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On bipartization of cubic graphs by removal of an independent set[☆]

Hanna Furmańczyk[a,*], Marek Kubale[b], Stanisław Radziszowski[c]

[a] Institute of Informatics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

[b] Department of Algorithms and System Modeling, Technical University of Gdańsk, Narutowicza 11/12, 80-233 Gdańsk, Poland

[c] Department of Computer Science, Rochester Institute of Technology, Rochester, NY 14623, United States

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ABSTRACT

We study a new problem for cubic graphs: bipartization of a cubic graph Q by deleting sufficiently large independent set I. It can be expressed as follows: *Given an integer k and a connected n-vertex tripartite cubic graph $Q = (V, E)$ with independence number $\alpha(Q)$, does Q contain an independent set I of size k such that $Q - I$ is bipartite?* We are interested for which values of k the answer to this question is affirmative. We prove constructively that if $\alpha(Q) \geq 2n/5$, then the answer is positive for each k satisfying $\lfloor (n-\alpha(Q))/2 \rfloor \leq k \leq \alpha(Q)$. It remains an open question if a similar construction is possible for $\alpha(Q) < 2n/5$.

We also show that this problem with $\alpha(Q) \geq 2n/5$ and k satisfying $\lfloor n/3 \rfloor \leq k \leq \alpha(Q)$ can be related to semi-equitable graph 3-coloring, where one color class is of size k, and the subgraph induced by the remaining vertices is equitably 2-colored. This means that Q has a coloring of type $(k, \lceil (n-k)/2 \rceil, \lfloor (n-k)/2 \rfloor)$.

1. Some preliminaries

There are many challenging and interesting problems involving independent sets and cubic graphs. One of the most known is the problem of independence, IS(Q, k):

Given a connected cubic graph $Q = (V, E)$ and an integer k, does Q contain an independent set of size at least k?

An *independent set* of a graph Q is a subset I of the vertices of Q, $I \subseteq V(Q)$, such that no two vertices in I are joined by an edge in Q. The size of the largest independent set is called the *independence number* of Q, and it is denoted by $\alpha(Q)$. The problem of finding the value of $\alpha(Q)$ is widely discussed in the literature. In general, the problem IS(Q, k) is NP-complete for cubic graphs, and even for planar cubic graphs [5]. A comprehensive survey of results on the IS problem, including cubic graphs, was presented in [1,8,10].

The second type of problems is connected with decycling sets of cubic graphs (also known as feedback-vertex sets). For a graph Q, a subset $S \subseteq V(Q)$ is a *decycling set* of Q if and only if $Q - S$ is acyclic, where by $Q - S$ we mean the subgraph of Q induced by the vertices in $\overline{S} = V(Q) \setminus S$. Although the decycling set decision problem is NP-complete in general, it is polynomially solvable for cubic graphs [11].

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* Corresponding author.
E-mail addresses: hanna@inf.ug.edu.pl (H. Furmańczyk), kubale@eti.pg.gda.pl (M. Kubale), spr@cs.rit.edu (S. Radziszowski).

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The third group contains problems connected with *bipartization* of cubic graphs. Given a graph, the task is to find a smallest set of vertices whose deletion makes the remaining graph bipartite. Choi et al. [4] showed that the bipartization decision problem is NP-complete for cubic graphs. Some approximation algorithms were given in [9].

In this paper we combine the above approaches and define the Bipartization IS problem BIS(Q, k), as follows:

> Given a connected cubic graph $Q = (V, E)$ and integer k, does Q contain an independent set I of size at least k such that $Q - I$ is bipartite?

We are interested for which values of k the answer to this question is affirmative. This problem can be seen as a task of finding independent odd decycling sets.

We say that a graph G is *t-colorable* if its vertex set can be partitioned into t independent sets-color classes. The smallest value of t admitting t-colorability of graph G is named the *chromatic number* of G and denoted by $\chi(G)$. Let us recall Brooks' theorem:

Theorem 1 ([2]). *For any connected graph G with maximum degree Δ, the chromatic number $\chi(G)$ of G is at most Δ, unless G is a clique or an odd cycle.*

This implies that

$$2 \le \chi(Q) \le 3$$

for all cubic graphs except K_4.

It is obvious that for 2-chromatic cubic graphs and $k \le |V(Q)|/2$ the answer to BIS(Q, k) is affirmative. Hence, in the sequel we consider only connected cubic graphs Q with $\chi(Q) = 3$. This means that $V(Q)$ can be partitioned into three independent sets and Q is not bipartite. The class of such cubic graphs will be denoted by $\mathcal{Q}_3$. Its subclass of graphs on n vertices will be denoted by $\mathcal{Q}_3(n)$.

A graph is *equitably* t-colorable if and only if its vertex set can be partitioned into independent sets $V_1, V_2, \ldots, V_t$ such that $\big||V_i| - |V_j|\big| \le 1$ for all $i, j = 1, 2, \ldots, t$. The smallest value of t admitting such coloring of the graph G is named *equitable chromatic number* of G and denoted by $\chi_=(G)$.

In the case of cubic graphs we know that

$$\chi_=(Q) = \chi(Q), \tag{1}$$

where $\chi_=(Q)$ is the equitable chromatic number of Q. This follows from

Theorem 2 (*Chen, Lih, and Wu, 1994, [3]*). *Every proper coloring of connected cubic graph can be made equitable without adding new colors.*

Chen et al.'s [3] algorithm relies on repeatedly decreasing the width of coloring (the difference between the cardinality of the largest and smallest color class) by one until the difference is not greater than one.

In this paper we are also interested in equitable coloring of $Q - I$. We will present an algorithm which, given an independent set of size $k \ge 2n/5$, constructs an appropriate independent set I of size k for the BIS(Q, k) problem with $Q \in \mathcal{Q}_3(n)$. We will also prove that such cubic graphs have colorings of type $(k, \lceil (n-k)/2 \rceil, \lfloor (n-k)/2 \rfloor)$, which means that $Q - I$ has an equitable 2-coloring. Such type of coloring is called *semi-equitable*, i.e. the coloring in which exactly one color class is of any size while the cardinalities of the remaining color classes differ by at most 1. Colorings of this kind are useful in a problem of scheduling identical jobs on three parallel uniform processors [7]. In such a model of scheduling one of processors is faster than the remaining two, while the two slower processors are of the same speed and the conflict graph is cubic.

2. Main results

Our main result concerning BIS(Q, k) is as follows.

Theorem 3. *If $Q \in \mathcal{Q}_3(n)$ and $\alpha(Q) \ge 2n/5$, then there exists an independent set I of size k in Q such that $Q - I$ is bipartite for $\lfloor n/3 \rfloor \le k \le \alpha(Q)$.*

Note, that this leaves the problem open for $\lceil n/3 \rceil \le \alpha(Q) < 2n/5$.

Before we prove Theorem 3, we need some auxiliary concepts.

We consider connected cubic graphs $Q \in \mathcal{Q}_3(n)$ with independence number $\alpha(Q) \ge 2n/5$, and let I be an independent set of size at least $2n/5$. If $Q - I$ is not bipartite, then the subgraph $Q - I$ consists of two parts: a 2-chromatic part of all bipartite components and a 3-chromatic part containing odd cycles (possibly with chords, bridges, pendant edges, etc.).

Definition 1. For $Q \in \mathcal{Q}_3$, the *residuum $R(I)$ of Q with respect to an independent set I* is the set of all odd cycles in the graph $Q - I$.

For example, for the graph in Fig. 1 and given I, $R(I) = \{v_1v_2v_3, v_4v_5v_6\}$.

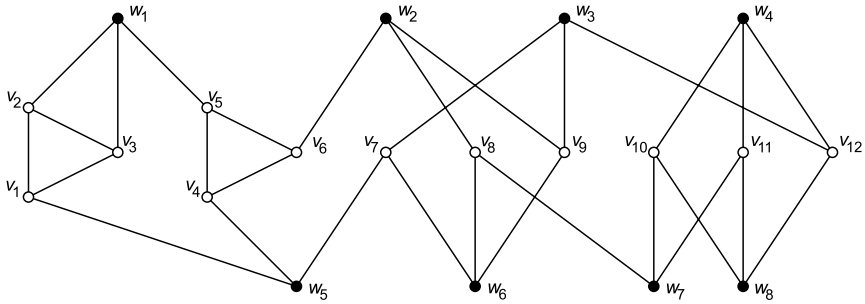

Fig. 1. Example of a cubic graph in $\mathcal{Q}_3(20)$ with independent set I of size 8. The vertices of I are marked in black. $R(I) = \{v_1v_2v_3, v_4v_5v_6\}$.

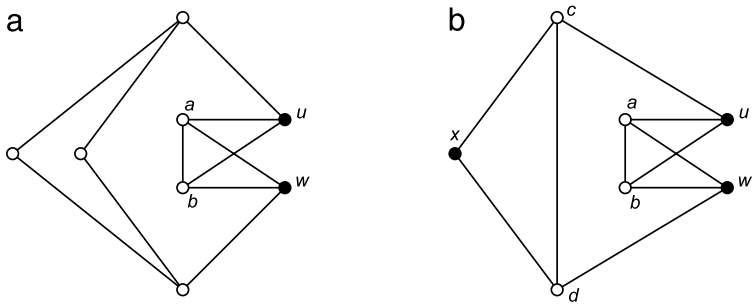

Fig. 2. Subgraph of Q containing: (a) diamond $K_4 - e$ with pseudo-free vertices u and w of type 1; $u, w \in I$; (b) diamond $K_4 - e$ with pseudo-free vertices u and w of type 2; $u, w, x \in I$.

Definition 2. Vertex $w \in I$ is a *free vertex* in $Q \in \mathcal{Q}_3$ with respect to independent set I if and only if its removal from I (but not from $V(Q)$) results in the same residuum, i.e. $R(I) = R(I \setminus \{w\})$. The set of all free vertices in I will be denoted by F_0.

For the graph in Fig. 1 and given independent set I, $F_0 = \{w_2, w_3, \ldots, w_8\}$. Vertex w_1 is not free because moving it from I to $Q - I$ creates a new odd cycle in $Q - I$, namely $w_1v_2v_3$. Clearly, $F_0 \subset I$.

Definition 3. A *diamond* in Q with respect to independent set I is a subgraph D on vertices $\{u, w, a, b\} \subseteq V(Q)$ isomorphic to $K_4 - e$, where $u, w \in I$.

Definition 4. Vertices $u, w \in I$ are *pseudo-free vertices of type 1* in Q with respect to independent set I if and only if there is a diamond D in Q on vertices $\{u, w, a, b\}$, and there is no odd cycle C of length at least 5 with vertices in $\bar{I} \cup \{u, w\}$ such that $|V(C) \cap V(D)| = 3$, (cf. Fig. 2(a)). The set of all pseudo-free vertices of type 1 will be denoted by F_1.

Note that $F_1 \subset I$ and F_1 is a disjoint union of pairs of vertices $\{u, w\}$ satisfying Definition 4.

Definition 5. Vertices $u, w \in I$ are *pseudo-free vertices of type* 2 in Q with respect to independent set I if and only if there is a diamond D in Q on vertices $\{u, w, a, b\}$, and there is a cycle C_5 with vertices in $\bar{I} \cup \{u, w\}$ such that $|V(C_5) \cap V(D)| = 3$, and the two vertices $\{c, d\} = V(C_5) \setminus V(D)$ have a common neighbor x in I (cf. Fig. 2(b)). The set of all pseudo-free vertices of type 2 will be denoted by F_2.

Note that vertices c and d are consecutive on C_5, and $F_2 \subset I$ is a disjoint union of pairs of vertices $\{u, w\}$ satisfying Definition 5.

Let $F(I) = F_0 \cup F_1 \cup F_2$. Clearly, $F_i \cap F_j = \emptyset$ for $i, j = 0, 1, 2$ and $i \neq j$.

The following main auxiliary lemma implies that, under the assumptions of Theorem 3, if $R(I)$ is nonempty, then so is $F(I)$.

Lemma 1. *If $Q \in \mathcal{Q}_3(n)$ has an independent set I of size at least $2n/5$ and $R(I) \neq \emptyset$, then there exists a free or pseudo-free (of type 1 or 2) vertex in I.*

We will prove Lemma 1 in Section 3.

Proof of Theorem 3. First, we will prove that our theorem holds for k satisfying $2n/5 \leq k \leq \alpha(Q)$.

Let I be any independent set of size at least $2n/5$. Assume that $R(I) \neq \emptyset$. We will show that there exists another independent set, say J, such that $|J| \geq |I|$ and $R(J) \subsetneq R(I)$.

Let C be an odd cycle belonging to $R(I)$. Any vertex $v \in V(C)$ must be of degree 2 or 3 in $Q - I$. If there exists $v \in V(C)$ of degree 3 in $Q - I$, then we set $J = I \cup \{v\}$. The new residuum $R(J)$ is a subset of $R(I) \setminus \{C\}$. Otherwise, if each $v \in V(C)$ is of

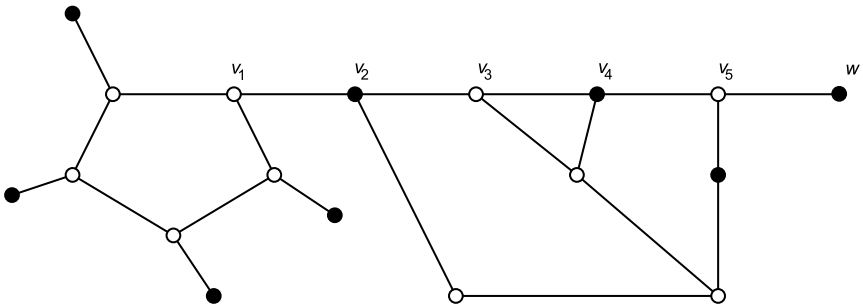

Fig. 3. Example of a subgraph of Q with alternating path $P = v_1v_2v_3v_4v_5w$, where vertex v_5 is as v_i of Subcase 1.2.

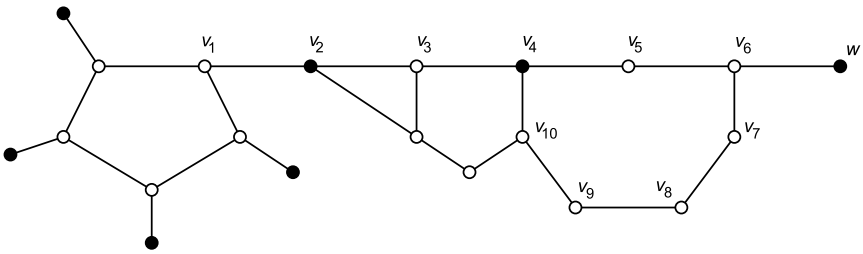

Fig. 4. Example of a subgraph of Q and path $P = v_1v_2v_3v_4v_5v_6w$, which is not alternating. Vertices $v_2, v_4, w \in I$, vertex v_5 is as v_j in Case 2. After applying the procedure described in Case 2: $v_1, v_3 \in I'$ while $v_2, v_4 \in \overline{I'}$ and there is a cycle $C' = v_4v_5v_6v_7v_8v_9v_{10}$ with $v_t = v_6$ (Subcase 2.2).

degree 2 in $Q - I$, then let $v_1 \in V(C)$ and $P = v_1v_2 \dots v_p$ be the shortest path from v_1 to any $v_p \in F(I)$ in Q such that none of vertices $v_1, v_2, \dots, v_{p-1}$ is free or pseudo-free. We know from Lemma 1 that $F(I)$ is nonempty. We consider two cases.

Case 1: *P is a path alternating between $\bar{I}$ and I.*
This means that $v_1, v_3, \dots, v_{p-1} \in \bar{I}$ and $v_2, v_4, \dots, v_p \in I$, $v_p = w$.

Subcase 1.1: *Each vertex of $v_3, v_5, \dots, v_{p-1}$ has exactly two neighbors in I.*
Then we interchange even and odd vertices between I and $\bar{I}$ along the path P so that a new independent set

$$J = \begin{cases} I \cup \{v_1, v_3, \dots, v_{p-1}\} \setminus \{v_2, v_4, \dots, v_p\} & \text{if } w \in F_0, \\ I \cup \{v_1, v_3, \dots, v_{p-1}, a\} \setminus \{v_2, v_4, \dots, v_p, u\} & \text{if } w \in F_1 \cup F_2, \end{cases}$$

of the same size is obtained, and $R(J) \subset R(I) \setminus \{C\}$.

Subcase 1.2: *There is a vertex in $\bar{I}$ on path P such that all (three) of its neighbors belong to I.*
In this case we choose among such vertices one with the smallest index, say v_i, $3 \le i \le p-1$ (see vertex v_5 in Fig. 3). Let us observe that vertex $v_{i-1} \in I$ can belong to odd cycles in $Q - (I \setminus \{v_{i-1}\})$, including the edge $\{v_{i-2}, v_{i-1}\}$, but there is no odd cycle in $Q - (I \setminus \{v_{i-1}\})$ passing through $\{v_{i-1}, v_i\}$. We interchange even and odd vertices along the prefix subpath $v_1v_2 \dots v_{i-1}$ of P, so that $J = I \cup \{v_1, v_3, \dots, v_{i-2}\} \setminus \{v_2, v_4, \dots, v_{i-1}\}$, and $R(J) \subset R(I) \setminus \{C\}$.

Case 2: *P is not an alternating path.*
This means that there is a vertex in $Q - I$ on P such that its successor on path P is also in $\bar{I}$. We choose among such vertices one with the smallest index, say v_j (see vertex v_5 in Fig. 4).
We consider the alternating part of the path P up to vertex v_{j-1} as in Case 1. Let I' be a new independent set obtained after applying the procedure from Case 1. Now, we have one of the following subcases:

Subcase 2.1: *There is no odd cycle in $Q - I'$ containing the edge $\{v_{j-1}, v_j\}$.*
Cycle C is broken and no new odd cycle is created. $J = I'$.

Subcase 2.2: *There is an odd cycle C' in $Q - I'$ containing the edge $\{v_{j-1}, v_j\}$.*
If there exists $v \in V(C')$ of degree 3 in $Q - I'$, then we set $J = I' \cup \{v\}$ (the new residuum $R(J)$ is a subset of $R(I') \setminus \{C'\}$; of course, $R(J) \subsetneq R(I)$). Otherwise, let v_t be the vertex belonging to both C' and P whose index t is the highest (see vertex v_6 in Fig. 4). Note, that $v_{t+1} \in I'$. In this case we consider the alternating part of the path P starting with vertex v_t as in Case 1, and finally obtain J, which clearly satisfies $R(J) \subsetneq R(I)$.

In fact, one can see that a single iteration of the algorithm breaks all odd cycles in $R(I)$ containing vertex v. If the new independent set has nonempty residuum, we repeat our algorithm iteratively (with another cycle C and path P). There is at least one odd cycle broken in each iteration of the algorithm. Consequently, after s iterations of the algorithm, we obtain a sequence of independent sets $J_1, J_2, \dots, J_s$ of non-decreasing sizes, $R(J_s) = \emptyset$, and hence $Q - J_s$ is bipartite.

Therefore, by Lemma 1, if a cubic graph $Q \in \mathcal{Q}_3$ has an independent set I of size $k \ge 2n/5$, then it also has an independent set J_s of size at least k such that $Q - J_s$ is bipartite. Due to Chen et al.'s [3] constructive proof of Theorem 2 thus obtained

3-coloring of Q can be equitalized to $(\lfloor n/3\rfloor, \lfloor (n+1)/3\rfloor, \lfloor (n+2)/3\rfloor)$ by decreasing the width of the coloring one by one, which completes the proof for all k, $\lfloor n/3\rfloor \le k \le \alpha(Q)$. $\square$

A single iteration of our bipartization construction of Theorem 3 clearly runs in $O(n^2)$ time. The naive worst case bound may require $|R(I)|$ such iterations, and $|R(I)|$ is bounded by the number of odd cycles in the original graph, which in turn can be exponential. On the other hand, one can easily note that the size of the set of vertices on all cycles in $R(I)$ is decreasing. Thus, $O(n)$ iterations are sufficient to complete the algorithm and the computational complexity of the whole algorithm is $O(n^3)$.

3. Proof of Lemma 1

Lemma 1. *If $Q \in \mathcal{Q}_3(n)$ has an independent set I of size at least $2n/5$ and $R(I) \neq \emptyset$, then there exists at least one free or pseudo-free (of type 1 or 2) vertex in I.*

Proof. We need to prove that $F(I) = F_0 \cup F_1 \cup F_2 \neq \emptyset$. First, we assume that n is divisible by 10 and let I be an independent set of size $2n/5$. Let $|L| = l$, where L denotes the set of isolated vertices in $Q - I$. We are interested in the structure of $Q - I$, including the value of l. This is a graph with $3n/5$ vertices and $3n/10$ edges. Let us notice that if $Q - I$ has no isolated vertices, $Q - I$ must define a perfect matching, in which case we have $3n/10$ K_2's. If $Q - I$ contains components with more than one edge, then we have some number of isolated vertices. For example, a cycle C_p in $Q - I$ "implies" p isolated vertices, and a path P_p "implies" $p - 2$ vertices. In general, a component of $Q - I$ with m' edges and n' vertices "implies" $2m' - n'$ isolated vertices. $Q - I$ can contain as components $K_1, K_2, P_p, C_{p'}$ $(p, p' \ge 3)$, and components which have at least one vertex of degree 3. Let Q_l denote the part of $Q - I$ excluding K_1's and K_2's, i.e. the part which "implies" the isolated vertices. For given Q and I, the subgraph $Q - I$ consists of Q_l, l isolated vertices K_1 and k_2 isolated edges K_2.

We consider two cases:

Case 1: $l > 2n/15$.

Since $3l > |I|$, there must exist at least one vertex in I, say u, which is adjacent to at least two vertices in L. Note that u cannot be on any cycle together with vertices of $Q - I$. This means that $u \in F_0$ is a free vertex, and $F_0 \neq \emptyset$.

Case 2: $l \le 2n/15$.

Define γ_1, γ_2 as the number of non-free vertices in I implied by k_2 isolated edges in $Q - I$, and others, respectively. We have $|I| = |F(I)| + \gamma_1 + \gamma_2$.

In this case, where $l \le 2n/15$, $F_0 \subset F(I)$ may be empty. We will show that if $F_0 = F_1 = \emptyset$, then $F_2 \neq \emptyset$.

First, we will prove that there exists K_2 among all k_2 isolated edges such that it is a subgraph of a diamond $(K_4 - e)$. We introduce some additional notation. Let $\mathcal{K}^i$ denote the set of all such K_2's in $Q - I$, whose endvertices have exactly i common neighbors in I, and let $|\mathcal{K}^i| = k_2^i$, $i = 0, 1, 2$. Of course, $k_2 = k_2^0 + k_2^1 + k_2^2$. Moreover, let us notice that $k_2^2 K_2$'s result in desirable k_2^2 diamonds with respect to I.

Claim. There is a diamond with respect to I in Q, i.e. $k_2^2 > 0$.

Proof of Claim. For a contradiction to the Claim, let us assume that the endvertices of each K_2 have at most one common neighbor in I. This means that k_2 isolated edges cause at most k_2 non-free vertices.

Since $F_0 = \emptyset$, then all vertices in I are non-free. Note that any vertex in I is adjacent to at most one isolated vertex in $Q - I$, since otherwise it would be free.

We will show that

$$3n/10 - 2l + 3 \le k_2 \le 3n/10 - 3l/4. \tag{2}$$

Indeed, let us consider the structure of Q_l implying the minimal number of K_2's. It is easy to see that such Q_l must contain C_3 and $(l-3)P_3$, with $3(l-2)$ vertices and $2(l-2)+1$ edges. This implies that $k_2 \ge (3n/5 - l - 3(l-2))/2 = 3n/10 - 2l + 3$. On the other hand, the structure of Q_l maximizing k_2 must contain the minimal number of vertices equal to $3n/5 - 2k_2 - l$ with $3n/10 - k_2$ edges. Hence k_2 satisfies $3(3n/5 - 2k_2 - l) \ge 3n/5 - 2k_2$, which implies the upper bound in (2).

We will bound from above the maximum number of non-free vertices in I for which an odd cycle (resulting from the fact that they are non-free vertices) is formed by vertices of $Q_l \cup I$ (this bound is denoted by $\gamma_{\max}$). One can check that its maximal value is equal to $3l/2 - 7/2$ and it is achieved by $Q_l = C_3 \cup (l-3)/p\, P_{p+2}$ for even p. The number of vertices in such Q_l is maximal for $p = 2$. In this case $k_2 = 3n/10 - 3l/2 + 3/2$.

Hence, the maximum number of non-free vertices in I implied by vertices of $Q - I$, assuming that $F_0 = \emptyset$ and that the endvertices of each K_2 have at most one common neighbor in I, is $\gamma_{max} + k_2$ and we have

$$\gamma_{\max} + k_2 \le (3l/2 - 7/2) + (3n/10 - 3l/2 + 3/2) < 2n/5. \tag{3}$$

End of proof of Claim.

Now, let us assume that $F_1 = \emptyset$ (with $F_0 = \emptyset$). This means that three vertices of each diamond D formed by edges from $\mathcal{K}^2$ lie on an odd cycle of length at least 5 (due to Definition 4). We note that such odd cycles can be

caused by joining vertices from $I \cap V(D)$ to endvertices of K_2 from $\mathcal{K}^1 \cup \mathcal{K}^0$. Observe that diamonds connected in this way to $K_2 \in \mathcal{K}^1$ result in pseudo-free vertices of type 2.

Finally, assume that $F_2 = \emptyset$. This implies that the endvertices of each $K_2 \in \mathcal{K}^2$ are joined to vertices of Q_l or to endvertices of $K_2 \in \mathcal{K}^0$. Since Q is cubic and connected, there is at most one diamond joined to each of $K_2 \in \mathcal{K}^0$.

Since $|V(Q_l)| + l + 2k_2 = 3n/5$, and $k_2 = k_2^0 + k_2^1 + k_2^2$, and by (2), we have

$$|V(Q_l)| + l + k_2^0 + k_2^1 + k_2^2 = 3n/5 - k_2 \leq 3n/10 + 2l - 3.$$

Since $\gamma_2 \leq |V(Q_l)|/2$, we get

$$2\gamma_2 + l + k_2^0 + k_2^1 + k_2^2 \leq 3n/10 + 2l - 3.$$

Because clearly

$$\gamma_2 + 2k_2^2 + k_2^1 = 2n/5 = |I|,$$

and hence $k_2^1 = 2n/5 - \gamma_2 - 2k_2^2 - k_2^1$, we have:

$$k_2^2 \geq \gamma_2 + n/10 + k_2^0 + 3 > \gamma_2 + k_2^0. \tag{4}$$

Since $F_1 = \emptyset$, inequality (4) means that at least one diamond with respect to I is joined to $K_2 \in \mathcal{K}^1$. This implies $F_2 \neq \emptyset$, a contradiction. □

4. Bipartization and equitable colorings

In the paper we have posed a new problem for cubic graphs $Q \in \mathcal{Q}_3(n)$ with an independent set I of size k. We answered the question about existence of appropriate bipartizing independent set for $\lfloor n/3 \rfloor \leq k \leq \alpha(Q)$ and $\alpha(Q) \geq 2n/5$.

On the other hand, Frieze and Suen [6] showed that the independence number of almost all cubic graphs on n vertices satisfies $\alpha(Q) \geq 2.16n/5 - \epsilon n$, for any constant $\epsilon > 0$. Moreover, they gave a simple greedy algorithm which finds an independent set I of that size in almost all cubic graphs. In practice this means that a graph from $\mathcal{Q}_3(n)$ is very likely to have an independent set of size $k \geq 2n/5$. Certainly, such a set I need not be bipartizing (cf. [7], Fig. 3).

Taking into consideration the structure of the bipartized subgraph $Q - I$, it turns out that such a subgraph can be colored in equitable way with two colors. Let us assume that $|I| = 2n/5$. Notice that $3n/5$ vertices of $Q - I$ induce binary trees (some of them may be trivial) and/or graphs whose 2-core is equibipartite (an even cycle possibly with chords). Note that deleting an independent set I of cardinality $2n/5$ from a cubic graph Q means also that we remove $6n/5$ edges from the set of all $3n/2$ edges of Q. The resulting graph $Q - I$ has $3n/5$ vertices and $3n/10$ edges. Let s_i, $0 \leq i \leq 3$, be the number of vertices in $Q - I$ of degree i, $\Sigma_{i=0}^{3} s_i = 3n/5$. Since the number of edges is half the number of vertices, the number of isolated vertices, s_0, is equal to $s_2 + 2s_3$. If $s_0 = 0$, then $Q - I$ is a perfect matching and its equitable coloring is obvious.

Suppose that $s_0 > 0$. Consider the part of $Q - I$ without isolated vertices and its 2-coloring. Each vertex of degree 3 causes the difference between cardinalities of color classes equal to at most 2 ($K_{1,3}$), similarly each vertex of degree 2 causes the difference at most 1 ($K_{1,2}$). The difference between the cardinalities of color classes in any coloring satisfying these conditions is at most $s_2 + 2s_3$ in $Q - I - L$, and an appropriate assignment of colors to isolated vertices L makes the graph $Q - I$ equitably 2-colored. Hence, we have:

Proposition 4. *If $Q \in \mathcal{Q}_3(n)$ has an independent set I of size $|I| = 2n/5$, then it has a semi-equitable coloring of type $(4n/10, \lceil 3n/10 \rceil, \lfloor 3n/10 \rfloor)$.*

Note, that if an n-vertex cubic graph Q has an independent set I of cardinality $|I| \geq 2n/5$ and consequently, by Theorem 2, there exists independent set J of the same cardinality such that $Q - J$ is bipartite, then we have more isolated vertices in $Q - J$ and a partition of $Q - J$ into V_1 and V_2 such that $\big||V_1| - |V_2|\big| \leq 1$ is possible. Hence we have

Corollary 5. *If $Q \in \mathcal{Q}_3(n)$ has an independent set I of size $|I| \geq 2n/5$, then it has a semi-equitable coloring of type $(|I|, \lceil (n - |I|)/2 \rceil, \lfloor (n - |I|)/2 \rfloor)$.*

Taking into account above considerations Theorem 3 can be extended as follows.

Theorem 6. *If $Q \in \mathcal{Q}_3(n)$ and $\alpha(Q) \geq 2n/5$, then there exists an independent set I of size k in Q such that $Q - I$ is bipartite for $\lfloor (n - \alpha(Q))/2 \rfloor \leq k \leq \alpha(Q)$.* □

The problem BIS(Q, k) for $k < \lfloor (n - \alpha(Q))/2 \rfloor$ in $\mathcal{Q}_3(n)$ with $\alpha(Q) \geq 2n/5$ stays open as well as the problem for cubic graphs with $\alpha(Q) < 2n/5$. We think that our algorithm is effective also for solving many instance of open cases.

Finally, note that Theorem 6 cannot be generalized to all 3-colorable graphs, since the sun S_3 graph[1] is a counterexample.

[1] The 3-sun S_3 is a graph on 6 vertices partitioned into two subsets $U = \{u_0, u_1, u_2\}$ and independent set $W = \{w_1, w_2, w_3\}$, where vertices from U form a cycle and each vertex $w_i \in W$ has exactly two neighbors, u_{i-1} and $u_{i \bmod 3}$.

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