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Anna Perelomova[a)]

AFFILIATIONS

Faculty of Applied Physics and Mathematics, Gdansk University of Technology, 11/12 Gabriela Narutowicza Street, Gdansk 80-233, Poland

[a)]Author to whom correspondence should be addressed: anna.perelomova@pg.edu.pl

ABSTRACT

Magnetosonic periodic perturbations in a uniform and infinite plasma model are considered. Damping due to compressional viscosity, electrical resistivity, and thermal conduction are taken into account, as well as some heating–cooling function, which may destroy the isentropicity of wave perturbations. The wave vector forms arbitrary angle θ with the equilibrium straight magnetic field, and all perturbations are functions of time and longitudinal coordinate. Variable θ and plasma-β bring essential difficulties in the description of magnetosonic perturbations, which may be fast or slow. Wave damping of each kind depends differently on θ and plasma-β. Longitudinal velocity, which is periodic at any distance from an exciter, is analytically constructed. It approximates the exact solution with satisfactory accuracy.

I. INTRODUCTION

Nonlinear acoustics usually deals with the Burgers equation, which describes the propagation of finite-magnitude perturbations in a planar flow with Newtonian attenuation and thermal conduction.[1–3] It may be rearranged into the diffusion equation and solved exactly. Taking into account the inhomogeneity of the background thermodynamic parameters, bulk flows, external forces, and dispersive properties of a flow produces more complex equations.[3,4] Only a small part of them has an analytical solution. Especial interest is paid to the flows in open systems with destroyed adiabaticity. Among others, this is connected with progress in experimental and theoretical studies in astrophysics and flows of unmagnetized gases with nonequilibrium chemical reactions and vibrational relaxation.[5–10] Heating and cooling effects destroy the adiabaticity of a flow and may lead to enhancement of sound, that is, to the acoustical activity of a medium. This is conditioned by some special type of generic heating–cooling function. Evolution of perturbations in initially uniform unbounded flows, when heating/cooling applies alone, without mechanical and thermal damping, is well understood. The leading-order equation may be rearranged into the purely nonlinear equation, which in turn may be exactly solved by the method of characteristics.[1,7,11,12] Perturbations in acoustically active media amplify until suppressed by nonlinear damping. This leads to discontinuities in the waveforms.[13–15] Involving damping complicates the dynamic equation for wave perturbations in a medium. The exact solutions to it are not found yet. The asymptotic behavior of perturbations far from a wave driver in a thermally conducting plasma has been analyzed by Chin *et al.*[16] It has been proven that perturbations in acoustically active flow develop into autowaves, that is, waveforms dependent on equilibrium plasma parameters but not on the initial magnitude of perturbations. The authors concluded that the autowaves exist only for the wavelength larger than the parameter depending on the ratio of thermal conduction and degree of non-adiabaticity introduced by some heating–cooling function. The evaluations concern nearly saw-tooth profiles of velocity. Magnetosonic periodic and monopolar waveforms with discontinuities in weakly attenuating plasma also develop into self-similar waveforms at any distance from an exciter. The periodic wave is stationary if nonlinear attenuation balances the inflow of energy. This particular stationary waveform and exact dynamics of initially saw-tooth and monopolar triangular velocity profiles were discussed in Ref. 17.

In this study, we derive the dynamic equation for longitudinal particle velocity in magnetosonic waves by taking into account mechanical damping, electrical resistivity, and thermal conduction and discover various contributions of every damping factor in dependence on the angle between the equilibrium magnetic field and the wave vector and plasma-β. The complete dynamical equation considers also weak nonlinearity and the heating–cooling function (Sec. II). The approximate solution to it is suggested and analyzed in Sec. III B. Section III A recalls periodic saw-tooth waves in the purely nonlinear flow of a magnetic gas.

II. EVOLUTIONARY EQUATION IN THE FINITE-MAGNITUDE FLOW WITH DAMPING DUE TO MECHANICAL AND THERMAL LOSSES AND ELECTRICAL RESISTIVITY

We make use of a set of MHD equations describing homogeneous fully ionized gas with finite electrical conductivity, thermal conduction, and mechanical viscosity. It includes the continuity equation, momentum equation, energy balance equation, and electrodynamic equations in the differential form[18,19]

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0,$$
$$\rho\frac{D\vec{v}}{Dt}=-\vec{\nabla}p+\mathrm{Div}\pi+\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B},$$
$$\frac{Dp}{Dt}-\gamma\frac{p}{\rho}\frac{D\rho}{Dt}=(\gamma-1)\left[L(p,\rho)+\vec{\nabla}\cdot(k\vec{\nabla}T)+\frac{1}{\sigma}\left(\frac{\vec{\nabla}\times\vec{B}}{\mu_0}\right)^2+\mathrm{Grad}\vec{v}:\pi\right],$$
$$\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\times(\vec{v}\times\vec{B})+\frac{\Delta\vec{B}}{\mu_0\sigma},$$
$$\vec{\nabla}\cdot\vec{B}=0, \tag{2.1}$$

where ρ and $\vec{v}$ are the density of a plasma and its velocity. The magnetic field is designated by $\vec{B}$, and μ_0 is the permeability of the free space. σ denotes the electrical conductivity of a plasma (reciprocal of electrical resistivity), and k is its thermal conduction. The third equation in the set (1) follows from the continuity and energy equations. It refers to an ideal gas with the ratio of specific heats (per unit mass) under constant pressure and constant density γ, $\gamma=C_P/C_V$. T is the temperature of a plasma, obeying an ideal gas state $T=\frac{p}{(C_P-C_V)\rho}$. The fourth equation is the induction equation of a gas with the electrical conductivity, and the fifth one is the Maxwell equation reflecting the solenoidal character of $\vec{B}$. The generic heating–cooling function $L(p,\rho)$ may destroy the non-isentropicity of a flow.[16,20] It incorporates the effects of heating and radiative cooling of a plasma. While heating may vary in dependence on the physical conditions, the radiative cooling occurs due to optically thin radiation. Nakariakov *et al.*[20] reviewed physically meaningful kinds of the heating function in the context of astrophysical applications, in particular in the high-temperature atomic plasma and cold molecular interstellar gas (coronal current dissipation, heating by Alfvén mode/mode conversion, constant heating per unit mass, heating by cosmic rays, and grain photoelectrons).

The momentum equation contains mechanical losses, which are described by Divπ, where the Navier–Stokes form of the viscous stress is

$$\pi=\eta_0(\mathrm{Grad}\vec{v}+(\mathrm{Grad}\vec{v})^T-\frac{2}{3}(\vec{\nabla}\cdot\vec{v})E),$$

where upper index T denotes transpose, E is the unit tensor, and η_0 designates the so-called compressional viscosity.[19,21] We retain only the first term in the Braginskii' expression for the viscous tensor. Typically, it strongly dominates all other terms. In the solar corona, it is at least five orders of magnitude larger and at least two orders in magnitude larger in the upper chromosphere.[22] The term $\frac{1}{\sigma}(\frac{\vec{\nabla}\times\vec{B}}{\mu_0})^2=\frac{\vec{j}^2}{\sigma}$ indicates heat losses by Joule heating ($\vec{j}$ is the current density). This rate is usually small but should be included in resistive MHD energy balance to be consistent with the resistivity in the Ohm law. The transport parameters depend on the temperature and the angle between the magnetic field and particle velocity. The classical transport theory concludes that heat conduction parallel to the magnetic field is much larger than the perpendicular one, that is, $k_{||}\gg k_{\perp}$, where $k=k_{\perp}\sin^2(\theta)+k_{||}\cos^2(\theta)$ ($k_{||}\sim T^{5/2}$). The electrical resistivity results from collisions between electrons and ions and depends on temperature as $T^{-3/2}$.[23] It is anisotropic with at most $\sigma_{\perp}^{-1}\approx 3.4\sigma_{||}^{-1}$, where the electrical conductivity along the magnetic field $\sigma_{||}$ is the Spitzer value. In resistive MHD, it is customary not to distinguish parallel and transverse electrical resistivities but make use of an isotropic σ, so as $\frac{\vec{j}}{\sigma}=\frac{\vec{j}_{||}}{\sigma_{||}}+\frac{\vec{j}_{\perp}}{\sigma_{\perp}}$.[19] In general, the transport parameters also depend on the microturbulent processes in a plasma. It is convenient not to specify transport parameters. We follow the ideas of Chin *et al.* and Kumar *et al.* in this issue.[16,24] The terms reflecting the variation of damping coefficients with coordinates are not considered. We assume that the wave vector of a planar flow is directed along the z axis and forms constant angle θ ($0\le\theta\le\pi$) with the straight equilibrium magnetic field $\vec{B}_0$. All perturbations in a flow are functions of t and z. The y-component of $\vec{B}_0$ equals zero, so as

$$B_{0,x}=B_0\sin(\theta),\quad B_{0,z}=B_0\cos(\theta),\quad B_{0,y}=0.$$

We accept the geometry used in the studies of Nakariakov *et al.* and Chin *et al.*[16,20]

In order to analyze the linear flow, all thermodynamic quantities are expanded around the equilibrium thermodynamic state as $f(z,t)=f_0+f'(z,t)$. A plasma is motionless in equilibrium: $\vec{v}_0=\vec{0}$. The leading-order system includes quadratic nonlinear terms

$$
\begin{aligned}
&\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_z}{\partial z} = \rho' \frac{\partial v_z}{\partial z} - v \frac{\partial \rho'}{\partial z},\\
&\frac{\partial v_x}{\partial t} - \frac{B_{0,z}}{\rho_0 \mu_0} \frac{\partial B_x}{\partial z} - \frac{\eta_0}{\rho_0} \frac{\partial^2 v_x}{\partial z^2} = -v_z \frac{\partial v_x}{\partial z} - \frac{B_{0,z}}{\rho_0^2 \mu_0} \rho' \frac{\partial B_x}{\partial z},\\
&\frac{\partial v_y}{\partial t} - \frac{B_{0,z}}{\rho_0 \mu_0} \frac{\partial B_y}{\partial z} - \frac{\eta_0}{\rho_0} \frac{\partial^2 v_y}{\partial z^2} = -v_z \frac{\partial v_y}{\partial z} - \frac{B_{0,z}}{\rho_0^2 \mu_0} \rho' \frac{\partial B_y}{\partial z},\\
&\frac{\partial v_z}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{B_{0,x}}{\rho_0 \mu_0} \frac{\partial B_x}{\partial z} - \frac{4\eta_0}{3\rho_0} \frac{\partial^2 v_z}{\partial z^2} = \frac{\rho'}{\rho_0^2} \frac{\partial p'}{\partial z} + \frac{B_{0,x}}{\rho_0^2 \mu_0} \rho' \frac{\partial B_x}{\partial z} - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\frac{B_x^2 + B_y^2}{2\mu_0} \right) - v_z \frac{\partial v_z}{\partial z},\\
&\frac{\partial p'}{\partial t} + c^2 \rho_0 \frac{\partial v_z}{\partial z} - (\gamma - 1)(L_p p' + L_\rho \rho') - \frac{k}{C_V \rho_0} \frac{\partial^2 p'}{\partial z^2} + \frac{k p_0}{\rho_0^2 C_V} \frac{\partial^2 \rho'}{\partial z^2} = -\gamma p' \frac{\partial v_z}{\partial z} - v_z \frac{\partial p'}{\partial z},\\
&\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial z}(B_{0,x} v_z - B_{0,z} v_x) - \frac{1}{\mu_0 \sigma} \frac{\partial^2 B_x}{\partial z^2} = -B_x \frac{\partial v_z}{\partial z} - v_z \frac{\partial B_x}{\partial z},\\
&\frac{\partial B_y}{\partial t} - \frac{\partial}{\partial z}(B_{0,z} v_y) - \frac{1}{\mu_0 \sigma} \frac{\partial^2 B_y}{\partial z^2} = -B_y \frac{\partial v_z}{\partial z} - v_z \frac{\partial B_y}{\partial z},
\end{aligned}
\tag{2.2}
$$

where $L_p = \frac{\partial L}{\partial p}$, $L_\rho = \frac{\partial L}{\partial \rho}$ are evaluated at the equilibrium state (p_0,ρ_0). It represents in fact Taylor series expansions of Eq. (2.1) in powers of the magnetosonic Mach number M. The Mach number is a dimensionless quantity, which is determined as the ratio of typical magnetosonic velocity magnitude to the speed of magnetosonic perturbations. It is usually small. This ensures weak nonlinear distortions of a wave. The terms of order M are collected on the left of Eq. (2.2), and the terms of order M^2 are collected on the right. Some linear terms are proportional to L_p, L_ρ, k, η_0, and σ^{-1}, that is, they are responsible for linear wave damping or amplification. These effects may be described by the general small dimensionless parameter, say, λ, which ensures weak variations of perturbation magnitude in the course of wave propagation. Hence, these terms are of order $M\lambda$. We treat λ and M as parameters of the comparable smallness. The terms $O(M^2\lambda)$, $O(M\lambda^2)$ along with $O(M^3)$ are discarded in the expansions of Eqs. (2.1) and (2.2). The resulting model describes small-signal magnetosonic perturbations and accounts for combined effects of weak nonlinearity and weak damping/amplification on magnetosonic waves. The dynamic equation for longitudinal velocity in a magnetosonic wave contains terms of order M, $M\lambda$, and M^2. $O(M\lambda)$ terms may be obtained from the linear analysis (Sec. II A), and $O(M^2)$ terms follow from the nonlinear analysis with damping/amplification linear phenomena discarded (Sec. II B).

A. Linear analysis

This is the case of small-magnitude perturbations, which is described by Eq. (2.2) with zero nonlinear terms in the right-hand side of equations. The dispersion relations follow from Eq. (2.2), if one looks for solution in the form of a sum of planar waves proportional to $\exp(i\omega(k_z)t - ik_z z)$

$$
f'(z,t) = \int_{-\infty}^{\infty} \tilde{f}(k_z) \exp(i\omega(k_z)t - ik_z z) dk_z,
$$

where k_z designates the wave number. Four relations are inherent to the wave motion, which rely on compressibility

$$
\omega_j = C_j k_z - i D_j C_j + i \frac{\alpha_j}{2} k_z^2, \quad j = 1, \ldots, 4, \tag{2.3}
$$

where

$$
\begin{aligned}
&D_j = \frac{C_j (C_j^2 - C_A^2)(\gamma - 1)}{2c_0^2 (C_j^4 - c_0^2 C_{A,z}^2)} (c_0^2 L_p + L_\rho),\\
&\alpha_j = \alpha_{\eta,j} \frac{4\eta_0}{3\rho_0} + \alpha_{k,j} \frac{k\left(\frac{1}{C_V} - \frac{1}{C_p}\right)}{\rho_0} + \alpha_{\sigma,j} \frac{1}{\mu_0 \sigma},\\
&\alpha_{\eta,j} = \frac{C_j^4 + C_j^2 (6c_0^2 - C_A^2) - 3c_0^2 (c_0^2 + C_A^2)}{4c_0^2 (2C_j^2 - c_0^2 - C_A^2)},\\
&\alpha_{k,j} = \frac{C_j^2 - C_A^2}{2C_j^2 - c_0^2 - C_A^2}, \quad \alpha_{\sigma,j} = \frac{C_j^2 - c_0^2}{2C_j^2 - c_0^2 - C_A^2}.
\end{aligned}
\tag{2.4}
$$

C_j is the magnetosonic speed satisfying the equation

$$
C_j^4 - C_j^2 (c_0^2 + C_A^2) + c_0^2 C_{A,z}^2 = 0, \tag{2.5}
$$

C_A and c_0

$$
C_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}, \quad c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}
$$

designate the Alfvén speed and the sound speed in unmagnetized gas in equilibrium, $C_{A,z} = C_A \cos(\theta)$. The real part of frequency will experience modification in the full, not asymptotic case. This would cause wave dispersion, as it has recently been shown in the infinite field approximation for slow magnetosonic waves by Zavershinskii *et al.* in Ref. 25.

There are two dispersion relations specifying the Alfvén waves, and one relation corresponds to the entropy mode. The dispersion relations Eqs. (2.3) and (2.5) have been established by Nakariakov *et al.*[16,20] (along with an appropriate evolutionary equation for longitudinal velocity in a plasma) in the flows of perfectly conducting plasma with zero mechanical viscosity, that is, with infinite σ and zero η_0. Equation (2.3) is leading-order and valid with accuracy up to terms proportional to the first powers of L_p, L_ρ, η_0, k, and σ^{-1}. We consider perturbations slowly varying over the characteristic wavelength. This imposes the smallness of effects associated with the heating–cooling function and damping

$$|D_j| \ll k_z, \quad \frac{\alpha_j}{2}k_z \ll |C_j|.$$

The magnetosonic perturbations may enhance if a linear flow is adiabatically unstable[26,27]

$$c_0^2 L_p + L_\rho > 0, \tag{2.6}$$

and the total damping is weak

$$D_j C_j > \frac{\alpha_j}{2}k_z^2.$$

The latter condition depends on the spectrum of perturbations.

Equation (2.5) readily determines the linear dynamic equation for magnetosonic perturbations, in particular for $v_z(z,t)$, bearing in mind that $i\omega$ corresponds to $\partial/\partial t$ and ik_z corresponds to $-\partial/\partial z$. It takes the form

$$\frac{\partial v_z}{\partial t} + C_j\frac{\partial v_z}{\partial z} - D_jC_jv_z - \frac{\alpha_j}{2}\frac{\partial^2 v_z}{\partial z^2} = 0. \tag{2.7}$$

The ordering number of the magnetosonic mode will be omitted in Sec. II B.

B. Nonlinear dynamic equation

The term of order M^2 in the dynamic equation was first derived by Nakariakov *et al.*[20] The author obtained a similar one in the studies of the magnetosonic heating by establishing the following links of magnetosonic perturbations:

$$\begin{aligned}
\rho' &= \frac{\rho_0}{C}v_z + \frac{(c_0^2 + C^4(\gamma-4) - C^2C_A^2(\gamma-3))\rho_0}{4C^4(c_0^2+C_A^2-2C^2)}v_z^2,\\
v_x &= \frac{C_{A,z}}{C_{A,x}}\left(\frac{c_0^2}{C^2}-1\right)v_z + \frac{c_0^2(C^2-c_0^2)(c_0^4 - C^2(2c_0^2+(\gamma-1)C_A^2)+\gamma C^4)}{2C^5(C^2-C_A^2)(c_0^2+C_A^2-2C^2)}\frac{C_{A,z}}{C_{A,x}}v_z^2,\\
v_y &= 0, \quad B_y' = 0,\\
p' &= \frac{c_0^2\rho_0}{C}v_z + \frac{c_0^2(c_0^4 - 3\gamma C^4 + C^2(2c_0^2(\gamma-1)+C_A^2(\gamma+1))\rho_0}{4C^4(c_0^2+C_A^2-2C^2)}v_z^2\\
B_z' &= \frac{(C^2-c_0^2)B_0}{CC_AC_{A,x}}v_z + \frac{B_0(C^2-c_0^2)(C^6 - C^2c_0^4 - C^4(3C_{A,z}^2 + c_0^2(\gamma-3)) + c_0^2C_{A,z}^2(C_{A,z}^2(\gamma+1)-c_0^2))}{4C^2(C^4-c_0^2C_{A,z}^2)C_AC_{A,x}^3}v_z^2.
\end{aligned} \tag{2.8}$$

Relations (2.8) are reproduced from Ref. 28. Substituted into Eq. (2.2), they yield the equivalent dynamic quadratic nonlinear equations for the longitudinal velocity v_z, bearing in mind that $v_z\frac{\partial v_z}{\partial t} \approx -Cv_z\frac{\partial v_z}{\partial z}$. This dynamic equation sounds

$$\frac{\partial v_z}{\partial t} + C\frac{\partial v_z}{\partial z} + \varepsilon v_z\frac{\partial v_z}{\partial z} = 0, \tag{2.9}$$

where ε is responsible for nonlinear distortions

$$\varepsilon = \frac{3c_0^2 + (\gamma+1)C_A^2 - (\gamma+4)C^2}{2(c_0^2 - 2C^2 + C_A^2)}.$$

The resulting equation, which incorporates weak nonlinearity and weak damping/amplification [Eqs. (2.7) and (2.9)], takes the form

$$\frac{\partial v_z}{\partial t} + C\frac{\partial v_z}{\partial z} + \varepsilon v_z\frac{\partial v_z}{\partial z} - DCv_z - \frac{\alpha}{2}\frac{\partial^2 v_z}{\partial z^2} = 0. \tag{2.10}$$

Equation (2.10) does not consider the nonlinear interaction of modes but only nonlinear "self-action" of an individual magnetosonic mode. This supposes that the perturbations associated with this dominant mode are much larger than those of other wave and non-wave modes. Equation (2.10) refers to both slow and fast modes. The evolutionary equation has been derived by Nakariakov *et al.*[20] in the case with zero mechanical viscosity and electrical resistivity of a gas. In the case of unmagnetized gas and $D=0$, Eq. (2.10) rearranges into the Burgers equation

$$\frac{\partial v_z}{\partial t} + c_0\frac{\partial v_z}{\partial z} + \varepsilon_0 v_z\frac{\partial v_z}{\partial z} - \frac{\alpha_0}{2}\frac{\partial^2 v_z}{\partial z^2} = 0 \tag{2.11}$$

with

$$\varepsilon_0 = \frac{\gamma+1}{2}, \quad \alpha_0 = \frac{4\eta_0}{3\rho_0} + \frac{(1/C_V - 1/C_P)k}{\rho_0},$$

where η_0 designates the Newtonian shear viscosity of a gas. It may look unexpected that the coefficients by compressional viscosity, thermal conduction, and electrical resistivity not only vary with θ and plasma-β but vary differently, in contrast to that in Eq. (2.11) where they are unit, with exception of $\alpha_{\sigma,0}$

$$\alpha_{\eta,0} = \alpha_{k,0} = 1, \quad \alpha_{\sigma,0} = 0.$$

Plasma-β is determined as the ratio of thermodynamic and magnetic pressures

$$\beta = \frac{2}{\gamma}\frac{c_0^2}{C_A^2}.$$

Figure 1 shows coefficients α_η, α_k, and α_σ as functions on θ and β in the cases of fast and slow magnetosonic modes. In all evaluations of this study, $\gamma = \frac{5}{3}$.

In both fast and slow modes, α_η varies from $\frac{3}{4}$ till 1, and α_k and α_σ vary from 0 till 1.

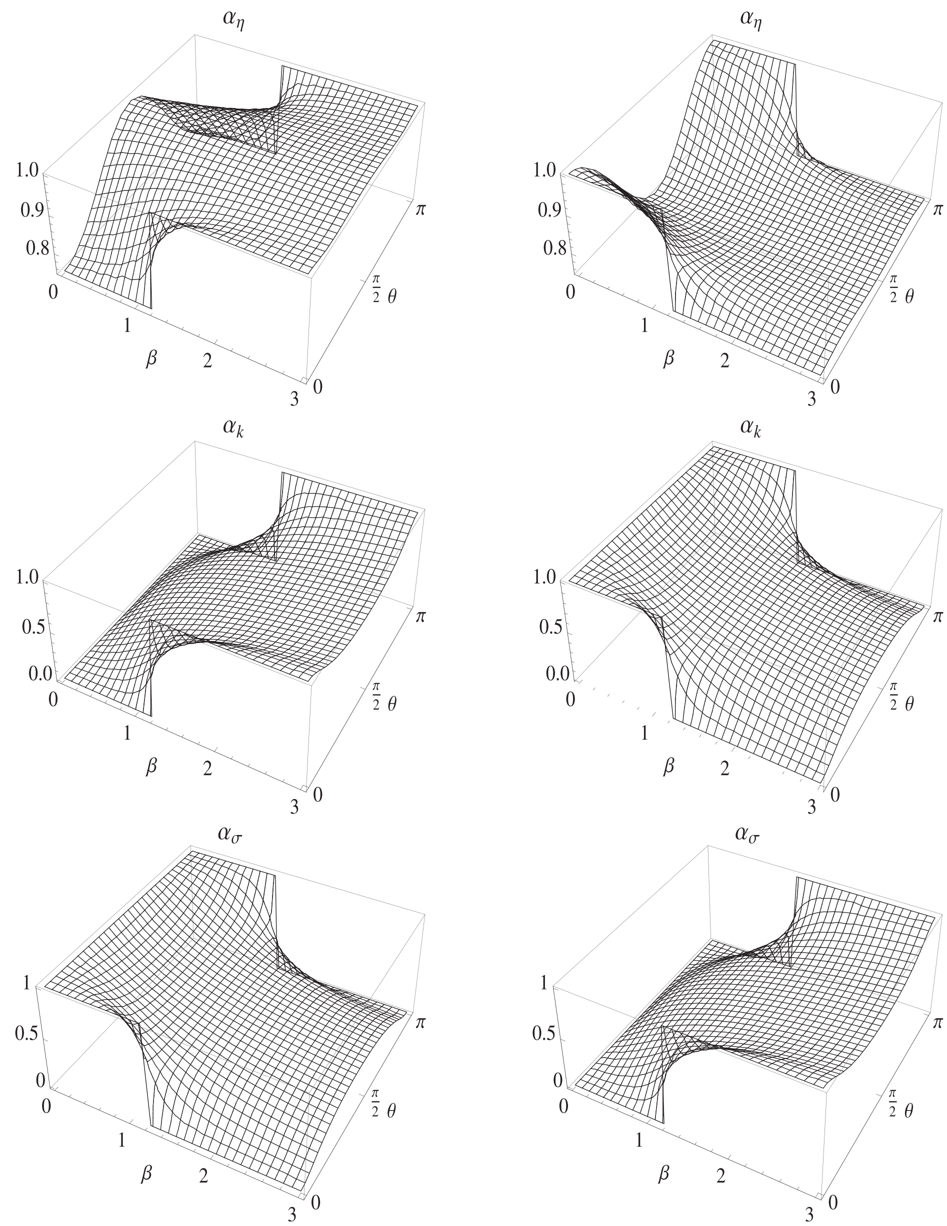

FIG. 1. Coefficients in different terms of Eq. (2.4). Factors by $\frac{4\eta_0}{\rho}$ (that is, α_η), $\frac{(1/C_V-1/C_P)k}{\rho_0}$ (α_k), and $\frac{1}{\mu_0\sigma}$ (α_σ) in the total damping in a plasma [Eq. (2.4)]. Left panels: fast magnetosonic modes and right panels: slow magnetosonic modes.

III. DYNAMICS OF PERIODIC PERTURBATIONS IN A PLASMA

A. Periodic magnetosonic saw-tooth waves in the case $\alpha = 0$ (recall)

We remind briefly the results of Ref. 17. For definiteness, we consider modes with $C>0$, that is, waves, slow or fast, propagating in the positive direction of the axis z. Hence, the sign of D coincides with the sign of $c_0^2 L_p + L_\rho$. Equation (2.10) readily rearranges in the new variables (for non-zero D and C)

$$V = v_z \exp(-Dz), \quad \zeta = \frac{e^{Dz}-1}{D}, \quad \tau = t - z/C \tag{3.12}$$

into the leading-order equation

$$\frac{\partial V}{\partial \zeta} - \frac{\varepsilon}{C^2} V \frac{\partial V}{\partial \tau} = 0. \tag{3.13}$$

ζ is always positive for non-zero D. Equation (3.13) is well studied in the nonlinear wave theory.[1,2] It may be solved by the method of characteristics. A discontinuity in the harmonic at an exciter waveform with the period T_0 always forms in acoustically active media (that is, in the case of $D>0$) and in the case of negative D if $\frac{DC^2T_0}{2\varepsilon\pi} > -1$.[7] We consider the saw-tooth shaped waves at an exciter waveform. One period of perturbations of velocity at an exciter situated at $Z=0$ is determined by the formula

$$\frac{V}{V_0} = -2\frac{\tau}{T_0}, \quad -\frac{T_0}{2} < \tau \leq \frac{T_0}{2}. \tag{3.14}$$

The method of characteristics results in a solution to Eq. (3.13) in the form of saw-tooth impulse series of the constant period T_0 and variable magnitude, which depends on the distance from an exciter

$$V(\zeta,\tau) = -\frac{2\tau}{T_0}\cdot\frac{V_0}{1+\dfrac{2\varepsilon V_0\zeta}{C^2 T_0}} \tag{3.15}$$

and

$$v_z(z,\tau) = -2\frac{\tau}{T_0}\cdot\frac{v_0 e^{Dz}}{1+\dfrac{2\varepsilon v_0(e^{Dz}-1)}{DC^2T_0}}, \quad -\frac{T_0}{2} < \tau \leq \frac{T_0}{2}. \tag{3.16}$$

If $D = 0$, the solution takes the form

$$v_z(z,\tau) = -2\frac{\tau}{T_0}\cdot\frac{v_0}{1+\dfrac{2\varepsilon v_0 z}{C^2T_0}}, \quad -\frac{T_0}{2} < \tau \leq \frac{T_0}{2}. \tag{3.17}$$

This is the case where $\zeta = z$ and $V = v_z$. The shape remains saw-tooth at any distance from an exciter for arbitrary D. The amplitude of velocity at large distances depends on the sign of D: it tends to zero if $D < 0$ and tends to $\frac{DC^2T_0}{2\varepsilon}$ in acoustically active flow. Hence, the limiting magnitude in acoustically active flow does not depend on the initial magnitude of an impulse v_0.[17]

B. Impact of mechanical damping, thermal conduction, and electrical resistivity

We start from the Burgers equation, which accounts for nonlinear and viscous terms

$$\frac{\partial V}{\partial\zeta} - \frac{\varepsilon}{C^2}V\frac{\partial V}{\partial\tau} - \frac{\alpha}{2C^3}\frac{\partial^2 V}{\partial\tau^2} = 0. \tag{3.18}$$

It is well studied in the wave theory and may be transformed into the linear diffusion equation by the Hopf-Cole transformation. The simple periodic exact solution at

$$-\frac{T_0}{2} < \tau \leq \frac{T_0}{2}$$

may be constructed making use of the solution to the purely nonlinear equation (3.13), that is, Eq. (3.15) and the shock smooth front

$$V = \tanh\left(\frac{V_0\varepsilon C\tau}{\alpha}\right), \tag{3.19}$$

which is a stationary solution to Eq. (3.18). It satisfies the equation

$$\frac{\varepsilon}{C^2}V\frac{\partial V}{\partial\tau} + \frac{\alpha}{2C^3}\frac{\partial^2 V}{\partial\tau^2} = 0.$$

The exact solution to Eq. (3.18) incorporates Eqs. (3.15) and (3.19) and accounts for variations of amplitude of the shock front with ζ as $\frac{V_0}{1+\frac{2\varepsilon V_0\zeta}{C^2T_0}}$. It takes the form[2]

$$V = \frac{V_0}{1+2\varepsilon V_0\zeta C^{-2}T_0^{-1}}\left(-2\frac{\tau}{T_0} + \tanh\left(\frac{V_0\varepsilon C\tau}{\alpha(1+2\varepsilon V_0\zeta C^{-2}T_0^{-1})}\right)\right). \tag{3.20}$$

These preliminary remarks are useful when we take a look at Eq. (2.10) rearranged in the retarded time $\tau = t - z/C$ and coordinate z. Its leading-order form differs from the Burgers equation by the term Dv_z

$$\frac{\partial v_z}{\partial z} - \frac{\varepsilon}{C^2}v_z\frac{\partial v_z}{\partial\tau} - Dv_z - \frac{\alpha}{2C^3}\frac{\partial^2 v_z}{\partial\tau^2} = 0. \tag{3.21}$$

This suggests us to try to construct a solution to it over one period of oscillations $-T_0/2 < \tau \leq T_0/2$ on the model of Eqs. (3.16) and (3.20), namely,

$$v_z = \frac{v_0\exp(Dz)}{1+2\varepsilon v_0(\exp(Dz)-1)D^{-1}C^{-2}T_0^{-1}} \times\left(-2\frac{\tau}{T_0} + \tanh\left(\frac{\varepsilon C\tau v_0\exp(Dz)}{\alpha(1+2\varepsilon v_0(\exp(Dz)-1)D^{-1}C^{-2}T_0^{-1})}\right)\right). \tag{3.22}$$

It turns out that Eq. (3.22) is not the exact solution to Eq. (3.21) but gives a discrepancy

$$\Delta = \frac{C^5D^3\exp(2Dz)\varepsilon T_0^2\tau v_0^2}{\alpha(C^2DT_0+2(\exp(Dz)-1)\varepsilon v_0)^2}\cosh^{-2} \times\left(\frac{C^3D\exp(Dz)\varepsilon T_0\tau v_0}{\alpha(C^2DT_0+2(\exp(Dz)-1)\varepsilon v_0)}\right).$$

The discrepancy is a result of substitution of Eq. (3.22) into (3.21). It should be small in magnitude compared with the terms on the left of Eq. (3.21). All of them are of comparable order. It seems reasonable to compare magnitudes of discrepancy and a small term, which distinguishes Eq. (3.21) from the Burgers equation, Dv_z. This suggests us to introduce the dimensionless parameter, which equals a ratio of absolute values of the discrepancy and Dv_0. It indicates the smallness of the discrepancy, which is conditioned by inequality

$$\frac{|\Delta|}{|D|v_0} \ll 1.$$

In dimensionless variables,

$$\Theta = \frac{\tau}{T_0}, \quad Z = \frac{z}{CT_0}, \quad X = \frac{e^{DdZ}}{1+(e^{DdZ}-1)G},$$

the discrepancy sounds as

$$\delta = \frac{|\Delta|}{|D|v_0} = X^2R|\Theta|\cosh^{-2}(XR\Theta), \tag{3.23}$$

where

$$Dd = DCT_0, \quad G = 2\varepsilon M/Dd, \quad R = \frac{\varepsilon MT_0C^2}{\alpha}, \quad M = \frac{v_0}{C},$$

$$X = \frac{e^{DdZ}}{1+(e^{DdZ}-1)G}.$$

where M, $|Dd|$, $\frac{\alpha}{T_0C^2}$ are much less than unity. This is conditioned by weak nonlinearity and weak variations of magnitudes of MHD perturbations during one period of oscillations (in order to support the wave process). The magnetosonic Reynolds number R measures the impact of nonlinear to damping effects. The approximate solution (3.22) in the dimensionless variables rearranges as

$$\frac{v_z}{v_0} = X(-2\Theta + \tanh(XR\Theta)). \tag{3.24}$$

In the case $Dd < 0$, X quickly tends to zero as Z enlarges. For assistance, Fig. 2 shows functions $y\cosh^{-2}(y)$ and X sketchy. X represents a dimensional amplitude of longitudinal velocity.

In view of limited $XR\Theta\cosh^{-2}(XR\Theta)$, which does not exceed 0.448 for any $XR\Theta$, Eq. (3.22) is a good approximation for solution to Eq. (3.18) independently on R. Evaluations for Z ensuring the smallness of absolute discrepancy δ may be done by expanding X in series in the vicinity $Z = 0$, $X \approx 1 + Dd(1-G)Z$, so as

$$1 - (|Dd| + 2\varepsilon M)Z \ll (0.448)^{-1} \approx 2.23,$$

which may be recognized as correct at zero Z and even more so at $Z > 0$. The most interesting case is positive D, when enlargement of perturbations due to heating counteracts damping and cooling. X is a limited function for any Z, if $G \geq 1$, $G^{-1} < X \leq 1$

$$G \geq 1, \quad \delta \leq 0.448, \tag{3.24a}$$

and δ decreases as Z grows. This condition may be considered as of satisfying the smallness of δ at any Z, R, and Θ. In the case $0 < G < 1$ $(1 \leq X < G^{-1})$, we may make estimations for all Z

$$0 < G < 1, \quad \delta \leq X^2 R|\Theta| \leq \frac{R}{2G^2} \ll 1, \tag{3.24b}$$

since $|\Theta| \leq 0.5$.

The case of especial importance is $X = 1$ for any Z, that is, $G = 1$. This gives the stationary waveform independent of the distance from a wave driver and the discrepancy $R|\Theta|\cosh^{-2}(R\Theta)$, which does not exceed 0.448 and may be set arbitrarily small by choice of domain of the Reynolds numbers. This particular case is discussed in Summary and Remarks.

The profiles of one period at various R, G are shown in Fig. 3. They depend on the dimensional distance from a wave driver, which indicates each curve. The magnitude of velocity may enlarge in a flow with weak damping and strong non-adiabaticity, which ensures isentropic instability (this is the case $R = 0.05$, $G = 0.5$ in Fig. 3). In the case $D > 0$ $(G > 0)$, the profile takes the following shape as z tends to infinity:

$$v_z = \frac{DC^2 T_0}{2\varepsilon}\left(-2\frac{\tau}{T_0} + \tanh\left(\frac{DC^3 T_0 \tau}{2\alpha}\right)\right) = \frac{MC}{G}\left(-2\Theta + \tanh\left(\frac{R\Theta}{G}\right)\right). \tag{3.25}$$

The limiting v_z does not depend on the initial magnitude v_0, that is, the autowave develops. The ratio $\frac{\tau_{max}}{T_0}$, where v_z achieves maximum $v_{z,max}$, and the dimensionless maximum $\frac{G}{MC}v_{z,max}$ are shown in Fig. 4 as functions of $\frac{G}{R}$. Both plots refer to the distances far from the exciter.

The smooth shock front forms at large distances from an exciter if $0 < \frac{G}{R} < 0.5$. In other cases, the saw-tooth wave develops.

IV. SUMMARY AND REMARKS

The main results of this study are the dynamic equation (2.10) (and its modified form (3.21)) and its approximate analytic solution (3.22) along with the analysis of this solution. The theory concerns fast and slow magnetosound waves. Equation (2.10) incorporates weak nonlinearity and weak damping/amplification due to the joint impact of the heating–cooling function, thermal conduction, compressional viscosity, and electrical resistivity of a plasma. The evolutionary equation expands the previous result by Chin *et al.*[16] by inclusion of compressional viscosity and electrical resistivity. The advantage of an approximate solution is its simple analytical form. The nonlinear regime is not well studied. The progress in studies of nonlinear MHD flows has been achieved mostly by numerical simulations of very particular cases. A plasma is considered as an open system with some generic function $L(p, \rho)$ describing unspecified heating and radiative cooling.[5,20,29] The theory may potentially find applications in astrophysical plasmas and laboratory plasmas (in the case of stable laboratory plasma affected by a nearly straight magnetic field). The theory is also of interest to the analysis of edge localized modes in laboratory plasma devices, in particular of the phenomenon of multifaceted asymmetric radiation from the edge (MARFE).[30,31] Studies of MHD waves in the coronal loops are important in the context of understanding of transfer of energy through the transition region and into the corona. The mechanism of dynamic flows in a corona and coronal heating is still an unresolved problem. Ofman and Wang were first who interpret SUMER oscillations in hot coronal loops associated with flares in terms of standing damped slow magnetosonic waves.[32] The decayless oscillations have also been detected in flaring coronal loops. This indicates the heating mechanism, which ensures the isentropical instability of a plasma flow. Solar flares are relatively small and local, taking place in the low solar atmosphere, and may be

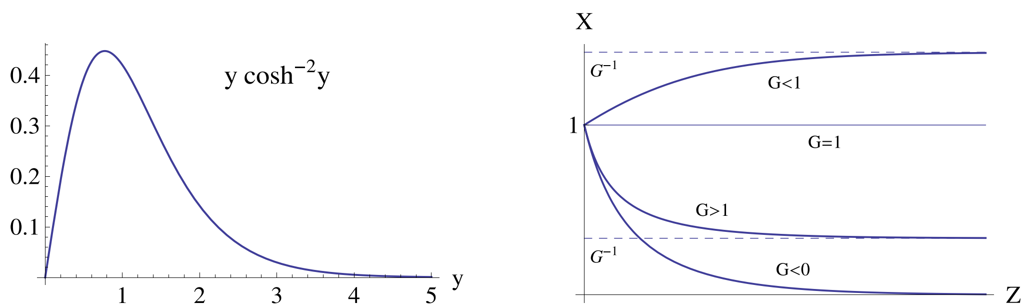

FIG. 2. Functions $y\cosh^{-2}(y)$ and $X = \frac{e^{DdZ}}{1+(e^{DdZ}-1)G}$.

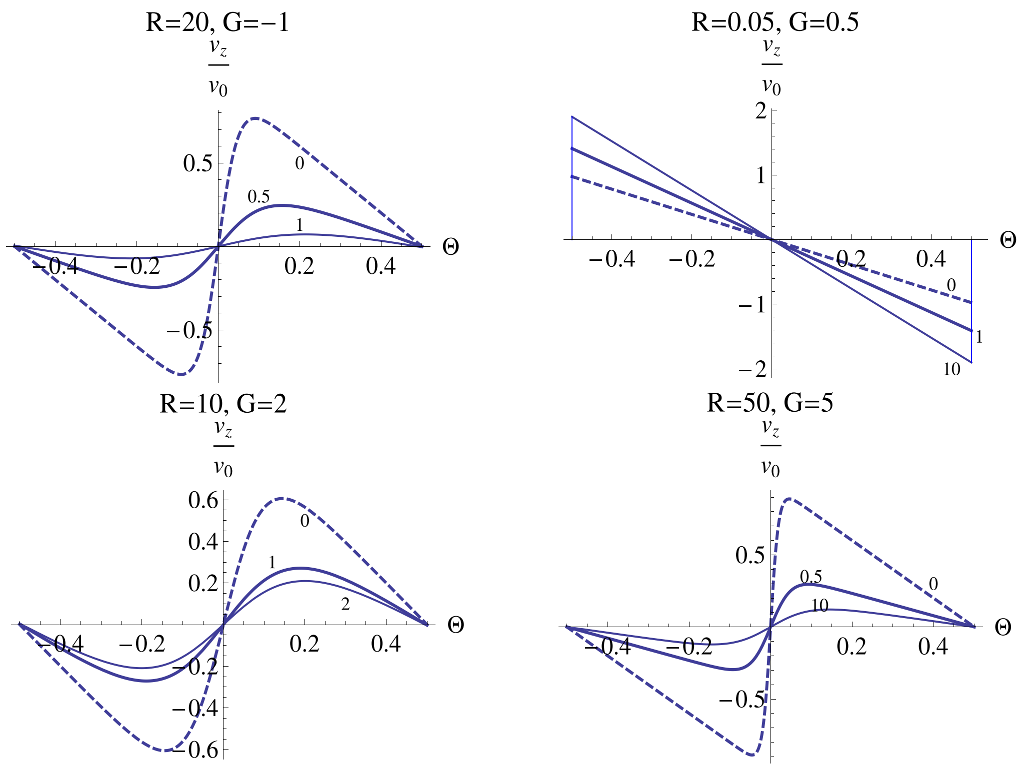

FIG. 3. The profiles of one period of velocity at different distances from an exciter, $|D|z$ at various R and G. Dimensionless distances are designated by numbers. $R = 20$, $G = -1$ (top row, left panel), $R = 0.05$, $G = 0.5$ (top row, right panel), $R = 10$, $G = 2$ (bottom row, left panel), and $R = 50$, $G = 5$ (bottom row, right panel).

considered as a possible reason for instability. Nakariakov *et al.* interpreted perturbations in the cool coronal loops as slow magnetosonic waves and proved the possibility of isentropic instability (that is, potential growth of wave perturbations in weakly damping plasma) in the solar atmosphere and in the cold intersellar molecular gas (ISM). Recent high-resolution ground-based and spaceborne observational instruments show the ubiquity of magnetoacoustic wave processes in the solar atmosphere. The dynamic equation (2.10) covers isentropically stable, neutral, and unstable flows and its approximate solution (3.22). In particular, the results may be addressed to a high temperature atomic plasma with $T > 10^4 K$ and cold interstellar molecular gas with $T < 10^3 K$.[20] In these kinds of plasmas, the magnetosound waves may amplify due to some heating regime. The results may be useful in remote diagnostics of astrophysical plasmas. There is no restrictions on the plasma-β (that is, on the magnitude of the equilibrium magnetic field) and an angle between the equilibrium magnetic field and the wave vector. The results may be addressed to both gases with low β as cold plasma of the inner atmosphere and that with finite β as rarified plasma of the outer atmosphere.

Equation (2.10) describing velocity in the fast or slow magnetosonic wave contains parameters depending on θ and plasma-β.

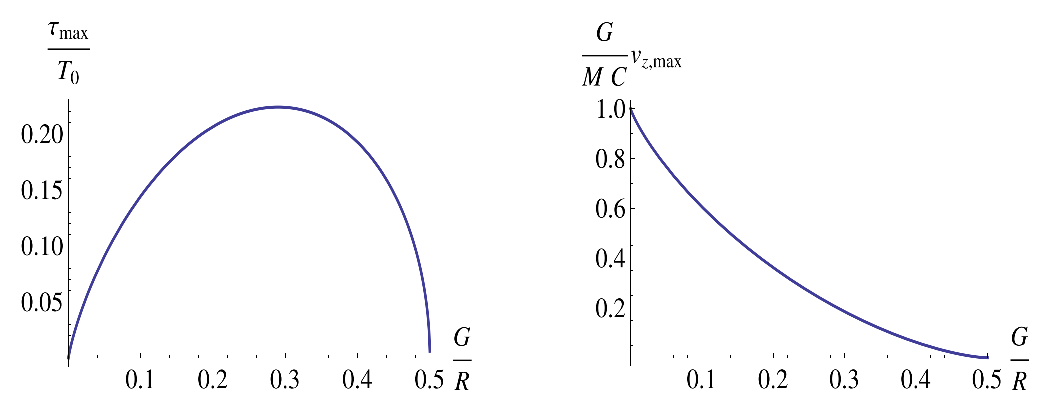

FIG. 4. The ratio $\frac{\tau_{max}}{T_0}$ where v_z achieves maximum far from the exciter (left panel), and the dimensionless maximum $\frac{G}{MC} v_{z,max}$ at various $\frac{G}{R}$ values (right panel).

This concerns linear magnetosonic speed, the parameter of nonlinearity, and individual factors by damping terms. The coefficients in the terms responsible for thermal conduction, compressional viscosity, and electrical resistivity of a plasma vary differently with plasma-β and θ. Similar equations but with constant parameters determined by the equilibrium thermodynamic state and the heating–cooling function arise in various non-magnetic flows with destroyed adiabaticity. In particular, they describe perturbations in gases with excited oscillatory degrees of freedom in molecules and in gases with chemical reactions.[7–10] The total damping of magnetosonic waves consists of parts associating with compressional viscosity, electrical resistivity, and thermal conduction. The relative importance of compressional viscosity and thermal conduction is determined by the ratio $\frac{Pr}{\beta}$, where $Pr = \frac{C_p\eta_0}{k}$ denotes the Prandtl number.[33] In the coronal plasma, $\beta = 0.016$, $Pr = 10^{-2}$, so that $\frac{Pr}{\beta} \approx 1$.[34] Hence, thermal conduction and viscosity contribute equally. The magnetic Prandtl number Pr_m is determined as a ratio of viscous to magnetic effects, $\frac{\eta_0\mu_0\sigma}{\rho_0}$. For the solar corona, Ruderman *et al.* obtained $Pr_m = 10^{10}$,[33] and electrical resistivity has only very small impact.

The approximate solution to (2.10), Eq. (3.22), is determined by the parameters R (this is the magnetosonic Reynolds number, which equals the ratio of nonlinear and damping factors) and G (the ratio of nonlinearity and the degree of non-adiabaticity). The case $G<0$ relates to a flow giving off energy. Along with losses due to other factors, this leads to quick damping of wave perturbations. $G>0$ is the case with energy recharge, which counteracts damping due to mechanical friction, thermal conduction, and electrical resistivity. The scenario of evolution and the shape of the waveform depend on the balance of nonlinearity, non-adiabaticity, and damping. The magnitude of velocity may enlarge or get smaller, and the shape of the waveform takes smooth or saw-tooth shape (Fig. 3). The especial case $G=1$ yields the stationary waveform, which does not vary with the distance from a wave driver.

Examples of stationary waveforms for different R values are shown in Fig. 5. They also may take a smooth or saw-tooth shape and $R=2$ in accordance with Eq. (3.24), and Fig. 3 shows the limiting value between these two species. In the case of acoustically active flow, the waveform far from the exciter is described by Eq. (3.25). It does not depend on the initial magnitude of velocity at an exciter.

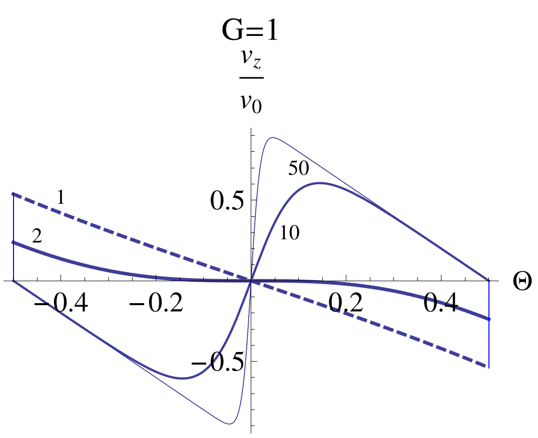

FIG. 5. The stationary profile of velocity at various R values (designated by numbers), $G=1$.

Chin *et al.* have analyzed the dynamics of the harmonic initially signal numerically for some parameters of a flow in thermoconducting nonlinear active flow in Ref. 16. The authors established stationary asymptotic autowaves analytically in the form of the saw-tooth wave despite that thermal conduction prevents the formation of discontinuities. Equation (3.22) includes this case for infinitely large distances from an exciter and damping tending to zero.[17] It describes a waveform at each distance from an exciter and accounts for electrical resistivity and compressible viscosity of a plasma. Hence, it may find more applications. It takes simple analytical form. Autowaves, that is, waves with parameters determined exclusively by the equilibrium parameters of a medium, were analytically studied in the context of solitary perturbations in magnetic flux tubes.[35] They are of growing interest in the fluid mechanics since this is an asymptotic case to which waveforms usually develop independently on the initial conditions. The autowaves observations may be especially useful in remote interpretation of plasma's features and processes in it.

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