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On convergence and stability of a numerical scheme of Coupled Nonlinear Schrödinger Equations

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Abstract

We consider the numerical solution of Coupled Nonlinear Schrödinger Equations. We prove the stability and convergence in the L_2 space for an explicit scheme the estimations of which are used for the implicit scheme and compare both methods. As a test we compare the numerical solutions of the Manakov system with known analytical solitonic solutions and as an example of the general system — evolution of two impulses with different group velocity (model of interaction of pulses in optic fibers). As a last example, a rectangular pulse evolution, shows asymptotic behavior typical for Nonlinear Schrödinger Equation asymptotics with the same initial conditions.

Keywords: CNLSE; Soliton collision; Crank–Nicolson; Finite difference

1. Introduction

A growing interest in encoded electromagnetic pulse propagation for over long distances and processing is noted [1,2]. A lot of realistic models are based on the Coupled Nonlinear Schrödinger Equations (CNLSE): they are developed and used in many nonlinear optics problems such as polarization modes interaction [2–4]. For such modes some numerical schemes and codes are elaborated beginning with the celebrated integrable Ablowitz-Ladik one [5,6].

In papers [7–12], the authors investigate numerical methods for solving CNLS equations based on finite difference schemes. Such difference schemes are applied in [13] to model vector spatial soliton behavior in nonlinear waveguide arrays.

There are numerical schemes which are unconditionally unstable, conditionally stable, and unconditionally stable [14]. Generally it could be said that we have two useful schemes: conditionally stable (Euler type) schemes and unconditionally stable (Crank–Nicolson type) schemes; we focus on these types of schemes. The most important thing is to use conservation laws for CNLS equations while the scheme is being constructed. There are a lot of possibilities in defining conservation laws [8], but in this paper we choose the standard one given by [10,9]. This choice fits the

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correspondent matrix norm definition in the finite-difference approximation and allows next to provide estimations via related spectral norms when stability and convergence analyses are performed.

Authors of [9,11,7] compare the explicit method with the implicit [14] one underlying it, notig that both of these methods have good and bad sides and there is no uniform point of view. Conditionally stable methods do not need matrix formulation, but they need smaller space and time steps to assure stability. Unconditionally stable methods need to solve the system of equations, but a bigger time step could be used. We would like to compare numerical solutions for the implicit and the explicit method under the scope of stability proof for the explicit method. In paper [9] authors prove convergence and stability for NLS equations but do not analyze CNLS equations and how stability and convergence depends on initial amplitude (energy) of impulses.

We consider a finite-difference scheme for the CNLSE which extends in a sense the results concerning linear Schrödinger equations [15], following the ideas successfully applied for the coupled Korteveg -de Vries (KdV) systems in [16,17].

The most important aim which we would like to achieve is to estimate stability and convergence parameters as a function of initial amplitude of pulses (energy) for the CNLS equations on a base of the norm originating from L_2 space. Such a result allows us to estimate time and space steps for the convergence regime inside which we could consider a solution as stable.

The system of the CNLSE is considered in the form:

$$\mathrm{i}\partial_t U + \mathrm{i}\sigma\partial_x U + k\partial_{xx} U + \left[a|U|^2 + b|V|^2\right] U = 0, \tag{1a}$$

$$\mathrm{i}\partial_t V - \mathrm{i}\sigma\partial_x V + k\partial_{xx} V + \left[c|V|^2 + d|U|^2\right] V = 0, \tag{1b}$$

where k parameter describes the dispersion in the optic fiber and functions U and V are differentiable upto the second order and the following notations are used

$$\begin{aligned}
&U = U(x,t),\\
&V = V(x,t),\\
&\partial_t V \equiv U_t \equiv \frac{\partial U(x,t)}{\partial t},\\
&\partial_{xx} V \equiv U_{xx} \equiv \frac{\partial^2 U(x,t)}{\partial x^2},
\end{aligned}$$

and (for the example of two modes excited in the waveguide [18,19]) the parameter σ is defined as

$$\sigma = \frac{1}{2}\left(k'_{01} - k'_{11}\right),$$

where $k'_{lp} = 1/V^g_{lp}$ and V^g_{lp} is the group velocity for the suitable mode (lp).

This parameter describes the difference between group velocities of the modes. For simplicity of notation, but without loss of generality, we take $a, b, c, d, k, \sigma \geq 0$ (in the other case we should put everywhere an estimation of the norm absolute value of these coefficients which could make equations more illegible).

2. Conservation laws

Let us prove first that the solutions of the system (1) satisfy the following conservation laws

$$\int_{-\infty}^{\infty} |U|^2 \mathrm{d}x = \text{const},$$

$$\int_{-\infty}^{\infty} |V|^2 \mathrm{d}x = \text{const}.$$

If one writes the conjugate to the first of the equations (1) (equation for U amplitude)

$$-\mathrm{i}\partial_t \overline{U} + i\delta\partial_x \overline{U} + k\partial_{xx}\overline{U} + \left[a|U|^2 + b|V|^2\right]\overline{U} = 0, \tag{2}$$

next multiply the result (2) by $-U$ and the Eq. (1) by $\overline{U}$, it yields

$$\mathrm{i}\overline{U}\partial_t U - \mathrm{i}\delta\overline{U}\partial_x U + k\overline{U}\partial_{xx}U + \left[a|U|^2 + b|V|^2\right]\overline{U}U = 0, \tag{3a}$$

$$\mathrm{i}U\partial_t\overline{U} - i\delta U\partial_x\overline{U} - kU\partial_{xx}\overline{U} - \left[a|U|^2 + b|V|^2\right]U\overline{U} = 0, \tag{3b}$$

with obvious relations

$$U\overline{U} = \overline{U}U = |U|^2.$$

As a next step, adding the Eq. (3a) to (3b)

$$\mathrm{i}\partial_t U\overline{U} - \mathrm{i}\delta\partial_x U\overline{U} + k\partial_x\left[\overline{U}\partial_x U - U\partial_x\overline{U}\right] = 0,$$

and integrating the result, one arrives at:

$$\begin{aligned}&\mathrm{i}\partial_t\int_{-\infty}^{\infty}(U\overline{U})\mathrm{d}x - i\delta\int_{-\infty}^{\infty}\partial_x(U\overline{U})\mathrm{d}x + k\int_{-\infty}^{\infty}\partial_x\left[\overline{U}\partial_x U - U\partial_x\overline{U}\right]\mathrm{d}x\\ &= \mathrm{i}\partial_t\int_{-\infty}^{\infty}U\overline{U}\mathrm{d}x - i\delta|U|^2\Big|_{-\infty}^{+\infty} + k\left[\overline{U}\partial_x U - U\partial_x\overline{U}\right]\Big|_{-\infty}^{+\infty} = 0,\end{aligned}$$

Suppose the boundary conditions at both infinities

$$\lim_{x\to\pm\infty} U = 0,$$

are imposed, then

$$\mathrm{i}\partial_t\int_{-\infty}^{\infty}|U|^2\mathrm{d}x = 0,$$

we obtain the first conservation law in the form

$$\int_{-\infty}^{\infty}|U|^2\mathrm{d}x = \text{const.}$$

For V the conservation law is derived in the same way

$$\int_{-\infty}^{\infty}|V|^2\mathrm{d}x = \text{const.}$$

Such conservation laws give us a possibility to define the basic norm in the space of columns $K = \binom{U}{V} \in L_2$ space as

$$\|K\|^2 = \int_{-\infty}^{\infty}(|U|^2 + |V|^2)\mathrm{d}x.$$

This norm definition in the finite-difference scheme construction allows us to provide estimations using the theorems from [20] via transition to the spectral norm. We will denote other norms by the same notation unless it leads to confusion.

3. Discretization of the CNLSE system

We use here standard discretization for partial differential equations [14]. Dividing time and space and introducing time step τ and space step h, we obtain a discrete difference grid in which node we calculate functions: $U(ih, j\tau)$ and $V(ih, j\tau)$. For the discrete time step notation we use index $j = 1, \dots, P$ and for the space step $i = 1, \dots, N$.

3.1. Explicit scheme

Choose the explicit scheme in a simple form [14] with a second order discretization with respect to time and a third order one to space:

$$\imath\frac{U_i^{j+1} - U_i^j}{\tau} + \imath\sigma\frac{U_{i+1}^j - U_{i-1}^j}{2h} + k\frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} + \left(a|U_i^j|^2 + b|V_i^j|^2\right)U_i^j = 0. \tag{4}$$

Let us derive the conservation law for the discrete CNLSE system (where $\overline{U}$ is complex conjugate of U). Multiply Eq. (4) by $(\overline{U}_i^{j+1} + \overline{U}_i^j)$ and the complex conjugateof this equation by $(U_i^{j+1} + U_i^j)$. If we apply zero boundary conditions, we have

$$\sum_{i=1}^{N} |U_i^{j+1}|^2 = \sum_{i=1}^{N} |U_i^j|^2, \tag{5a}$$

$$\sum_{i=1}^{N} |V_i^{j+1}|^2 = \sum_{i=1}^{N} |V_i^j|^2, \tag{5b}$$

where $i = 1, \ldots, N$.

3.2. Implicit scheme

This six point scheme [10]

$$\imath \frac{U_i^{j+1} - U_i^j}{\tau} + \imath\sigma \frac{U_{i+1}^{j+1} + U_{i+1}^j - U_{i-1}^{j+1} - U_{i-1}^j}{4h} + k \frac{U_{i+1}^{j+1} + U_{i+1}^j - 2U_i^{j+1} - 2U_i^j + U_{i-1}^{j+1} + U_{i-1}^j}{2h^2}$$
$$+ \left(a|U_i^{j+1/2}|^2 + b|V_i^{j+1/2}|^2\right) \frac{U_{i+1}^j + U_i^j}{2} = 0,$$

is exactly Crank–Nicolson one [14]. This implicit scheme is based on the same finite difference as the explicit scheme; we expect that the implicit scheme converges to an exact solution more quickly than the explicit scheme.

Elements in a node $1/2$ are calculated by iterations.

4. Stability

We can separate the real and the imaginary part of U and V by putting

$$U = \xi + \imath\eta,$$
$$V = \alpha + \imath\beta.$$

Substituting this into (1) yields four equations with real amplitudes

$$\xi_t + \sigma\xi_t + k\eta_{xx} + \left[a(\xi^2 + \eta^2) + b(\alpha^2 + \beta^2)\right]\eta = 0, \tag{6a}$$

$$-\eta_t - \sigma\eta_t + k\xi_{xx} + \left[a(\xi^2 + \eta^2) + b(\alpha^2 + \beta^2)\right]\xi = 0, \tag{6b}$$

$$\alpha_t - \sigma\alpha_t + k\beta_{xx} + \left[c(\alpha^2 + \beta^2) + d(\xi^2 + \eta^2)\right]\beta = 0, \tag{6c}$$

$$-\beta_t + \sigma\beta_t + k\alpha_{xx} + \left[c(\alpha^2 + \beta^2) + d(\xi^2 + \eta^2)\right]\alpha = 0. \tag{6d}$$

If we apply the explicit scheme (4) to the system (6), we can build a time evolution matrix as

$$\tilde{\mathbf{T}}^{j+1} = \begin{pmatrix} \tilde{T}_{11}^{j+1} & \tilde{T}_{12}^{j+1} & 0 & 0 \\ \tilde{T}_{21}^{j+1} & \tilde{T}_{22}^{j+1} & 0 & 0 \\ 0 & 0 & \tilde{T}_{33}^{j+1} & \tilde{T}_{34}^{j+1} \\ 0 & 0 & \tilde{T}_{43}^{j+1} & \tilde{T}_{44}^{j+1} \end{pmatrix} = \begin{pmatrix} \tilde{T}_{11}^{j+1} & \tilde{T}_{12}^{j+1} \\ \tilde{T}_{21}^{j+1} & \tilde{T}_{22}^{j+1} \end{pmatrix} \bigoplus \begin{pmatrix} \tilde{T}_{33}^{j+1} & \tilde{T}_{34}^{j+1} \\ \tilde{T}_{43}^{j+1} & \tilde{T}_{44}^{j+1} \end{pmatrix},$$

where

$$\tilde{T}^j_{11} = \tilde{T}^j_{22} = \delta_{i,r} - \frac{\tau\sigma}{2h}\left(\delta_{i+1,r} - \delta_{i-1,r}\right), \tag{7a}$$

$$\tilde{T}^j_{12} = -\tilde{T}^j_{21} = -\frac{\tau k}{h^2}\left(\delta_{i+1,r} - 2\delta_{i,r} - \delta_{i-1,r}\right) - \tau\delta_{i,r}\left\{a[(\xi^j)^2 + (\eta^j)^2] + b[(\alpha^j)^2 + (\beta^j)^2]\right\}, \tag{7b}$$

$$\tilde{T}^j_{33} = \tilde{T}^j_{44} = \delta_{i,r} + \frac{\tau\sigma}{2h}\left(\delta_{i+1,r} - \delta_{i-1,r}\right), \tag{7c}$$

$$\tilde{T}^j_{34} = -\tilde{T}^j_{43} = -\frac{\tau k}{h^2}\left(\delta_{i+1,r} - 2\delta_{i,r} - \delta_{i-1,r}\right) - \tau\delta_{i,r}\left\{c[(\alpha^j)^2 + (\beta^j)^2] + d[(\xi^j)^2 + (\eta^j)^2]\right\}, \tag{7d}$$

that acts in the vector space R_w of the columns:

$$\mathbf{W}^j = \begin{pmatrix} \xi^j \\ \eta^j \\ \alpha^j \\ \beta^j \end{pmatrix},$$

$$\mathbf{W}^{j+1} = \tilde{\mathbf{T}}^{j+1}\mathbf{W}^j = \tilde{\mathbf{T}}^{j+1}\tilde{\mathbf{T}}^j\mathbf{W}^{j-1} = \cdots = \prod_{k=j+1}^{1}\tilde{\mathbf{T}}^k\mathbf{W}^0.$$

Now we will prove a stability with respect to small perturbations of initial conditions $\mathbf{dW}$ [16,17]. It is important to remark that matrix $\tilde{T}$ is nonlinear and depends on the initial conditions. In this case we should obtain a new matrix dT by evaluation of the differential from the matrix $\tilde{T}$ for $\mathrm{d}\xi$, $\mathrm{d}\eta$, $\mathrm{d}\alpha$ and $\mathrm{d}\beta$

$$\mathbf{dW}^{j+1} = \prod_k \mathbf{dT}^k\mathbf{dW}^0.$$

For the stability conditions, we require that the operator $\prod_k \mathbf{dT}^k$ must be bounded by a constant in a sense of spectral matrix norm [20]

$$\left\|\prod_k \mathbf{dT}^k\right\| \le C. \tag{8}$$

For stability, a sufficient condition could be written in the form

$$\|\mathbf{dT}^k\| < \exp(\rho(\tau,h)\tau),$$

where ρ is a constant, but in the case with $|\rho(\tau,h)| \le \text{const} < \infty$ this condition is also sufficient for stability under condition $\tau, h \to 0$ and a dependence between τ and h.

In the matrix $\mathbf{dT}^j$, all elements are matrices with an index of spatial grids. Now we should upper estimate the function $\rho(\tau,h)$ via upper estimation of all matrix elements using the matrix spectral norm [17]:

$$\|dT^j_{11}\|^2 \le 1 + \frac{\tau\sigma}{h}, \tag{9a}$$

$$\|dT^j_{12}\|^2 \le \frac{4\tau k}{h^2} + \tau a \max_i[3(\xi^j_i)^2 + (\eta^j_i)^2] + \tau b \max_i[(\alpha^j_i)^2 + (\beta^j_i)^2] \tag{9b}$$

$$+\ 2\tau(a|\xi^j_i\eta^j_i| + d|\beta^j_i\xi^j_i| + d|\alpha^j_i\xi^j_i|), \tag{9c}$$

$$\|dT^j_{21}\|^2 \le \frac{4\tau k}{h^2} + \tau a \max_i[(\xi^j_i)^2 + 3(\eta^j_i)^2] + \tau b \max_i[(\alpha^j_i)^2 + (\beta^j_i)^2] \tag{9d}$$

$$+\ 2\tau(a|\xi^j_i\eta^j_i| + d|\beta^j_i\eta^j_i| + d|\alpha^j_i\eta^j_i|), \tag{9e}$$

$$\|dT^j_{22}\|^2 \le 1 + \frac{\tau\sigma}{h}, \tag{9f}$$

$$\|dT^j_{33}\|^2 \le 1 + \frac{\tau\sigma}{h}, \tag{9g}$$

$$\|dT_{34}^j\|^2 \le \frac{4\tau k}{h^2} + \tau c \max_i[3(\alpha_i^j)^2 + (\beta_i^j)^2] + \tau d \max_i[(\xi_i^j)^2 + (\eta_i^j)^2] \tag{9h}$$

$$+\ 2\tau(b|\alpha_i^j \eta_i^j| + b|\alpha_i^j \xi_i^j| + c|\alpha_i^j \beta_i^j|), \tag{9i}$$

$$\|dT_{43}^j\|^2 \le \frac{4\tau k}{h^2} + \tau c \max_i[(\alpha_i^j)^2 + 3(\beta_i^j)^2] + \tau d \max_i[(\xi_i^j)^2 + (\eta_i^j)^2] \tag{9j}$$

$$+\ 2\tau(b|\beta_i^j \eta_i^j| + b|\beta_i^j \xi_i^j| + c|\alpha_i^j \beta_i^j|), \tag{9k}$$

$$\|dT_{44}^j\|^2 \le 1 + \frac{\tau\sigma}{h}. \tag{9l}$$

Let us divide the matrix $\mathbf{dT}^j = \mathbf{S}^j + \mathbf{A}^j$ for symmetric $\mathbf{S}^j$ and antisymmetric part $\mathbf{A}^j$. We use Schwartz and triangle inequalities to estimate mixed terms (like $\beta\eta$) and the commutator $[\mathbf{S}^{j+1}, \mathbf{A}^{j+1}]$. It is perceptible that the submatrices dT_{ii} (7) are divided to three matrices. The identity matrix $\mathbf{E}$, the symmetric one (but without the identity matrix part $\mathbf{S}^{j+1} - \mathbf{E}$) and the antisymmetric one $\mathbf{A^{j+1}}$ (with nonzero elements for $r = i - 1$ and $r = i + 1$) yields

$$\begin{aligned}
\|\mathbf{dT^{j+1}}\|^2 &= \|\mathbf{dT^{j+1}}^*\mathbf{dT^{j+1}}\| = \|(\mathbf{S^{j+1}} + \mathbf{A^{j+1}})(\mathbf{S^{j+1}} - \mathbf{A^{j+1}})\| \\
&\le 1 + 2\|\mathbf{S}^{j+1} - \mathbf{E}\| + \|\mathbf{A}^{j+1}\|^2 + \|[\mathbf{S}^{j+1}, \mathbf{A}^{j+1}]\| + \|\mathbf{S}^{j+1} - \mathbf{E}\|^2 \\
&\le 1 + 2\tau\left\{\max(a,d)\max_i[(\xi_i^j)^2 + (\eta_i^j)^2] + \max(b,c)\max_i[(\alpha_i^j)^2 + (\beta_i^j)^2]\right\} \\
&\quad + \left\{\frac{\tau\sigma}{h} + \frac{4\tau k}{h^2} + 3\tau\max(a,d)\max_i[(\xi_i^j)^2 + (\eta_i^j)^2] + 3\tau\max(b,c)\max_i[(\alpha_i^j)^2 + (\beta_i^j)^2]\right\}^2 \\
&\quad + 4\tau\left\{\max(a,d)\max_i[(\xi_i^j)^2 + (\eta_i^j)^2] + \max(b,c)\max_i[(\alpha_i^j)^2 + (\beta_i^j)^2]\right\} \\
&\quad \times\left\{\frac{\tau\sigma}{h} + \frac{4\tau k}{h^2} + 3\tau\max(a,d)\max_i[(\xi_i^j)^2 + (\eta_i^j)^2] + 3\tau\max(b,c)\max_i[(\alpha_i^j)^2 + (\beta_i^j)^2]\right\} \\
&\quad + 4\tau^2\left\{\max(a,d)\max_i[(\xi_i^j)^2 + (\eta_i^j)^2] + \max(b,c)\max_i[(\alpha_i^j)^2 + (\beta_i^j)^2]\right\}^2.
\end{aligned} \tag{10}$$

Taking into account the finite-difference conservation laws (5a), we write

$$I_u = \sum_{i=1}^{N}|U_i^j|^2 = \sum_{i=1}^{N}|U_i^0|^2, \tag{11a}$$

$$I_v = \sum_{i=1}^{N}|V_i^j|^2 = \sum_{i=1}^{N}|V_i^0|^2, \tag{11b}$$

now we can upper estimate norm (10) as

$$\|\mathbf{dT^{j+1}}\|^2 \le 1 + 2\tau M I_A + \frac{\tau^2\sigma^2}{h^2} + 8\frac{\tau^2\sigma k}{h^3} + 16\frac{\tau^2 k^2}{h^4} + 5\tau^2 M I_A\left[5 M I_A + 2\frac{\sigma}{h} + 8\frac{k}{h^2}\right] \le \exp(\rho\tau),$$

where

$$M = \max(a, b, c, d),$$
$$I_A = I_U + I_V.$$

Finally we can write ρ as

$$\rho = 2 M I_A + \frac{\tau\sigma^2}{h^2} + 8\frac{\tau\sigma k}{h^3} + 16\frac{\tau k^2}{h^4} + 5\tau M I_A\left[5 M I_A + 2\frac{\sigma}{h} + 8\frac{k}{h^2}\right]. \tag{12}$$

For stability, we require that $\tau \to 0$ much faster than $h \to 0$ in case of Eq. (12) we write

$$\tau < (\text{constant})h^4.$$

It is important to remark that the stability of this system of equation depends on the initial energy quantity. All these parameters: σ, k, M and I_A are constant during numerical calculations.

5. Convergence

In this section, we prove that the terms at l.h.s of equations in the finite difference form (4) converge to the Eq. (1), the solutions of which are differentiable with respect to time and (twice) to x. We put into Eq. (4) a solution in the form

$$\mathbf{W}_i^j = \mathbf{ES}_i^j + \mathbf{V}_i^j = \begin{pmatrix} e\xi_i^j + d\xi_i^j \\ e\eta_i^j + \mathrm{d}\eta_i^j \\ e\alpha_i^j + d\alpha_i^j \\ e\beta_i^j + d\beta_i^j \end{pmatrix}, \tag{13}$$

where **ES** is an exact solution and **V** is the difference between a numerical solution and the exact solution of CNLS equations.

Below we show only one of the matrix components

$$\begin{aligned} &\frac{\mathrm{d}\xi_i^{j+1} - \mathrm{d}\xi_i^j}{\tau} + \sigma\frac{\mathrm{d}\xi_{i+1}^j - \mathrm{d}\xi_{i-1}^j}{2h} + k\frac{\mathrm{d}\eta_{i+1}^j - 2\mathrm{d}\eta_i^j + \mathrm{d}\eta_{i-1}^j}{h^2} + \frac{e\xi_i^{j+1} - e\xi_i^j}{\tau} + \sigma\frac{e\xi_{i+1}^j - e\xi_{i-1}^j}{2h} \\ &\quad + k\frac{e\eta_{i+1}^j - 2e\eta_i^j + e\eta_{i-1}^j}{h^2} + \left\{a\left[(e\xi_i^j + \mathrm{d}\xi_i^j)^2 + (e\eta_i^j + \mathrm{d}\eta_i^j)^2\right] + b\left[(e\alpha_i^j + d\alpha_i^j)^2 + (e\beta_i^j + d\beta_i^j)^2\right]\right\} \\ &\quad \times (e\eta_i^j + \mathrm{d}\eta_i^j) = 0. \end{aligned}$$

Let us use the conservation laws (5a) for the Eq. (1) and the finite difference equation (4), that allows us to write

$$\begin{aligned} &I_{eu} = \sum_{i=1}^{N}[(e\xi_i^j)^2 + (e\eta_i^j)^2] = \sum_{i=1}^{N}[(e\xi_i^0)^2 + (e\eta_i^0)^2], \\ &I_{ev} = \sum_{i=1}^{N}[(e\alpha_i^j)^2 + (e\beta_i^j)^2] = \sum_{i=1}^{N}[(e\alpha_i^0)^2 + (e\beta_i^0)^2], \\ &\mathrm{d}\xi_i^{j+1} - \mathrm{d}\xi_i^j + \tau\sigma\frac{\mathrm{d}\xi_{i+1}^j - \mathrm{d}\xi_{i-1}^j}{2h} + \tau k\frac{\mathrm{d}\eta_{i+1}^j - 2\mathrm{d}\eta_i^j + \mathrm{d}\eta_{i-1}^j}{h^2} \\ &\qquad + \tau\left\{a\left[(e\xi_i^j)^2 + (e\eta_i^j)^2\right] + b\left[(e\alpha_i^j)^2 + (e\beta_i^j)^2\right]\right\}\mathrm{d}\eta_i^j \\ &\qquad - \tau\left\{a\left[(e\xi_i^j)^2 + (e\eta_i^j)^2\right) + b\left[(e\alpha_i^j)^2 + (e\beta_i^j)^2\right]\right\}\mathrm{d}\eta_i^j \\ &\qquad - \tau\left\{a\left[(e\xi_i^j)^2 + (e\eta_i^j)^2\right] + b\left[(e\alpha_i^j)^2 + (e\beta_i^j)^2\right]\right\}e\eta_i^j \\ &\qquad + \tau\left\{a\left[(e\xi_i^j + \mathrm{d}\xi_i^j)^2 + (e\eta_i^j + \mathrm{d}\eta_i^j)^2\right] + b\left[(e\alpha_i^j + d\alpha_i^j)^2 + (e\beta_i^j + d\beta_i^j)^2\right]\right\}(e\eta_i^j + \mathrm{d}\eta_i^j) \\ &\quad = -\tau\left(\frac{e\xi_i^{j+1} - e\xi_i^j}{\tau} - \sigma\frac{e\xi_{i+1}^j - e\xi_{i-1}^j}{2h} - k\frac{e\eta_{i+1}^j - 2e\eta_i^j + e\eta_{i-1}^j}{h^2}\right) \\ &\qquad + \tau\left\{a\left[(e\xi_i^j)^2 + (e\eta_i^j)^2\right] + b\left[(e\alpha_i^j)^2 + (e\beta_i^j)^2\right]\right\}e\eta_i^j. \end{aligned}$$

The right side of the equation for the differentiable solutions is of the order $\Theta(\tau + h + h^2)$ [14], while to represent the left side we use the matrix **T**. Hence one arrives at

$$\begin{aligned}
\mathrm{d}\xi_i^{j+1} &= \mathbf{T}_{1i}^{j+1}\mathbf{V}_i^j + \tau\left\{a\left[(e\xi_i^j)^2+(e\eta_i^j)^2\right]+b\left[(e\alpha_i^j)^2+(e\beta_i^j)^2\right]\right\}(e\eta_i^j+\mathrm{d}\eta_i^j)\\
&\quad-\tau\left\{a\left[(e\xi_i^j+\mathrm{d}\xi_i^j)^2+(e\eta_i^j+\mathrm{d}\eta_i^j)^2\right]+b\left[(e\alpha_i^j+\mathrm{d}\alpha_i^j)^2+(e\beta_i^j+\mathrm{d}\beta_i^j)^2\right]\right\}\\
&\quad\times(e\eta_i^j+\mathrm{d}\eta_i^j)+\Theta(\tau+h+h^2).
\end{aligned}$$

Let us define the norm of the numerical solution as

$$\|U^j\| = h\left(\sum_i |U_i^j|^2\right)^{\frac{1}{2}}.$$

Now we upper estimate one element of the vector $\mathbf{V}^j$

$$\begin{aligned}
\|\mathrm{d}\xi^{j+1}\| &\le \|\mathbf{T}^{j+1}\|\|\mathbf{V}^j\| + \tau\,(aI_u+bI_v)\,\|\eta^j\| - \tau\,(aI_{ue}+bI_{ve})\,\|\eta^j\| + \Theta(\tau+h+h^2)\\
&\le \exp(j\tau\rho)\|\mathbf{V}^0\| + \tau\,(aI_u+bI_v)\,\|U^j\| - \tau\,(aI_{ue}+bI_{ve})\,\|U^j\| + \Theta(\tau+h+h^2),
\end{aligned}$$

where ρ is given by (12).

Next, we write the converge condition for all $\mathbf{V}^{j+1}$ matrix components up to the choice of the initial error $\|\mathbf{V}^0\| = 0$

$$\|\mathbf{V}^{j+1}\| \le Q + \Theta(\tau+h+h^2),$$

where we define

$$Q = 4|\max(a,b,c,d)(I_u+I_v)^{3/2} - \max(a,b,c,d)(I_{ue}+I_{ve})^{3/2}|.$$

6. Numerical results

6.1. Nonlinear Schrödinger equation

For a simple test, we put $c = d = 0$ and $k = 0.5$, where in this case we have the simple nonlinear Schrödinger equation. First we test the NLS equation with an initial condition $U(0,x) = \mathrm{sech}(x)$, the pulse of which should not change shape during propagation (Fig. 1(a)).

If we set the initial condition as:

$$\begin{aligned}
U(0,x) &= A_1\mathrm{sech}(x),\\
V(0,x) &= 0,
\end{aligned}$$

it leads to the single NLSE equation in the form

$$\mathrm{i}U_t + \frac{1}{2}U_{xx} + |U|^2U = 0, \tag{14}$$

which completes the problem to be considered.

When amplitude $A_1 = 1$ the exact solution of the Eq. (14) is

$$U(t,x) = A_1\mathrm{sech}(x)\exp(\mathrm{i}t/2).$$

Next we test the NLS equation with the amplitude $A_1 = 2$; the results are the same as in [2] (Fig. 1(b)). For this case the exact solution of the Eq. (14) takes a more complicated form [2]

$$U(t,x) = \frac{[\cosh(3x)+\exp(4\mathrm{i}t)\cosh(x)]\exp(\mathrm{i}t/2)}{\cosh(4x)+4\cosh(2x)+3\cos(4t)}.$$

The numerical solution for the second case deviates more than the first one because it has a bigger energy per pulse, and we use for both cases the same time and space steps (see Eq. (12)). (See Fig. 2.)

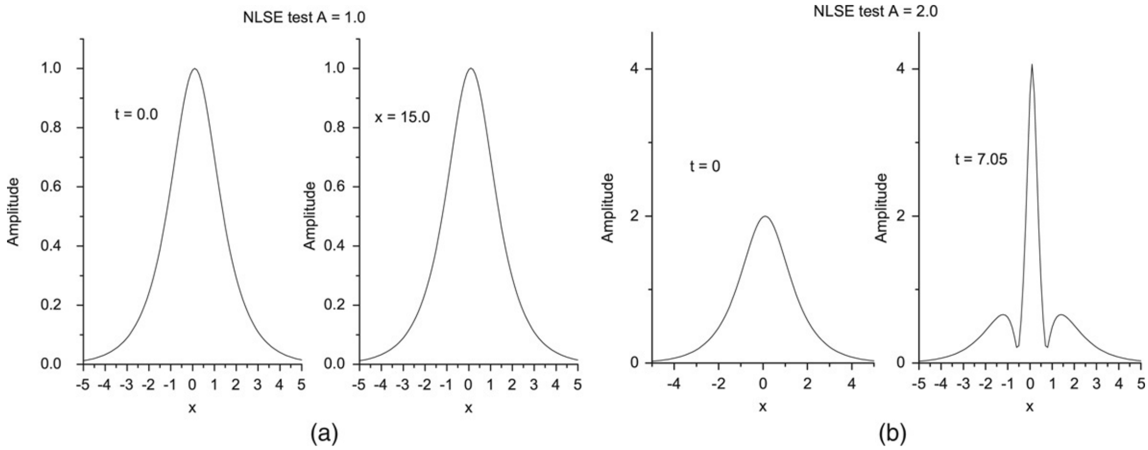


Fig. 1. NLS equation with initial amplitude (A_1) (a) 1 and (b) 2.

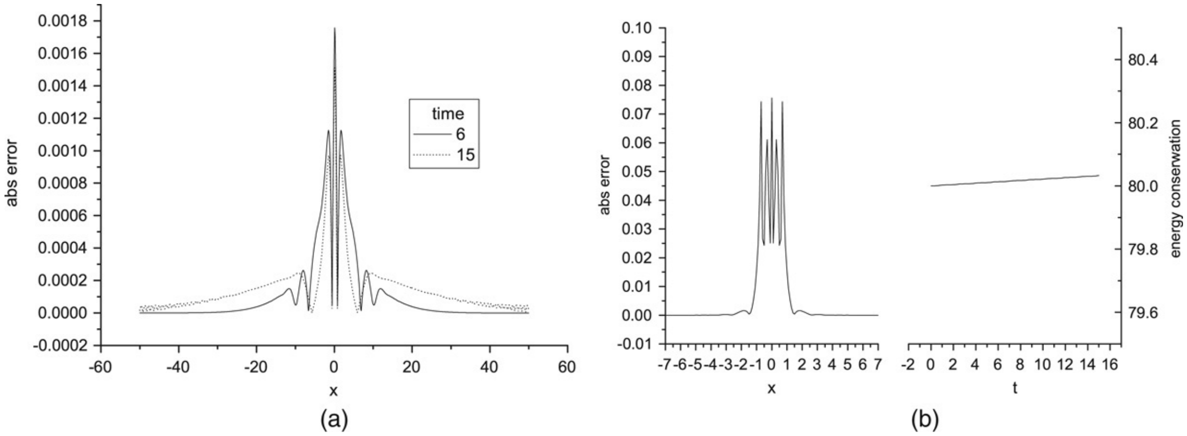


Fig. 2. Errors of numerical calculation for different time values (a) $A_1 = 1$ and (b) $A_1 = 2$.

6.2. Manakov solitons

Let us consider the soliton solutions of the Manakov system as examples to test the stability and the convergence of the explicit scheme (see Table 1). We start with the stability of the explicit scheme. At Fig. 3 six cases of energy conservation behavior are shown for the values of $\rho\tau$ which are bigger than 0.1: we observe very unstable results. When we decrease the time step, our solution is stabilized. It is very important to remark that ρ depends on the initial amplitude of pulses (see Fig. 4). We choose the initial conditions as

$$U(0, x) = A_1 \mathrm{sech}(x),$$
$$V(0, x) = A_2 \mathrm{sech}(x).$$

6.3. Collision of two solitons

We conduct this experiment to compare the results with the "experiment 1" of the paper [10]. The general properties of such collisions of two solitons are well known [21]; hence we do not focus on details of this type of soliton interaction. (See Fig. 5.)

6.4. Different group velocities

The examples to be studied in this section are most important for us, because we are interested in describing the modes' interaction in a nonlinear waveguide of different excited modes (different modes have different group

Table 1
Parameters for the numerical experiment "Manakov solitons"

x	$-50\ldots50$
Time	15
Sigma	0
a	1
b	1
c	1
d	1
A_1	1
A_2	1
Space steps	1000
Time steps	10 000–1000 000

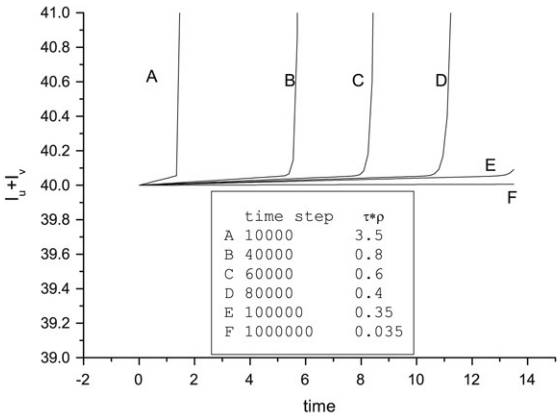


Fig. 3. Stability of explicit method for Manakov examples.

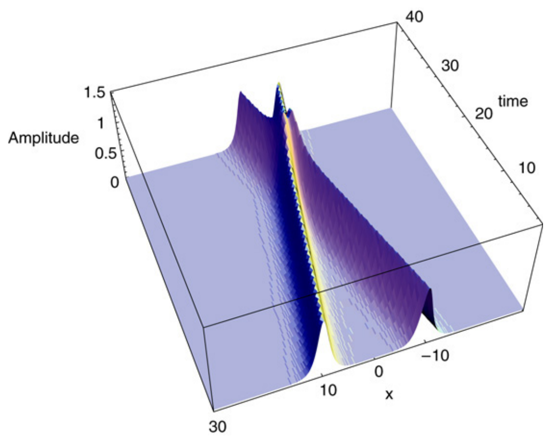
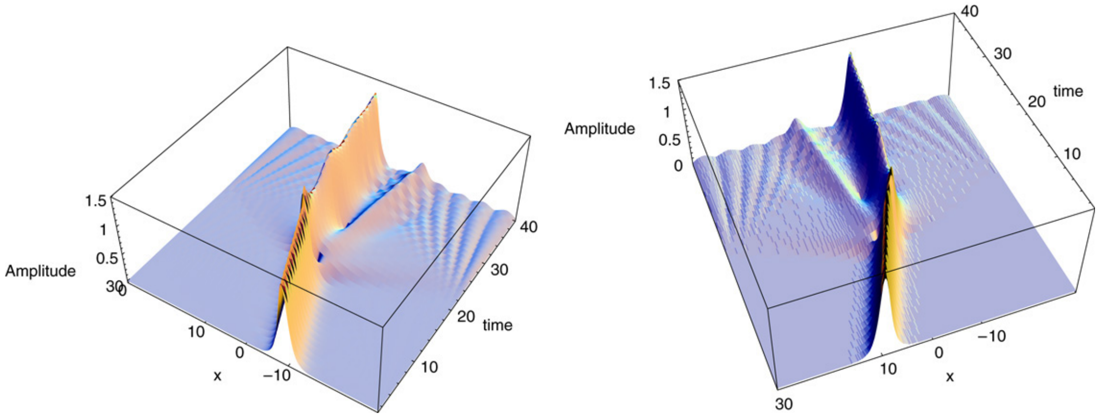


Fig. 4. Manakov case: $A_1 = A_2 = 1$, nonlinear coefficients $(a, b, c, d) = 1$ and $\sigma = 1$.

velocities) [18,19]. This investigation could be useful not only for the situation described below, but for soliton trains interactions as well [22].

As initial conditions, we took two sech-impulses and each equation has different nonlinear coefficients. This could appear in a waveguide when the two modes are excited with different group velocities. We set initial conditions for



(a) $|U|$.

(b) $|V|$.

Fig. 5. Inelastic collision of two solitons.

Table 2
Parameters for numerical experiment "Different group velocity"

	(a)	(b)
x	$-30\ldots30$	
Time	40	40
σ	0	0
a	1	1
b	1/3	1/3
c	1	1
d	1/3	1/3
A_1	1.2	1.2
A_2	1.4	1.4
v_1	0.7	0.95
v_2	0	0
h	0.2	0.2
τ	0.02	0.02

this case as:

$$U(0,x) = A_1 \operatorname{sech}(x + D_1)\exp(\mathrm{i}v_1 x),$$
$$V(0,x) = A_2 \operatorname{sech}(x + D_2)\exp(\mathrm{i}v_2 x).$$

Consider two solitons, one with zero velocity and second with velocity greater than zero; we show this situation in Fig. 6. These two cases of pulse interaction differ only by group velocity (parameters for this pulses are in the Table 2).

These two impulses start from the same position (e.g. two modes excited in the waveguide). When the velocity of the second pulse is 0.7 (for given parameters) the first pulses intercept part of the energy through nonlinear interactions and move with the second pulse with average velocity for both pulses (Fig. 6(a)). When the velocity of the second pulses is high enough we have two pulses which move with different group velocities; see Fig. 6(b) (nonlinear interaction between pulses happen only at the start of propagation).

6.5. *Difference between explicit and implicit scheme*

If we would like to have more space details (smaller h), in the explicit scheme, we must take an adequate smaller time step to assure stability of the numerical scheme. This has a very big influence on calculation time. In an implicit scheme, we could use an iteration method [14,7] and a shorter time of calculation is achieved. On the Fig. 7 we show

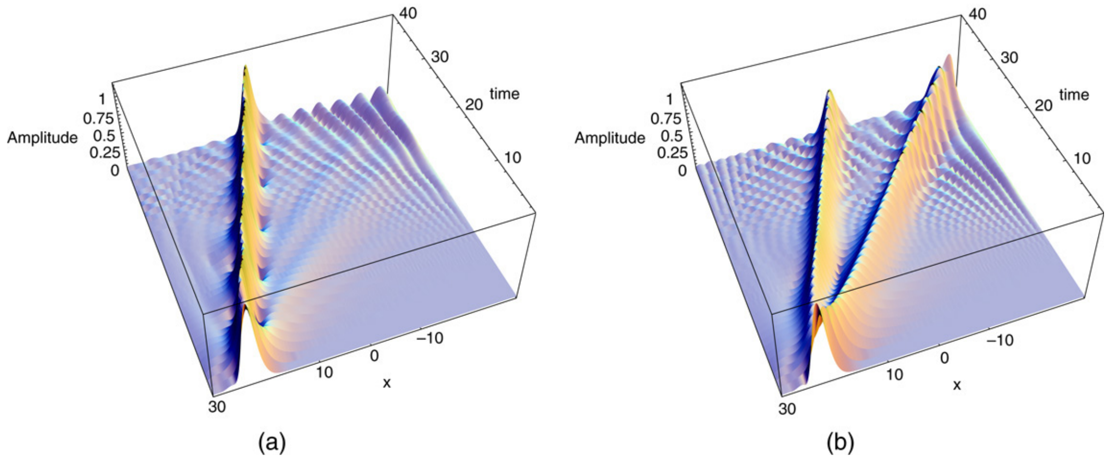


Fig. 6. Two cases of impulses with different group velocity (for parameters see Table 2).

Table 3
Parameters for numerical experiments for implicit and explicit methods

Parameter	Explicit	Implicit
Space step	300	300
Time step	1000 000	2000
Time	40	40
x	$-30\ldots30$	$-30\ldots30$
A_1	1.5	1.5
A_2	1.5	1.5
a	1	1
b	0.2	0.2
c	1	1
d	1.6	1.6
σ	0.3	0.3

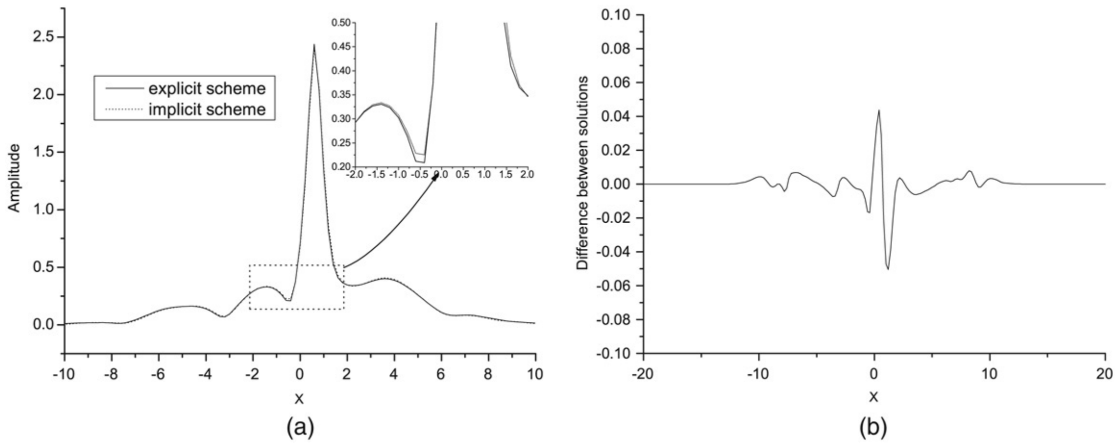


Fig. 7. Comparison between explicit and implicit methods.

numerical results for both methods (see Table 3). Note, the time step for the explicit scheme is much larger than for the implicit scheme, but each step in the implicit scheme needs 2–4 iterations.

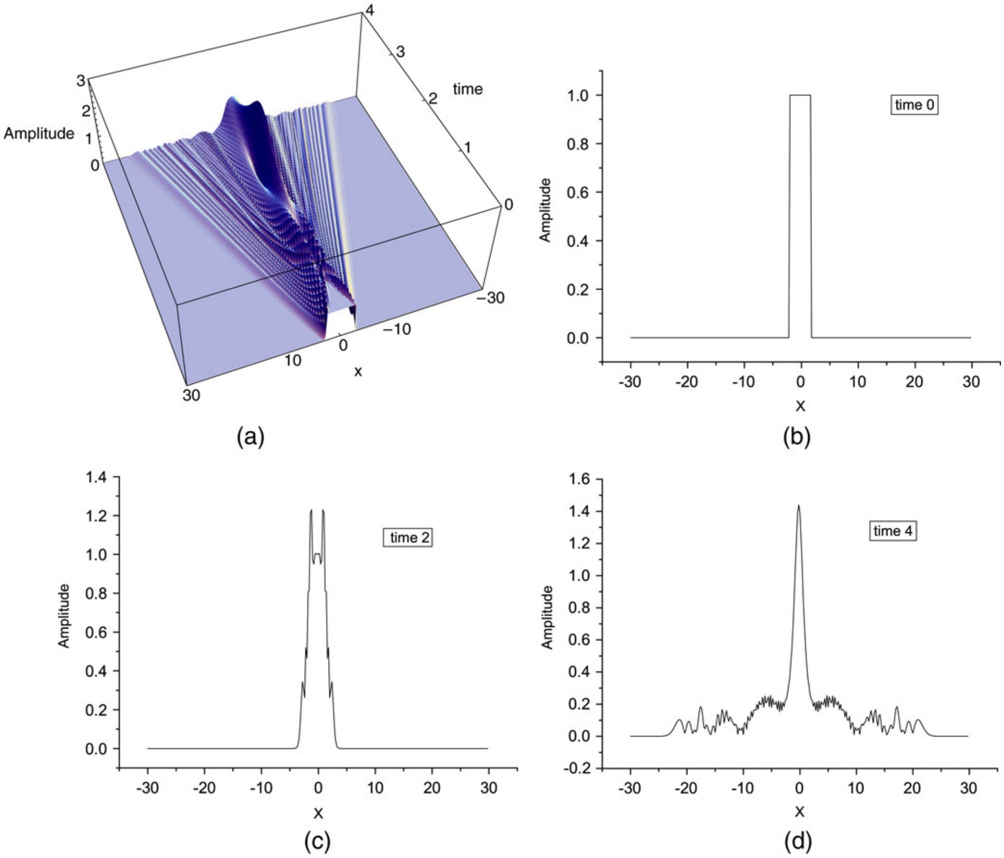


Fig. 8. Rectangular pulse evolution. On picture, (a) there is a 3D plot; on picture (b), (c) and (d) there are intersections of plot (a) in different times.

6.6. *Rectangular pulse decay*

In this subsection, we engage in the asymptotic solution of The Nonlinear Schrödinger Equation. As an initial condition, we choose rectangular pulse as in [23] (see Fig. 8(b)).

We test our method for this kind of initial pulse. First we make two calculations with different time steps (for the first $\tau = 2e^{-4}$ and $\tau = 2e^{-5}$ for the second one) and compare errors between (Fig. 9(a)). The difference between the calculations is of the order determined by $\tau\rho$. Next we check the energy conservation (see Fig. 9(b)).

For the presented time ($t = 4$), we expect that the transient stage from rectangular shape of the pulse to the asymptotic behaviour of the solution is realized. On the Fig. 10, the evolution (maximum amplitude) of rectangular pulses with different widths is presented. If the pulse has smaller width, then the decay is faster.

7. Conclusion

We study convergence and stability conditions which would be useful to estimate other numerical schemes for NLS as well. Additionally, these conditions could be compared with ones for numerical methods for other nonlinear equations [17], and show characteristics of CNLS equations (from the numerical side). The most important results which we showed in this paper concerned how to estimate a time step for the explicit method due to the initial energy of pulses. The results could be adapted to other methods based on this basic scheme which we use [8]. There is a possibility to enhance this method to third or higher orders in time, and this method could be used as a starting point (to calculate the lacked points of grid).

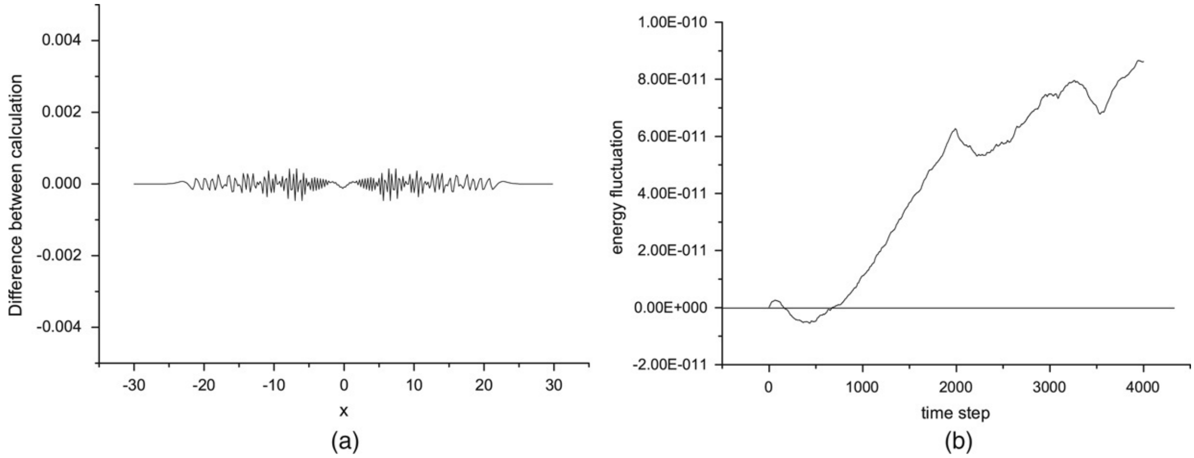


Fig. 9. (a) Difference between two numerical solutions with different time steps ($\tau = 2e^{-4}$ and $\tau = 2e^{-5}$. (b) The energy deviation (to the initial state) for the time step $\tau = 2e^{-5}$.

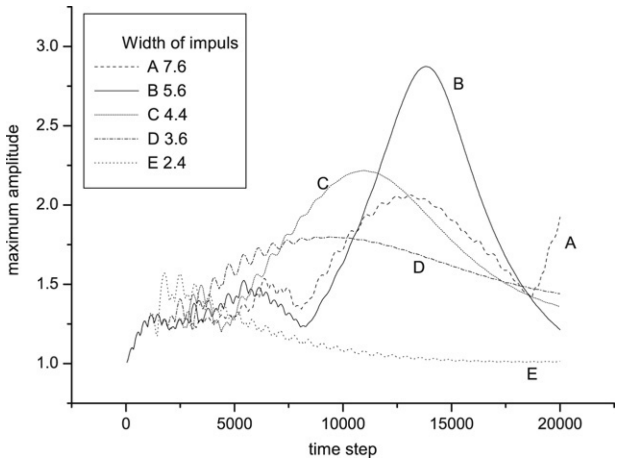


Fig. 10. Maximum amplitude for different pulse widths (in round brackets) for time step $\tau = 2e^{-4}$ (maximum time in dimensionless unit is equal 4).

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