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On coertia and inertia in aspects of Natanson's nonlinear extended thermodynamics

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Abstract

In this article, the previously underrepresented contributions of Natanson to the field of thermodynamics have been presented. In order to identify a source of irreversibility at Nature, Natanson introduced the concept of Coertia, which is similar to inertia. Natanson's Coertia is a fundamental property of space that is responsible for every irreversible phenomena in matter, as well as in the electromagnetic and gravitational fields. We focus on the mathematical reconstruction of a few of his principal ideas that until now have been neglected by the literature. To set these ideas in proper epistemological order, we thought it would be valuable to first revalue and reconstruct some missing parts of the proceedings process by Ladislavus Natanson constructed their thermodynamics. We also aimed to present Natanson's achievements against the background of modern continuum mechanics, exemplifying old but still relevant approaches. We propose that Natanson's ideas were ahead of their time by about one century. Give that scientist was educated in the scientific royal way: chemistry, through mechanic of solid and fluid, thermodynamics, electro-chemistry, electrodynamics, early quantum and relativistic mechanics, we can closely compare their conceptions and solutions. Natanson was in strong opposition with Newtonian mechanisms, the Maupertuis least action principle formed the basis of his activities, which they were developing as a sum of elementary quantum actions.

1 Introduction

This Cimmelli, Jou, Rugerri and Ván [1] have recently elaborated concise versions of the modern mathematical methods used in thermodynamics. To do this, they surveyed numerous results from classical, irreversible, extended and statistical thermodynamics in order to obtain a summary of current methods and their usefulness as scientific tools. Here we take a similar approach but instead outline the state of thermodynamics at the end of the 19th century. We begin by short as possible introducing the state-of-art for the period of 1870-1880 before describing the thermodynamic models developed by Natanson.

Here we aimed to reconstruct and revalorize extremely abstract, potential-based thermodynamic model that was created by Ladislavus Natanson (Cracow). We specifically wanted to show the important developments that took place at the end of the 19th century and restrict oversells to some selected not yet published mathematical ideas devoted to Natanson [see:2-5]. Thus, we can rediscover an original construction of a very general theory that starts from the formal unification of mechanics and thermodynamics. Sometimes we present the original results without explanation of the original symbols in the equation, if nowadays those are well-known. The authors would like to mention that a historical reconstruction of the abstract thermodynamics in the second half of the 19th century was given by S. Bordoni [6]. Natanson belongs to that historical context.

1.1 Towards Rational Thermodynamics

Most would agree that the tradition of classical field theory finally began with Lagrange's Analitique Mechanics. In analogy, we agree that Carnot's roots of rational thermodynamics also have the same origin. Lazare Carnot's statement, shared in a form of an advice to his son Sadi, is well-known : "In order to find a base for thermodynamics you have to prolongation the mathematics of Analytic Mechanics". Therefore, the overall aim of Sadi Carnot was to create a science of caloric balance, similar to and the formal structures of analytical mechanics. Thus, it is unsurprising that Carnot's equation of motion of substantial caloric possesses the same mathematical structures as the balance of entropy.

However, Carnot's abstractive approach to energy conversion from heating to working is very mystique. Truly speaking, Sadi Carnot provides a germ of a mathematical model that is actually a specific mechanical model, completely hidden in footnotes. This Carnot approach blossomed slowly over about ten decades across Britain and Europe. Owing to the efforts of Clapeyron, Lamè, Hoëné-Wroński, Ferdinand Reech, James Thomson (older brother of Lord Kelvin), some elements of Carnot's ideas were retained in the foundation of the Second Law of thermodynamics. Unfortunately, none of the three foundations of thermodynamic laws (Clausius, Thomson, Helmholtz) borrowed from Carnot's concept of rational thermodynamics.

After the 1850s, some researchers occasionally turned into towards rational thermodynamics (e.g., Mikhail Okatov, Francois Massieu, Josiah Willard Gibbs, Franz Neumann, Arthur von Oettingen and the young Max Planck). Thus, Pierre Duhem and Ladislavus Natanson were the first among chemists to explore the connections between the contents of thermo-chemistry and the formal structures of analytical mechanics.

1.2 Energetism

Energetism began with William Macquorn Rankine and Ferdinand Reech's researches in the mid-19th century. Unfortunately, this science was a kind of reaction to a powerful paradigm that assumed that all physical phenomena are essentially mechanical; several researchers were in fact actively engaged in the project of demonstrating this by reducing physical theory to mechanics. Both Duhem and Natanson were against a direct interpretation of the Rankine-Reech approach to a clear and adequate foundation for thermodynamics. They especially rejected the orthodox vision developed by the German school of energetism. It is important to note that in Germany, the three leading figures of Georg Helm, Wilhelm Ostwald and Ernst Mach, had been promoting a theory of energetic inspired by thermodynamics at the end of the 19th century, but with rather different motivations.

In 1911, Duhem [7] directly criticised the understanding of energetism. He disagreed with Ostwald's stance about treating energy as the only ultimate real object (also known as Helm's phenomenal view). Although Duhem was in agreement with Ostwald and Helm in opposing the universal reduction of everything to mechanics, he did not offer an alternative form of reduction and never appeals to their writings. However, Mach's concept of relativity, as well as his criticism of Newton, were appreciated by both Duhem [8] and Natanson [9].

1.3 Atomism – kinetic Theory of Gases

Basing on the atomistic version of nature and the philosophical fundamentals given by Democrit, Lucrecjus and Rudolf Clausius in the middle of 19th century, a new version of thermodynamics and traditions of research emerged from the kinetic theory of gases. Different "mechanical theories of heat" were presented during the last decades of the 19th century and very meaning of the adjective mechanical was at stake. Between 1860 and 1870, Krönig, Meyer, James Clerk Maxwell and Ludwig Boltzmann pursued the integration of thermodynamics with the kinetic theory of gases. Stefano Bordoni has recently proposed [6] a finer classification would require at least five streams, which can be sorted according to their conceptual distance from mechanics:

1. a purely phenomenological approach, where thermodynamics relied on its own foundations;

2.the energetism approach, where thermodynamics emerged as a specific implementation of a science of energy;

3.a macroscopic approach based on structural analogy with abstract mechanics;

4.the combination of macroscopic and microscopic approaches based on the same analogy;

5.a microscopic approach, where specific mechanical models of forces and/or collisions merged with statistical assumptions that did not belong to the tradition of mechanics

1.4 Irreversible Thermodynamics

The concept of irreversibility likely originated from Leonardo da Vinci, who was the first to make a distinction between first and second type perpetuum mobile. Unfortunately, from the extensive writings about the irreversible phenomena, only a few concepts have been translated into precise mathematical meaning. In the time of Duhem and Natanson, irreversibility, relative to reversibility, had a poor mathematical understanding. However, interest in irreversibility was renewed following discoveries by Jaumann, Lippmann, Eckart, Onsager, Prigogine, Meixner, Reik, Lohr, de Groot, Gyarmathy, Kluitenberg, Liukov, Ziegler, Biot, Machlup and others [10]. With regard to recent trends in rational thermodynamics and the thermodynamics of irreversible processes, the authors would like to mention the excellent paper of I. Muller and W.Weiss [10].

1.5 Extended Irreversible Thermodynamics

Although extended irreversible thermodynamics formally began with Maxwell's much celebrated paper, it was only developed after a resurrection inspired by Ingo Müller's 1969 dissertation. G. Lebon, David Jou, José Casas-Vázquez, Péter Ván, Vito Cimmelliand Tommaso Ruggeri are among the researchers that have addressed the various types of extended thermodynamics [1,11].

1.6 Thermodynamics of hidden parameters

The concept of hidden parameters is typically considered to have arisen from the efforts of such pioneers and veterans as Josef Kestin, Wolfgang Muschik, Miroslav Grmela, Gerard Maugin, Witold Kosiński, Lilliana Restuccia and Bogdan Maruszewski. Unfortunately, authors such as Herman Helmholtz, Edward Routh, Kálmán Szily von Nagy-Szigeth, Hans Reissner and Pierre Duhem are rarely recognized [12,13,14]. Contemporary related literature about internal variables are classified into two groups according to Maugin and Muschik [12,13]. Internal variables of state with a relaxation type evolution generated by thermodynamics and dynamic degrees of freedom with variational evolution. The two kind of evolution is actually the same when one uses dual internal variables and weakly nonlocal theory [14].

2 Natanson's nonlinear extended thermodynamics

Ladislavus Natanson initial interests were focused on the Maxwell kinetic theory of gases [15], which was the subject of his diploma [16] and doctoral theses [17] prepared at Dorpat under the supervisor of professor Arthur von Oettingen. His interest in the kinetic theory of gases increased after a long trip to the Cavendish Laboratory [18] and to Gratz University [19]. In 1980, Natanson (now 26-years-old) and living in Warsaw written his first book, entitled "Introduction to Theoretical Physics" [9]. This book was wholly original and the few last chapters were completely novel, containing an introduction to extended thermodynamics in a fully three-dimensional framework.

2.1 Natanson's fundamental equation

According to Maxwell, Natanson introduced the use of two kinds of velocity vectors: molar $\mathbf{u}$ and molecular $\mathbf{c}$ [9,20,21]:

$$\mathbf{u}+\mathbf{c}=\left(u\mathbf{e}_x+v\mathbf{e}_y+w\mathbf{e}_z\right)+\left(\xi\mathbf{e}_x+\eta\mathbf{e}_y+\zeta\mathbf{e}_z\right) \tag{1}$$

Taking into account the body force $\mathbf{f}$ Natanson, repeating Maxwell's original reasoning [15], was able to extended Maxwell fundamental equation [9,p.385,eq.12] :

$$\frac{d}{dt}(\bar{Q}n)+\mathrm{div}\left(n\overline{\mathbf{c}\otimes Q}\right)+\mathrm{div}(\mathbf{u})\bar{Q}n=\frac{\delta}{\delta t}(\bar{Q}n)+n\overline{\left(\mathbf{f}\cdot\frac{\delta Q}{\delta\mathbf{u}}\right)} \tag{2}$$

That is now the well-known starting point for the kinetic theory of gases.

2.2 Continuity equation

By repeating Maxwell's reasoning, step-wise, Natanson obtained a set of evolution equations for different balanced quantities Q (topological charges). By firstly taking $Q=m$ and the following identities [22]:

$$\overline{\mathbf{c}\otimes Q}=\overline{m\mathbf{c}}=0,\quad \bar{Q}n=\rho,\quad \bar{Q}=m,\quad \overline{\frac{\delta Q}{\delta\mathbf{u}}}=0 \tag{3}$$

Natanson arrived at the Euler mass continuity:

$$\frac{d}{dt}\rho+\rho\mathrm{div}\mathbf{u}=0 \tag{4}$$

Next, by setting (3) into (2), a non-conservative form of the fundamental equation was obtained [23,§1,eq.5] :

$$\rho\frac{d}{dt}\bar{Q}+\mathrm{div}\left(\rho\overline{\mathbf{c}\otimes Q}\right)=\rho\frac{\delta}{\delta t}\bar{Q}+\rho\overline{\left(\mathbf{f}\cdot\frac{\delta Q}{\delta\mathbf{u}}\right)} \tag{5}$$

where the d'Alembert-Euler material derivative is defined as:

$$\frac{d}{dt}(\cdot)_{X=\mathrm{const}}=\frac{\partial}{\partial t}(\cdot)_{x=\mathrm{const}}+\mathrm{grad}(\cdot)\mathbf{u} \tag{6}$$

2.3 Balance of linear momentum

Next, putting $Q=\mathbf{u}+\mathbf{c}$ and using the following identities:

$$\bar{Q} = \mathbf{u}, \quad \overline{\boldsymbol{c} \otimes Q} = \overline{\mathbf{c} \otimes \mathbf{c}}, \quad \frac{\delta}{\delta t}\bar{Q} = 0, \quad \frac{\delta Q}{\delta \mathbf{u}} = \mathbf{I} \tag{7}$$

Natanson obtains the evolution of the linear momentum [23, §4,eq.4] :

$$\rho \frac{d}{dt}\mathbf{u} + \operatorname{div}\left(\rho \overline{\mathbf{c} \otimes \mathbf{c}}\right) = \rho \mathbf{f} \tag{8}$$

where, according to Gabriel Stokes and the British tradition of the instance of the Cauchy tension tensor $\mathbf{t}$, Natanson introduced a Stokes-Reynolds pressure tensor, denoted as: $\mathbf{p} = \rho\left(\overline{\mathbf{c} \otimes \mathbf{c}}\right)$.

2.4. Balance of total energy

It was more difficult and time consuming for Natanson to balance the total energy. Taking $Q = (\mathbf{u} + \mathbf{c}) \cdot (\mathbf{u} + \mathbf{c}) = (u + \xi)^2 + (v + \eta)^2 + (w + \zeta)^2$ and a few not trivial identities:

$$\bar{Q} = \overline{\mathbf{u} \cdot \mathbf{u} + 2\mathbf{c} \cdot \mathbf{u} + \mathbf{c} \cdot \mathbf{c}} = \mathbf{u}^2 + \overline{\mathbf{c}^2} = \mathbf{u}^2 + \overline{\xi^2} + \overline{\eta^2} + \overline{\zeta^2} \tag{9}$$

$$\mathbf{c} \otimes \bar{Q} = \overline{\mathbf{cu} \cdot \mathbf{u}} + \overline{\mathbf{c} \cdot u\boldsymbol{c}} + \overline{\mathbf{u} \cdot \mathbf{uc}} + \overline{\mathbf{c} \cdot \mathbf{cc}} = 2\overline{\mathbf{u} \cdot \mathbf{cc}} + \mathbf{q} \tag{10}$$

$$\begin{aligned} \operatorname{div}\left(\rho \overline{\mathbf{c} \otimes Q}\right) &= \operatorname{div}(\rho \mathbf{q}) + 2\operatorname{div}\left(\rho \overline{\mathbf{c} \otimes \mathbf{c}} \cdot \mathbf{u}\right) = \\ &= \operatorname{div}(\rho \mathbf{q}) + 2\operatorname{div}\left(\rho \overline{\mathbf{c} \otimes \mathbf{c}}\right) \cdot \mathbf{u} + 2\left(\rho \overline{\mathbf{c} \otimes \mathbf{c}}\right) \cdot \mathbf{u} \otimes \nabla \end{aligned} \tag{11}$$

as well as:

$$\frac{\delta}{\delta t}\bar{Q} = \frac{\delta}{\delta t}\overline{\xi^2} + \frac{\delta}{\delta t}\overline{\eta^2} + \frac{\delta}{\delta t}\overline{\zeta^2} = 0 \tag{12}$$

(an assumption) and:

$$\rho \mathbf{f} \cdot \frac{\delta Q}{\delta \mathbf{u}} = 2\rho \mathbf{f} \cdot \mathbf{u}, \qquad \mathbf{p} = \rho \overline{\mathbf{c} \otimes \mathbf{c}} = \mathbf{p}^T, \ \operatorname{grad}\mathbf{u} \equiv \mathbf{u} \otimes \nabla, \qquad \mathbf{d} = \frac{1}{2}(\nabla \otimes \mathbf{u} + \mathbf{u} \otimes \nabla) \tag{13}$$

Natanson, defined the heat energy flux as $\mathbf{q} = \overline{\mathbf{c} \cdot \mathbf{cc}}$, arriving at:

$$\rho \frac{d}{dt}\left(\mathbf{u}^2 + \overline{\mathbf{c}^2}\right) + \operatorname{div}(\rho \mathbf{q}) + 2\operatorname{div}(\mathbf{p}) \cdot \mathbf{u} + 2\mathbf{p} \cdot \mathbf{d} = 2\rho \mathbf{f} \cdot \mathbf{u} \tag{14}$$

and after removing a contribution coming from momentum, ultimately arrived as: [22,§2, eq.4]:

$$\rho \frac{d}{dt}\left(\overline{\mathbf{c}^2}\right) + \operatorname{div}(\rho \mathbf{q}) + 2\mathbf{p} \cdot \mathbf{d} = 0 \tag{15}$$

2.5. Evolution of heat flux

Furthermore, by considering $Q = (\mathbf{u} + \mathbf{c}) \otimes (\mathbf{u} + \mathbf{c}) \otimes (\mathbf{u} + \mathbf{c})$ Natanson decided to take a source of energy flux as: $Q = (\mathbf{u} + \mathbf{c}) \cdot (\mathbf{u} + \mathbf{c}) \cdot (\mathbf{u} + \mathbf{c})$ by exploring the not-trivial identities:

$$\bar{Q} = \mathbf{uu}^2 + \mathbf{u}\overline{(\mathbf{c} \cdot \mathbf{c})} + \overline{\mathbf{cu} \cdot \mathbf{c}} + \overline{\mathbf{cc} \cdot \mathbf{u}} + \overline{\mathbf{cc} \cdot \mathbf{c}} \tag{16}$$

$$\overline{\mathbf{c} \otimes Q} = 2\overline{(\mathbf{c} \cdot \mathbf{u})\mathbf{c} \otimes \mathbf{u}} + \overline{\mathbf{c}^2\mathbf{c} \otimes \mathbf{u}} + \mathbf{u}^2\overline{\mathbf{c} \otimes \mathbf{c}} + 2\overline{\mathbf{c} \cdot \mathbf{uc} \otimes \mathbf{c}} + \overline{(\mathbf{c} \cdot \mathbf{c})\mathbf{c} \otimes \mathbf{c}} \tag{17}$$

$$\frac{\delta}{\delta t}\bar{Q} = \mathbf{u}\frac{\delta}{\delta t}\overline{\mathbf{c}^2} + \frac{\delta}{\delta t}\overline{(\mathbf{u}\cdot\mathbf{c})\mathbf{c}} + \frac{\delta}{\delta t}\mathbf{q} \tag{18}$$

$$\overline{\mathbf{f}\frac{\delta Q}{\delta \mathbf{u}}} = \overline{\mathbf{f}(\mathbf{u}+\mathbf{c})\cdot(\mathbf{u}+\mathbf{c})} + 2\mathbf{f}\big(\mathbf{u}\otimes\mathbf{u} + \overline{\mathbf{c}\otimes\mathbf{c}}\big) \tag{19}$$

Natanson [23,§3,eq.2] obtain the evolution equation for energy heat flux vector:

$$\begin{gathered}\rho\frac{d}{dt}\left[\left(\mathbf{u}^2+\overline{\mathbf{c}^2}\right)\mathbf{u}\overset{(1)}{+}2\overline{(\mathbf{u}\cdot\mathbf{c})\mathbf{c}}+\mathbf{q}\right]+\operatorname{div}\left[\begin{array}{l}2\rho\overline{(\mathbf{u}\cdot\mathbf{c})\mathbf{u}\otimes\mathbf{c}}\overset{(2)}{+}\rho\mathbf{u}\otimes\mathbf{q}+\rho\mathbf{u}^2\overline{\mathbf{c}\otimes\mathbf{c}}+\\ \overset{(3)}{+}2\overline{(\mathbf{u}\cdot\mathbf{c})\mathbf{c}\otimes\mathbf{c}}+\overline{(\mathbf{c}\cdot\mathbf{c})\mathbf{c}\otimes\mathbf{c}}\end{array}\right]\\ =\rho\frac{\delta}{\delta t}\left[\overline{\mathbf{c}^2}\mathbf{u}\overset{(4)}{+}2\overline{(\mathbf{c}\cdot\mathbf{u})\mathbf{c}}+\mathbf{q}\right]+\rho\mathbf{f}\left(\mathbf{u}^2+\overline{\mathbf{c}^2}\right)+2\mathbf{f}\left[\rho\mathbf{u}\otimes\mathbf{u}\overset{(5)}{+}\rho\overline{\mathbf{c}\otimes\mathbf{c}}\right]\end{gathered} \tag{20}$$

This fully geometrically nonlinear equation that appeared in Natanson's original paper required 26 pages. Natanson has found this equation steeply [21-25], in following papers [21,sign (1)], [22, sign (2)], [24,sign (3)], [25,sign (4)], [23,sign(5)]. It is worth noting that a linear version of this equation was discovered by Cattaneo in 1948:

$$\frac{\partial}{\partial t}\mathbf{q} + \frac{\mathbf{q}}{\tau} = \kappa \nabla T \tag{21}$$

Which is a prototype of eq.(21) that can be derived from Maxwell's eq. (167) of his celebrated paper [15].

2.6. Evolution of the linear momentum flux

Next, taking $Q = (\mathbf{u}+\mathbf{c}) \otimes (\mathbf{u}+\mathbf{c})$ and the appropriate identities:

$$\bar{Q} = \mathbf{u}\otimes\mathbf{u} + \overline{\mathbf{c}\otimes\mathbf{c}} \equiv \mathbf{u}\otimes\mathbf{u} + \mathbf{p} \tag{22}$$

$$\overline{\boldsymbol{c}\otimes Q} = \overline{\mathbf{c}\otimes\mathbf{u}\otimes\mathbf{c}} + \overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{u}} + \overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}} \tag{23}$$

$$\begin{gathered}\operatorname{div}\big(\rho\overline{\mathbf{c}\otimes Q}\big) = \operatorname{div}\big(\mathbf{p}\otimes\mathbf{u}^{2,3} + \mathbf{p}\otimes\mathbf{u} + \rho\overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}}\big) = \\ = (\mathrm{grad}\mathbf{p})\mathbf{u} + \mathbf{p}(\mathrm{div}\mathbf{u}) + (\mathrm{div}\mathbf{p})\otimes\mathbf{u} + \mathbf{p}\,\mathrm{grad}^T\mathbf{u} + \operatorname{div}\big(\rho\overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}}\big)\end{gathered} \tag{24}$$

$$\rho\frac{\delta}{\delta t}\bar{Q} = \frac{\delta}{\delta t}\mathbf{p},\quad \rho\left(\overline{\mathbf{f}\cdot\frac{\delta Q}{\delta\mathbf{u}}}\right) = \rho(\mathbf{u}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{u}) \tag{25}$$

Natanson obtains the nonlinear evolution equation for momentum flux:

$$\begin{gathered}\frac{d}{dt}\mathbf{p} + (\mathrm{grad}\mathbf{u})\mathbf{p} + \mathbf{p}\big(\mathrm{grad}^T\mathbf{u}\big) + (\mathrm{div}\mathbf{u})\mathbf{p} + \operatorname{div}\big(\rho\overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}}\big) + \\ +\mathbf{u}\otimes\big(\rho\delta_t\mathbf{u} + \rho\mathbf{u}\mathrm{grad}^{\mathrm{T}}\mathbf{u} + \mathrm{div}\mathbf{p}\big) + \big(\rho\delta_t\mathbf{u} + \rho\mathbf{u}\mathrm{grad}^{\mathrm{T}}\mathbf{u} + \mathrm{div}\mathbf{p}\big)\otimes\mathbf{u} = \\ = \frac{\delta}{\delta t}\mathbf{p} + \rho(\mathbf{u}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{u})\end{gathered} \tag{26}$$

By next omitting parts $\mathbf{u}\otimes(\cdot)$, $(\cdot)\otimes\mathbf{u}$ and $(\mathbf{u}\otimes\mathbf{f}+\mathbf{f}\otimes\mathbf{u})$:

$$\mathrm{grad}\mathbf{u} = \mathbf{u}\otimes\nabla = \frac{1}{2}(\nabla\otimes\mathbf{u} + \mathbf{u}\otimes\nabla) + \frac{1}{2}(\mathbf{u}\otimes\nabla - \nabla\otimes\mathbf{u}) = \mathbf{d} + \mathbf{w} \tag{27}$$

Natanson arrived at [26,eq.12]:

$$\frac{\partial}{\partial t}\mathbf{p} + (\mathrm{grad}\ \mathbf{p})\mathbf{u} + (\mathbf{d}+\mathbf{w})\mathbf{p} + \mathbf{p}\left(\mathbf{d}+\mathbf{w}^{\mathrm{T}}\right) + (\mathrm{div}\ \mathbf{u})\mathbf{p} + \mathrm{div}\left(\rho\overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}}\right) = \frac{\delta}{\delta t}\mathbf{p} \tag{28}$$

2.7. Evolution of mass flux vector

According to Maxwell, by considering an evolution equation for a diffusion flux $\mathbf{j} = \rho\mathbf{u}$, Natanson was able to formulate the following equation [27,**1**,§6,eq.4]:

$$\frac{\partial}{\partial t}\mathbf{j} + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u} + \rho\overline{\mathbf{c}\otimes\mathbf{c}}\right) = \frac{\delta}{\delta t}\mathbf{j} + \rho\mathbf{f}_A \tag{29}$$

where $\mathbf{f}_A$ is a diffusion force.

3 Hypothesis of Coertion and logical structure of extended thermodynamics (1901)

In order to identify a source of irreversibility at Nature, Natanson introduced the concept of Coertia, which is similar to inertia. Natanson's Coertia is a fundamental property of space that is responsible for every irreversible phenomena in matter, as well as in the electromagnetic and gravitational fields. Owing to this concept, the irreversible changes proposed in the Maxwell procedure can be described with appropriate relaxation times as [28]:

$$\frac{\delta}{\delta t}\mathbf{q} = -\frac{\mathbf{q}}{\tau_q}\ ;\ \frac{\delta}{\delta t}\mathbf{p} = -\frac{\mathbf{p}}{\tau_P};\quad \frac{\delta}{\delta t}\mathbf{j} = -\frac{\mathbf{j}}{\tau_j} \tag{30}$$

where τ_q, τ_p, τ_j are relaxation times for heat, momentum and mass fluxes.

By looking at the Maxwell procedure of finding moments of the fundamental equation, Natanson quickly realized the necessity for cutting of the moment, hereby setting appropriate closure equations. He proposed the following logical structure: taking Q as a balanced quantity and the $\mathbf{f}_Q$ flux of Q and $\mathbf{F}_Q$ as a super-flux of $\mathbf{f}_Q$, the set of equations were determined [23]:

- Balance equation:

$$\frac{\partial}{\partial t}\bar{Q} + \mathrm{div}\mathbf{f}_Q = 0 \tag{31}$$

- Evolution equation $\mathbf{f}_Q$:

$$\frac{\partial}{\partial t}\mathbf{f}_Q + \frac{1}{\tau_f}\mathbf{f}_Q + \mathrm{div}\mathbf{F}_Q = 0 \tag{32}$$

- Evolution equation $\mathbf{F}_Q$:

$$\mathbf{F}_Q = a^2\mathrm{grad}\bar{Q} \tag{33}$$

- Resulting equation for Q:

$$\frac{\partial^2}{\partial t^2}\bar{Q} + \frac{1}{\tau_f}\frac{\partial}{\partial t}\bar{Q} - a^2\mathrm{div}(\mathrm{grad}\bar{Q}) = 0 \tag{34}$$

In Natanson's opinion, the above governing equation describes a whole real phenomena of nature, where reversibility is entangled with irreversibility by the relaxation time only. Thus, if $\tau_f = \infty$ (inertia), it is a case of reversible, while if $\tau_f = 0$ (coertia) it is an irreversible phenomenon.

At around 1903 year, yet another professor of Jagellonian University, Stanisław Zaremba, starting from a general discussion on the possibility of Galilelian relativity extension to continua, other than Maxwellian electromagnetic aether. From this position, Zaremba proposed a further group of transformations beyond the Lorentzian. Zaremba started from a clear argument [29-31] that: “Natanson’s evolution equation does not fulfill the principle of Galillelian relativity”. Zaremba, was especially interested in Natanson’s fully nonlinear evolution equation for the linear momentum flux (eq. 28), rewritten in the new form:

$$\frac{d_{M-N}}{dt}\mathbf{p} + \mathrm{div}\left(\rho\overline{\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}}\right) = \frac{\delta}{\delta t}\mathbf{p} \tag{35}$$

where the Maxwell-Natanson (un-objective) time derivative is:

$$\frac{d_{M-N}}{dt}\mathbf{p} = \frac{\delta}{\delta t}\mathbf{p} + (\mathrm{grad}\mathbf{p})\mathbf{u} + (\mathbf{d}+\mathbf{w})\mathbf{p} + \mathbf{p}\left(\mathbf{d}+\mathbf{w}^{\mathrm{T}}\right) + (\mathrm{div}\mathbf{u})\mathbf{p} \tag{36}$$

Here our revalorization, called “The Maxwell-Natanson Derivative”, is in opposition to Truesdell’s reconstruction, which he called “The Maxwell-Zaremba Derivative” [32]. Such an expression erroneously suggests that the original Maxwell fundamental equation is objective; it only becomes an objective after Zaremba’s correction.

4 Conclusions

Among thermodynamics researchers, there is a deeply rooted belief in the impossibility of fittingly take the laws of thermodynamics to a precise mathematical framework. This leads to the impression that the mathematical foundations of our sciences have “shallow roots” [33-34]. Looking at the theoretical base from a mathematical perspective, and especially at the first law of thermodynamics, it is impossible to identify a unique framework among the collection of numerous incomplete, restrictions on real energy conversion and questionable mathematical equations [35-37]. There has been no experimental violation of the first law of thermodynamics for more than 200 years, yet there remains no intellectual ambition to develop a single mathematically consistent statement of the first and second law of thermodynamics. Therefore, the laws continue to be understood pragmatically and taken by us “on faith”.

Most investigators are not interested in the mathematical foundation of nature, simply because the foundations must first be extended and this is as yet beyond our knowledge limits. Referring to Josef Kestin, from a purely mathematical standpoint, we have a Babel-like understanding of the foundations. This problem Maxwell undertakes in his Mater and Motion, saying that: the foundation of the first law of thermodynamics requires a knowledge of the whole of physics, chemistry, biology and even sociology. While at the end of 19th century this was possible, given the current volume of the literature, it is today impossible to achieve such knowledge. The best examples of those that have come close to achieving such a pinnacle are the subject of this article Ladislavus Natanson. He invested much time and effort into building a mathematical framework of generalized thermodynamics. Here our aim was to reconstruct, in a way that is understandable for a contemporary reader, a sole example of his original mathematical approaches to thermodynamics.

Finally, we stress that he had a love of Aristotle, and he was overcome by Cartesian’s approach to the description of gravitation and light and both developed the Maxwellian electrodynamics into a thermodynamic frame (Natanson – the quantum thermodynamics). Ladislavus Natanson started as chemists in a private laboratory of Jerzy Bogucki (Marie Curie’s uncle) in Warsaw; as the excellent starting points. In Table 1 we present a scheme outlining the steps in his evolution of knowledge, from chemical practice to the thermodynamics, history of physics and, ultimately, to Aristotelian metaphysic and wisdom.

Table 1: The frame of knowledge and scientific activity of Natanson.

The King's Road to Happiness
Thermo-chemistry
↓
Continuum Thermodynamics
↓
Electromagnetic Thermodynamics
↓
Gravitation Thermodynamics
↓
General Thermodynamics
↓
History of Physics
↓
Metaphysics
↓
Wisdom

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References

[1] V.A.Cimmelli, D. Jou, T. Ruggeri, P. Van: Entropy production and recentresults in non-equilibrium theories, Entropy, 16, 1756-1807, 2014.

[2] J. Badur: Rozwój pojęcia energii [in Polish], Development of Energy Concept, IMP PAN Publishers, Gdansk, 2009.

[3] K. Gumiński: On the Natanson Principle of Irreversible Processes, Acta Physica Polonica, vol. A 58, 501-507, 1980.

[4] B. Średniawa: History of Theoretical Physics at Jagiellonian University in Cracow in XIXth Century and in the First Half of XXth Century, Zeszyty Naukowe Uniwersytetu Jagiellońskiego DCCXXVII, Prace fizyczne z. 24, 1985.

[5] K. Czapla: Władysław Natanson — fizyk i filozof [in Polish], Ladislavus Natanson - physicist and philosopher, Semina Scientiarum, 4, 63-82, 2005.

[6] S. Bordoni: Routes towards an abstract thermodynamics in the late nineteenth century, Eur. Phys. J. H, 38, 617–660, 2013.

[7] P. Duhem: Traite d'energetique ou thermodynamique generale. Tome 1. Conservation de l'energie. Mecaniquerationelle. Statique generale. Deplacement de l'equilibre- Tome II. Dynamiquegenerale. Conductibilite de la chaleur. Stabilite de l'equilibre [in French], Energy treatment or general thermodynamics. Volume 1. Conservation of energy. Rational mechanics. General static. Displacement of the equilibrium - Volume II. General dynamics. Conductibility of heat. Stability of equilibrium, Gauthier Villars, Paris,1911.

[8] P. Duhem: L'Evolution de la mecanique [in French], The Evolution of Mechanics, Rev. Gen. d. Sciences, 14, 1-320, 1903. translation: G. Oraves : The Evolution of Mechanics, Alphen aan den Rijn, Sijthoff & Noordhoff, 1980.

[9] L. Natanson: Wstęp do Fizyki Teoretycznej [in Polish], Introduction to Theoretical Physics, Wyd. Prac Matematyczno-Fizycznych, Warszawa, 1890.

[10] I. Muller, W. Weiss: Thermodynamics of irreversible processes – past and present, Eur. Phys. J. H, 37, 139–236, 2012.
[11] G. Lebon, D. Jou, J. Casas-Vazques: Understanding non-equilibrium thermodynamics, Springer, Berlin, 2008.
[12] G. Maugin, W. Muschik: Thermodynamics with internal variables. Part I. General concepts, Journal of Non-Equilibrium Thermodynamics 19, 217-249, 1994.
[13] G. Maugin, W. Muschik: Thermodynamics with internal variables. Part II. Applications, Journal of Non-Equilibrium Thermodynamics, 19, 250-289, 1994.
[14] P. Ván, A. Berezovski, J. Engelbrecht: Internal variables and dynamic degrees of freedom, Journal of Non-Equilibrium Thermodynamics, 33, 235-254, 2008.
[15] J. Maxwell: On the dynamical theory of gases, Phil. Trans. Royal Soc. London., 57, 49-88, 1866.
[16] L. Natanson: Űber die kinetische Theorie unvollkommener Gase [in German], About the kinetic theory of imperfect gases, Master thesis, Dorpat, 1887.
[17] L. Natanson: Űber die kinetische Theorie der Joule'schen Erscheinung [in German], On the kinetic theory of Joule's appearance, PhD thesis, Dorpat,1888.
[18] L. Natanson: Űber die Wärmeerscheinungen bei der Ausdehung der Gase [in German], About the heat phenomena in the expansion of the gases, Wied. Ann., 37, 341-352, 1889, Phil. Mag., 29, 18-30, 1890.
[19] L. Natanson: Sur les temperatures, des pressions et volumes caracteristiques [in French], On the temperatures, pressures and volumes characteristics, C.R. de l'Ac. D. Sc., 169, 890-893, 1889.
[20] L. Natanson: On the probability of molecular configuration, Phil. Mag., 34, 51-54, 1892.
[21] L. Natanson: Sur les potentiels thermodynamiques [in French], On thermodynamic potentials, Bull Int. de l'Acad de Cracove, 156-161, 1891.
[22] L. Natanson: Sur l'interprétation cinétique de la fonction de dissipation [in French], On the kinetic interpretation of the dissipation function, Bull Int. de l'Acad de Cracove, 338-357, 1893, C.R. de l'Ac. D. Sc., 117, 539-542, 1893.
[23] L. Natanson: On the laws of viscosity, Philosophical Magazine, 2, 342–356, 1901.
[24] L. Natanson: Sur l'énergie cinétique du mouvement de la chaleur et la fonction de dissipation correspondante [in French], On the kinetic energy of the movement of heat and the corresponding dissipation function, Rozpr. Wydz. Mat.-Przyr. PAU, Kraków, 29, 273-278, 1895; Zf. F. Phys. Chem., 16, 289-302, 1895; Phil. Mag., 39, 501-509, 1895; Bull Int. de l'Acad de Cracove, 295-300, 1894.
[25] L. Natanson: Sur les lois des phénomènes irreversibles [in French], On the laws of irreversible phenomena, Rozpr. Wydz. Mat.-Przyr. PAU Kraków, 30, 309-336, 1896; Bull Int. de l'Acad de Cracove, 117-145, 1896; Zf. f. Phys. Chem., 21, 193-217, 1896; Phil. Mag., 41, 385-406, 1986.
[26] L. Natanson: Sur la fonction dissipative d'un fluide visqueux [in French], On the dissipative function of a viscous fluid. Journ. de Phys. théor. et apppl, 2, 702-705, 1903.
[27] L. Natanson: Sur le lois de la diffusion [in French], On the law of diffusion, Rozpr. Wydz. Mat.-Przyr. PAU Kraków, 41, 447-461, 1901, Bull Int. de l'Acad de Cracove, 335-348, 1901.
[28] L. Natanson: Inertia and coercion, The journal of physical chemistry, 7, 118-135, 1903.
[29] S. Zaremba: Remarques sur les travaux de le Natanson reltifs à la thèorie de la viscositè [in French], Remarks on the work of Natanson related to the viscosity theory, Rozpr. Wydz. Mat.-Przyr. PAU Kraków, 43, 14-21, 1903; Bull Int. de l'Acad de Cracove, 85-93, 1903.
[30] S. Zaremba: Sur une généralisation de la teorieclasique de la viscosité [in French], On a generalization of the viscosity theory of viscosity, Rozpr. Wydz. Mat.-Przyr. PAU Kraków, 43, 223-246, 1903; Bull Int. de l'Acad de Cracove, 380-403, 1903.
[31] S. Zaremba: Le principle des mouvements relatifs et les équations de mécanique physique [in French], The principle of relative movements and the equations of physical mechanics. Rozpr. Wydz. Mat.-Przyr. PAU Kraków, 43, 503-510, 1903; Bull Int. de l'Acad de Cracove, 614-621, 1903.
[32] C. Truesdell, R. Toupin: The classical field theories, Principles of Classical Mechanics and Field Theory, Series: Encyclopedia of Physics (Prinzipien der Klassischen Mechanik und Feldtheorie, Handbuch der Physik) S. Flügge [Ed.] vol. 2 / 3 / 1, 226-858, Springer, Berlin, 1960.
[33] J. Badur, P. Ziółkowski, D. Sławiński: Duhem i Natanson – dwie mechaniki [in Polish], Duhem and Natanson – two mechanics, Biuletyn Polskiego Towarzystwa Mechaniki Teoretycznej i

Stosowanej, 127-162, 2015.
[34] J. Badur, M. Feidt, P. Ziółkowski: Without heat and work - futher remarks on the Gyftopoulos-Beretta exposition of thermodynamics, International Journal of Thermodynamics, 21(3), 180–184, 2018.
[35] W. Pietraszkiewicz: Refined resultant thermomechanics of shells, International Journal of Engineering Science, 49, 1112–1124, 2011.
[36] V. Eremeyev, W. Pietraszkiewicz: Phase transitions in thermoelastic and thermoviscoelastic shells, Archives of Mechanics, 61 (1), 41-67, 2009.
[37] P. Ziółkowski, J. Badur: On Navier slip and Reynolds transpiration numbers, Archives of Mechanics, 70, 269-300, 2018.