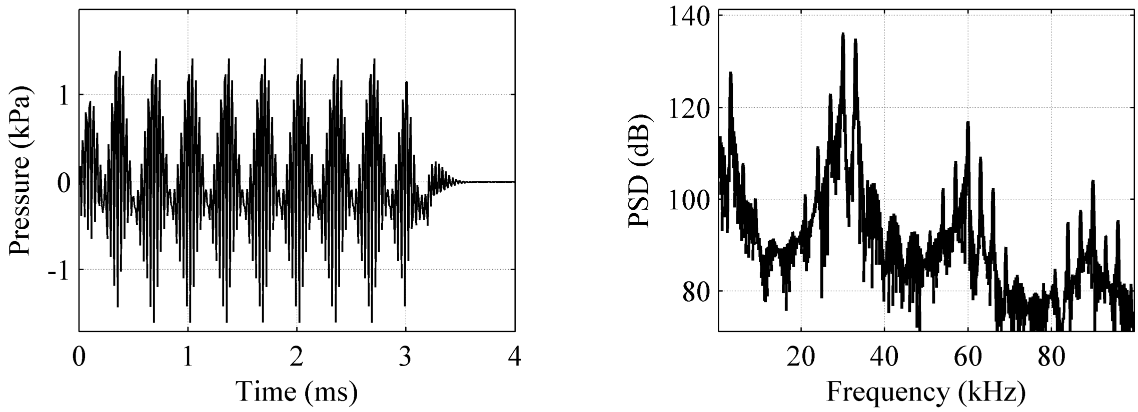

Fig. 10. Reflected wave and power spectral density: $\beta = 10^{-6}$, ρ_L = 1200 kg/m^3

3. CONCLUSIONS

The nonlinear acoustic waves propagation in one-dimensional bubbly liquid layer was considered and its mathematical model presented. The linear non-dissipative wave equation was solved numerically by employing the finite-difference method. The Rayleigh-Plesset equation was solved using classical Runge-Kutta method of order four. Some results of theoretical investigation were also discussed.

The proposed in this paper mathematical model can be used to study wave propagation for different signals propagated in media with different physical parameters.

It is worth mentioning that a correct choice of physical parameters as well as a choice of values of numerical parameters are very important in the process of theoretical investigation as they influence accuracy and correctness of results.

All presented in this paper results were obtained assuming that one bubble layer is surrounded by media with different physical properties. It is not difficult to extend this model to the case of more than one layer.

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