Opuscula Math. 38, no. 3 (2018), 409–425
https://doi.org/10.7494/OpMath.2018.38.3.409 Opuscula Mathematica

ON DOMINATION MULTISUBDIVISION NUMBER OF UNICYCLIC GRAPHS

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Communicated by Andrzej Żak

Abstract. The paper continues the interesting study of the domination subdivision number and the domination multisubdivision number. On the basis of the constructive characterization of the trees with the domination subdivision number equal to 3 given in [H. Aram, S.M. Sheikholeslami, O. Favaron, *Domination subdivision number of trees*, Discrete Math. 309 (2009), 622–628], we constructively characterize all connected unicyclic graphs with the domination multisubdivision number equal to 3. We end with further questions and open problems.

Keywords: domination number, domination subdivision number, domination multisubdivision number, trees, unicyclic graphs.

Mathematics Subject Classification: 05C69, 05C05, 05C38.

1. INTRODUCTION

For domination problems, multiple edges and loops are irrevelant, so we forbid them. Additionaly, in this paper we consider connected graphs only. We use $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G and denote $|V(G)| = n$, $|E(G)| = m$.

The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v.

We say that a vertex v is *a universal vertex* of G if it is a neighbour of every other vertex of a graph and v is a *leaf* of G if v has exactly one neighbour in G. A vertex v is called a *support vertex* if it is adjacent to a leaf. If v is adjacent to more than one leaf, then we call v a *strong support vertex*. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$.

A subset D of $V(G)$ is *dominating* in G if every vertex of $V(G) - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. A minimum dominating set of a graph G is called a $\gamma(G)$-set. A vertex $v \in V(G)$ is $\gamma(G)$-*critical* if $\gamma(G - v) < \gamma(G)$.

For a graph $G = (V, E)$ subdivision of the edge $e = uv \in E$ with vertex x leads to a graph with vertex set $V \cup \{x\}$ and edge set $(E - \{uv\}) \cup \{ux, xv\}$. Let $G_{e_1,e_2,\dots,e_k}$ denote the graph G with subdivided edges $e_1, e_2, \dots, e_k$, where each edge is subdivided once. Let $G_{e,k}$ denote graph G with subdivided edge e with k vertices (instead of edge $e = uv$ we put a path $(u, x_1, x_2, \dots, x_k, v)$). For $k = 1$ we write G_e.

The *domination subdivision number*, $\mathrm{sd}(G)$, of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. We consider subdivision number for connected graphs of order at least 3, since the domination number of the graph K_2 does not increase when its only edge is subdivided. The domination subdivision number was defined in [13] and studied, for example in [1, 3, 6].

There are also many papers concerning total domination subdivision number (see for example [5, 8]), roman domination subdivision number, paired domination subdivision number, double domination number any many more.

2. MOTIVATION AND RELATION TO PREVIOUS WORK

In this paper we continue the study of the domination multisubdivision number defined by Dettlaff, Raczek and Topp in [4]. Let $\mathrm{msd}(uv)$ be the minimum number of subdivisions of the edge uv such that $\gamma(G)$ increase. Let the *domination multisubdivision number* of a graph $G, m > 0$, denoted by $\mathrm{msd}(G)$, be defined as

$$\mathrm{msd}(G) = \min\{\mathrm{msd}(uv) : \ uv \in E(G)\}.$$

Domination multisubdivision number is well defined for all graphs with at least one edge. In [4] were also studied some complexity aspects regarding the domination subdivision and domination multisubdivision numbers of graphs. That is, there was studied the following decision problems. Given a graph $G = (V, E)$ with the domination number $\gamma(G)$: Is $\mathrm{sd}(G) > 1$? and, Is $\mathrm{msd}(G) > 1$? As a result, in [4], was obtained that these decision problems for the domination subdivision number, as well as for the domination multisubdivision number, are NP-complete even for bipartite graphs. In this sense, it is desirable to find or describe some families of graphs in which is possible to give the exact value for these parameters.

A sudy of similar parameter, namely total domination multisubdivision number was carried in [2].

A unicyclic graph is a graph containing precisely one cycle. A family of unicyclic graphs is widely studied by many authors in the theory of domination, see for example [7, 10, 11].

As it was proven in [4], $\mathrm{msd}(G) \in \{1, 2, 3\}$. Interesting problem about graphs in which subdividing any single edge two times does not increase its dominating number arises. What is their structure like? The class of all trees T with $\mathrm{msd}(T) = 3$ is already characterized and the next section sums up the results from [4] and [1] on this topic. Next we characterize all unicyclic graphs with the domination multisubdivision number equal to 3.

For any unexplained terms and symbols see [9].

3. TREES WITH THE MULTISUBDIVISION DOMINATION NUMBER EQUAL TO 3

It is possible to observe that if $\mathrm{msd}(G) = 3$, then G does not have a strong support vertex, since subdividing an edge incident with a leaf and a strong support vertex two times results in a graph with bigger domination number.

In order to describe all unicyclic graphs with $\mathrm{msd}(G) = 3$, we first recall the class of all trees with the domination multisubdivision number equal to 3.

The following constructive characterization of the family $\mathcal{T}$ of labeled trees T with $\mathrm{sd}(T) = 3$ was given in [1] by Aram, Sheikholeslami and Favaron. Dettlaff, Raczek and Topp in [4] have proven that for any tree T, $\mathrm{msd}(T) = \mathrm{sd}(T)$. Thus, this is also a characterization of all trees with $\mathrm{msd}(T) = 3$. In what follows we recall the characterization given in [1].

The label of a vertex v is also called a status and is denoted by $sta(v)$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the following two operations defined on a graph G.

Operation $\mathcal{T}_1$. Assume $sta(v) = A$. Then add a path (x, y, z) and the edge vx. Let $sta(x) = sta(y) = B$, and $sta(z) = A$.

Operation $\mathcal{T}_2$. Assume $sta(v) = B$. Then add a path (x, y) and the edge vx. Let $sta(x) = B$, and $sta(y) = A$.

Let $\mathcal{T}$ be the minimum family of trees obtained from P_4, where the two leaves have status A and the two support vertices have status B, by a finite sequence of Operations $\mathcal{T}_1$ or $\mathcal{T}_2$.

If $T \in \mathcal{T}$, we let $A(T)$ and $B(T)$ be the set of vertices of statuses A and B, respectively, in T.

Here we a few recall important properties of trees belonging to the family $\mathcal{T}$.

Observation 3.1 (Aram, Sheikholeslami, Favaron [1]). *Let $T \in \mathcal{T}$ and $v \in V(T)$.*

(1) *If v is a leaf, then $sta(v) = A$.*
(2) *If v is a support vertex, then $sta(v) = B$.*
(3) *If $sta(v) = A$, then $N(v) \subseteq B(T)$.*
(4) *If $sta(v) = B$, then v is adjacent to exactly one vertex of $A(T)$ and at least one vertex of $B(T)$.*
(5) *The distance between any two vertices in $A(T)$ is at least* 3.

Lemma 3.2 (Aram, Sheikholeslami, Favaron [1]). *If $T \in \mathcal{T}$, then $A(T)$ is a $\gamma(T)$-set.*

The following corollary is a consequence of the results contained in [4].

Corollary 3.3 (Dettlaff, Raczek, Topp [4]). *Let T be a tree. Then*

$$\mathrm{msd}(T) = 3 \text{ if and only if } T \in \mathcal{T}.$$

The *private neighbourhood of a vertex u with respect to a set $D \subseteq V(G)$*, where $u \in D$, is the set $pn[u, D] = N_G[u] - N_G[D - \{u\}]$.

Lemma 3.4 (Aram, Sheikholeslami, Favaron [1]). *Let $T \in \mathcal{T}$ and $u \in A(T)$. There is a $\gamma(T)$-set of T, say D, such that $u \in D$ and $pn[u, D] = \{u\}$.*

Lemma 3.5 (Sampathkumar, Neeralagi [12])**.** *A vertex $u \in V(G)$ is $\gamma(G)$-critical if and only if there exists a $\gamma(G)$-set D where $pn[u, D] = \{u\}$.*

Observation 3.1 altogether with Lemma 3.2, Lemma 3.4 and Lemma 3.5 imply the following result.

Corollary 3.6. *Let $T \in \mathcal{T}$ and $u \in V(T)$. Then $u \in A(T)$ if and only if*

$$\gamma(T - u) < \gamma(T).$$

4. UNICYCLIC GRAPHS WITH THE MULTISUBDIVISION DOMINATION NUMBER EQUAL TO 3

In this section we give a constructive characterization of all unicyclic graphs with the multisubdivision domination number equal to 3.

Let $T_{\mathcal{A}} \subseteq \mathcal{T}$ be the set of all tress T belonging to $\mathcal{T}$ and such that there exist vertices $u_1, u_2, u_3, v_1, v_2, v_3 \in V(T)$ with the following properties:

— u_1 and v_1 are leaves in T with $d_T(u_1, v_1) \geq 7$;
— u_2 and v_2 are the support vertices in T adjacent to u_1 and v_1, respectively;
— u_3 and v_3 are adjacent to u_2 and v_2, respectively where $d(u_3) = d(v_3) = 2$.

See Figure 1.

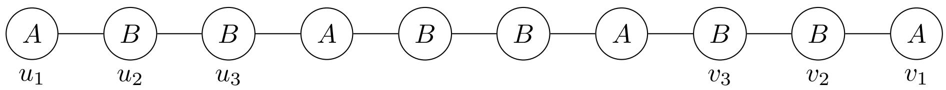

Fig. 1. A tree $T \in T_{\mathcal{A}}$

Denote by $\mathcal{A}$ the set of all unicyclic graphs G obtained from any tree $T \in T_{\mathcal{A}}$ by identifying: u_1 with v_1, u_2 with v_2 and u_3 with v_3. Denote the identified vertices by w_1, w_2 and w_3, respectively. Let the statuses of vertices of G correspond to the statuses of the revelant vertices of the tree T, in addition let $sta(w_1) = A$, $sta(w_2) = sta(w_3) = B$. See Figure 2.

Let $T_{\mathcal{B}} \subseteq \mathcal{T}$ be the set of all tress T belonging to $\mathcal{T}$ and such that there exist vertices $u_1, u_2, v_1, v_2 \in V(T)$ with the following properties:

— u_1 and v_1 are leaves in T with $d_T(u_1, v_1) \geq 5$;
— u_2 and v_2 are the support vertices adjacent to u_1 and v_1, respectively;
— on the (u_1, v_1)-path in T exists a vertex of degree greater than 2 labelled B;

See Figure 3.

Denote by $\mathcal{B}$ the set of all unicyclic graphs G obtained from any tree $T \in T_{\mathcal{B}}$ by identifying: u_1 with v_1 and u_2 with v_2. Denote the identified vertices by w_1 and w_2, respectively. Let the statuses of vertices of G correspond to the statuses of the relevant vertices of the tree T, in addition let $sta(w_1) = A$ and $sta(w_2) = B$. See Figure 4.

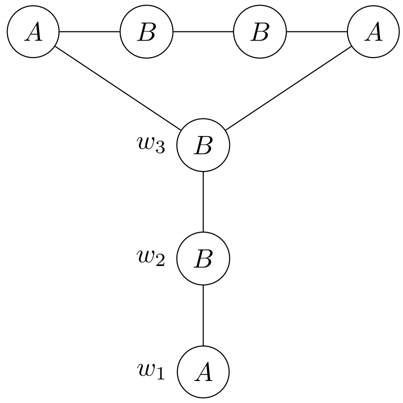

Fig. 2. A graph G obtained from the tree $T \in T_{\mathcal{A}}$ illustrated in Figure 1

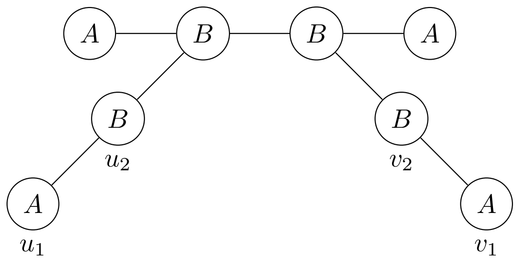

Fig. 3. A tree $T \in T_{\mathcal{B}}$

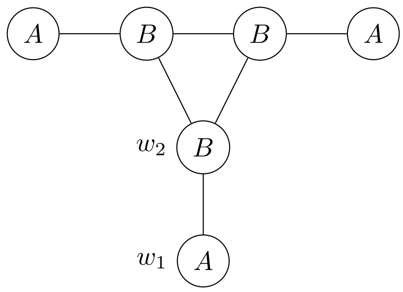

Fig. 4. A graph G obtained from the tree $T \in T_{\mathcal{B}}$ illustrated in Figure 3

We define $\mathcal{F}_0 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ to be the family of unicyclic graphs where $\mathcal{A}$ and $\mathcal{B}$ are classes of graphs defined above and let $\mathcal{C}$ be the subclass of all cycles $C_n = (c_1, c_2, \ldots, c_n, c_1)$ for $n = 4, 7, 10, \ldots$. Let every vertex of C_n has status A.

Before we prove that each graph of the family $\mathcal{F}_0$ has the domination multisubdivision number equal to 3, we formulate the properties of the family $\mathcal{F}_0$.

If $G \in \mathcal{F}_0$, we let $A(G)$ and $B(G)$ be the set of vertices of statuses A and B, respectively, in G.

Observation 4.1. *Let $G \in \mathcal{F}_0$ and $v \in V(G)$.*

(1) *If v is a leaf, then $sta(v) = A$.*
(2) *If v is a support vertex, then $sta(v) = B$.*
(3) *If $G \in \mathcal{A} \cup \mathcal{B}$ and $sta(v) = A$, then $N(v) \subseteq B(G)$.*
(4) *If $sta(v) = B$, then v is adjacent to exactly one vertex of $A(G)$ and at least one vertex of $B(G)$.*
(5) *If $G \in \mathcal{A} \cup \mathcal{B}$, then $d(x, y) \geq 3$ for any two vertices $x, y \in A(G)$.*

Lemma 4.2. *If G is a graph belonging to $\mathcal{A} \cup \mathcal{B}$, then $A(G)$ is a $\gamma(G)$-set.*

Proof. Let G be a graph belonging to $\mathcal{A} \cup \mathcal{B}$. Then by Observation 4.1, $A(G)$ is a dominating set of G, so $\gamma(G) \leq |A(G)|$.

On the other hand by Observation 4.1 (5), it is not possible for one vertex of $V(G)$ dominate two or more vertices of $A(G)$. Therefore any dominating set of G contains at least as many elements as $A(G)$. Thus $\gamma(G) \geq |A(G)|$ and hence $|A(G)|$ is a $\gamma(G)$-set. $\square$

The following result is an immediate consequence of Lemmas 3.2 and 4.2.

Corollary 4.3. *Let $G \in \mathcal{A}$ or $G \in \mathcal{B}$ be obtained from a tree $T \in T_{\mathcal{A}}$ or $T \in T_{\mathcal{B}}$, respectively. Then*

$$\gamma(G) = \gamma(T) - 1.$$

Lemma 4.4. *If G is a graph belonging to the family $\mathcal{F}_0$, then $x \in A(G)$ if and only if*

$$\gamma(G - x) < \gamma(G).$$

Proof. If G is a graph belonging to the subclass $\mathcal{C}$, then the statement is easily verifiable.

Assume first $G \in \mathcal{A}$ and let x be a vertex of $V(G) - \{w_1, w_2, w_3\}$. Then G is obtained from a tree $T \in T_{\mathcal{A}}$ and $x \in V(T)$. Let D_T be a $\gamma(T - x)$-set. Without loss of generality we may assume $u_2, v_2 \in D_T$. Then $(D_T - \{u_2, v_2\}) \cup \{w_2\}$ is a dominating set of $G - x$ of cardinality $\gamma(T - x) - 1$. Thus, $\gamma(G - x) \leq \gamma(T - x) - 1$.

On the other hand, let D_G be a $\gamma(G - x)$-set containing w_2 and not containing w_1, w_3. Such a set exists, since w_1 is a leaf and w_3 is not a support vertex. Then $(D_G - \{w_2\}) \cup \{u_2, v_2\}$ is a dominating set of $T - x$ of cardinality $\gamma(G - x) + 1$. Hence $\gamma(G - x) = \gamma(T - x) - 1$.

Aditionally assume $x \in A(G) - \{w_1\}$. Then $x \in A(T)$ and by Corollary 3.6, $\gamma(T - x) < \gamma(T)$. Therefore,

$$\gamma(G - x) = \gamma(T - x) - 1 < \gamma(T) - 1 = \gamma(G)$$

and hence in this situation $\gamma(G - x) < \gamma(G)$.

Next assume $x = w_1$. Then by Corollary 3.6, $\gamma(T - u_1) < \gamma(T)$. Let D_T be a $\gamma(T - u_1)$-set. Without loss of generality we may assume $u_3, v_2 \in D_T$. Then $(D_T - \{u_3, v_2\}) \cup \{w_3\}$ is a dominating set of $G - w_1$ of cardinality $\gamma(T - u_1) - 1$. Therefore we obtain that

$$\gamma(G - w_1) \leq \gamma(T - u_1) - 1 < \gamma(T) - 1 = \gamma(G)$$

and hence $\gamma(G - x) < \gamma(G)$ for every $x \in A(G)$.

Thus we assume $x \in B(G) - \{w_2, w_3\}$. Then $x \in V(T)$ and by Corollary 3.6, $\gamma(T - x) \geq \gamma(T)$. Therefore,

$$\gamma(G) = \gamma(T) - 1 \leq \gamma(T - x) - 1 = \gamma(G - x)$$

and hence in this situation $\gamma(G - x) \geq \gamma(G)$. Next assume $x = w_3$. Since $w_3 \in B(T)$, Corollary 3.6 implies that $\gamma(T) \leq \gamma(T - u_3)$. Let D_G be a $\gamma(G - w_3)$-set. Without loss of generality we may assume $w_2 \in D_G$. Then $(D_G - \{w_2\}) \cup \{u_2, v_2\}$ is a dominating set of $T - u_3$ of cardinality $\gamma(G - w_3) + 1$. Therefore we obtain that

$$\gamma(G) + 1 = \gamma(T) \leq \gamma(T - u_3) \leq \gamma(G - w_3) + 1$$

and hence $\gamma(G - x) \geq \gamma(G)$ for $x = w_3$.

Lastly assume $x = w_2$. Again by Corollary 3.6, $\gamma(T) \leq \gamma(T - u_2)$. By the construction of G and Observation 4.1, w_3 is adjacent to a vertex $y \in A(G)$. Since $\gamma(G - y) < \gamma(G)$, w_3 does not belong to any minimum dominating set of $G - y$. Thus there exists a minimum dominating set of G, say D_G, containing w_3. Then $w_1 \in D_G$. Therefore $(D_G - \{w_1, w_3\}) \cup \{u_1, u_3, v_2\}$ is a dominating set of $T - u_2$ or $(D_G - \{w_1, w_3\}) \cup \{v_1, v_3, u_2\}$ is a dominating set of $T - v_2$, both sets are of cardinality $\gamma(G - w_2) + 1$. Assuming the first case, we obtain that

$$\gamma(G) + 1 = \gamma(T) \leq \gamma(T - u_2) \leq \gamma(G - w_2) + 1$$

and hence $\gamma(G - x) \geq \gamma(G)$ for every $x \in B(G)$.

Assume now $G \in \mathcal{B}$ and let x be a vertex of $V(G) - \{w_1, w_2\}$. Then G is obtained from a tree $T \in T_{\mathcal{B}}$ and $x \in V(T)$. In a similar way as in case of $G \in \mathcal{B}$ we may justify that $\gamma(G - x) = \gamma(T - x) - 1$.

Assume aditionally $x \in A(G) - \{w_1\}$. Then $x \in V(T)$ and since $x \in A(T)$, $\gamma(T - x) < \gamma(T)$. Therefore,

$$\gamma(G - x) = \gamma(T - x) - 1 < \gamma(T) - 1 = \gamma(G)$$

and hence in this situation $\gamma(G - x) < \gamma(G)$.

Next assume $x = w_1$. Then by Corollary 3.6, $\gamma(T - u_1) < \gamma(T)$. Let D_T be a $\gamma(T - u_1)$-set. Without loss of generality we may assume $u_3, v_2 \in D_T$, where $u_3 \in B(T)$ and $u_2u_3 \in E(T)$. Then $D_T - \{v_2\}$ is a dominating set of $G - w_1$ of cardinality $\gamma(T - u_1) - 1$. Therefore we obtain that

$$\gamma(G - w_1) \leq \gamma(T - u_1) - 1 < \gamma(T) - 1 = \gamma(G)$$

and hence $\gamma(G - x) < \gamma(G)$ for every $x \in A(G)$.

Thus we assume $x \in B(G) - \{w_2\}$. Then $x \in V(T)$ and by Corollary 3.6, $\gamma(T - x) \geq \gamma(T)$. Therefore,

$$\gamma(G) = \gamma(T) - 1 \leq \gamma(T - x) - 1 = \gamma(G - x)$$

and hence in this situation $\gamma(G - x) \geq \gamma(G)$.

Lastly assume $x = w_2$. Again by Corollary 3.6, $\gamma(T) \leq \gamma(T - u_2)$. Denote by D_G a minimum dominating set of $G - w_2$. Then $w_1 \in D_G$ and $(D_G - \{w_1\}) \cup \{u_1, v_1\}$ is a dominating set of $T - u_2$ of cardinality $\gamma(G - w_2) + 1$. Thus we obtain that

$$\gamma(G) + 1 = \gamma(T) \leq \gamma(T - u_2) \leq \gamma(G - w_2) + 1$$

and hence $\gamma(G - x) \geq \gamma(G)$ for every $x \in B(G)$. $\square$

Lemma 4.5. *If G is a graph belonging to the family $\mathcal{F}_0$, then*

$$\mathrm{msd}(G) = 3.$$

Proof. Let G be a graph belonging to the family $\mathcal{F}_0$. The statement is easily verifiable when G belongs to the Subclass $\mathcal{C}$ of the family $\mathcal{F}_0$.

Now assume G belongs to the Subclass $\mathcal{A}$ of the family $\mathcal{F}_0$. Then G is obtained from a tree $T \in \mathcal{T}$. Suppose $\mathrm{msd}(G) \neq 3$, e.g. $\mathrm{msd}(G) = k$, where $k \in \{1, 2\}$. Then there exists an edge e such that $\gamma(G_{e,k}) \geq \gamma(G) + 1$. If $e \in E(G) - \{w_1w_2, w_2w_3\}$, then $e \in E(T)$ and Lemma 4.2 altogether with Corollary 4.3 imply that

$$\gamma(G_{e,k}) - 1 \geq \gamma(G) = \gamma(T) - 1 = \gamma(T_{e,k}) - 1.$$

Hence, $\gamma(G_{e,k}) \geq \gamma(T_{e,k})$. Let D_1 be a $\gamma(T_{e,k})$-set. Then $(D_1 - \{u_1, u_2, v_1, v_2\}) \cup \{w_2\}$ is a dominating set of $G_{e,k}$ of cardinality smaller than $\gamma(G_{e,k})$, a contradiction. Thus we assume that $e \in \{w_1w_2, w_2w_3\}$. Observe that graphs $G_{w_1w_2,k}$ and $G_{w_2w_3,k}$ are isomorphic, so without loss of generality we just consider the case of subdividing the edge w_1w_2. Denote by x_1 and x_2 the new vertices obtained by subdividing w_1w_2 when $k = 2$. In the case when $k = 1$, we assume x_1 does not exists, but the rest of our reasoning is the same. Then $G_{w_1w_2,k}$ may be obtained from $T_{u_1u_2,k}$ by identyfing x_2 with v_1, u_2 with v_2 and u_3 with v_3. Without loss of generality we may assume that v_2 belongs to the $\gamma(T_{u_1u_2,k})$-set, say D'. Then $D' - \{v_2\}$ is a dominating set of $G_{w_1w_2,k}$. Hence by Lemma 4.2 altogether with Corollary 4.3,

$$\gamma(G) = \gamma(T) - 1 = \gamma(T_{u_1u_2,k}) - 1 \geq \gamma(G_{u_1u_2,k}).$$

This implies that $\gamma(G) \geq \gamma(G_{u_1u_2,k})$, which lead us to a contradiction with the assumption that $\mathrm{msd}(G) \neq 3$.

At last assume G belongs to the Subclass $\mathcal{B}$ of the family $\mathcal{F}_0$. Then G is obtained from a tree $T \in \mathcal{T}$. Again we suppose $\mathrm{msd}(G) \neq 3$, e.g. $\mathrm{msd}(G) = k$, where $k \in \{1, 2\}$. Then there exists an edge e such that $\gamma(G_{e,k}) \geq \gamma(G) + 1$. If $e \in E(G) - \{w_1w_2\}$, then $e \in E(T)$ and Lemma 4.2 altogether with Corollary 4.3 imply that

$$\gamma(G_{e,k}) - 1 \geq \gamma(G) = \gamma(T) - 1 = \gamma(T_{e,k}) - 1.$$

Hence, $\gamma(G_{e,k}) \geq \gamma(T_{e,k})$. Let D_1 be a $\gamma(T_{e,k})$-set. Then $(D_1 - \{u_1, u_2, v_1, v_2\}) \cup \{w_2\}$ is a dominating set of $G_{e,k}$ of smaller cardinality than $\gamma(G_{e,k})$, a contradiction. Thus we conclude that $e = w_1w_2$. Denote by x_1 and x_2 the new vertices obtained by subdividing w_1w_2 when $k = 2$. In case when $k = 1$, we assume x_1 does not exists, but the rest of our reasoning is the same. Then $G_{w_1w_2,k}$ may be obtained from $T_{u_1u_2,k}$ by identyfing x_2 with v_1 and u_2 with v_2. Since on the (u_1, v_1)-path in T exists a vertex of degree greater than 2 labelled B, we may obtain a dominating set D' such that either

- D' is a $\gamma(T_{u_1u_2,k})$-set such that v_2 belongs to D' and v_2 does not have a private neighbour in $V(T_{u_1u_2,k}) - \{v_1, v_2\}$ with respect to D', or
- D' is a $\gamma(T_{v_1v_2,k})$-set such that u_2 belongs to D' and u_2 does not have a private neighbour in $V(T_{v_1v_2,k}) - \{u_1, u_2\}$ with respect to D'.

In both cases $D' - \{v_2\}$ is a dominating set of $G_{w_1w_2,k}$. Hence by Lemma 4.2 altogether with Corollary 4.3,

$$\gamma(G) + 1 = \gamma(T) = \gamma(T_{u_1u_2,k}) \geq \gamma(G_{u_1u_2,k}) + 1.$$

This implies that $\gamma(G) \geq \gamma(G_{u_1u_2,k})$, which lead us to a contradiction with the assumption $\mathrm{msd}(G) \neq 3$. □

Now we introduce a family of unicyclic graphs $\mathcal{F}$ which contains all graphs of the family $\mathcal{F}_0$ and graphs that can be obtained as follows. Let G_0 be an element of $\mathcal{F}_0$. If k is a positive integer, then G_k can be obtained recursively from G_{k-1} by one of the operations $\mathcal{T}_1$ or $\mathcal{T}_2$ described previously.

If $G \in \mathcal{F}$, we let $A(G)$ and $B(G)$ be the set of vertices of statuses A and B, respectively, in G. It is an easy observation that if $G \in \mathcal{F}$ and $x \in B(G)$, then x is adjacent to a vertex of $A(G)$.

Lemma 4.6. *Let G be a graph such that $w, y, z \in V(G)$ induce a path $P_3 = (w, y, z)$ in G, where $d_G(z) = 1$, $d_G(y) = d_G(w) = 2$. Then*

$$\gamma(G - \{w, y, z\}) = \gamma(G) - 1.$$

Proof. Let G, w, y, z be as in the assumption of the lemma. Denote $G' = G - \{w, y, z\}$.

Any $\gamma(G')$-set may be expanded to a dominating set of G by adding to it y. Thus, $\gamma(G) \leq \gamma(G') + 1$.

On the other hand, there exists a $\gamma(G)$-set D containing y and not containing z, w. Then $D - \{y\}$ is a dominating set of G' and hence $\gamma(G') \leq \gamma(G) - 1$.

Therefore $\gamma(G') = \gamma(G) - 1$. □

Lemma 4.7. *If G is a graph belonging to the family $\mathcal{F}$, then $x \in A(G)$ if and only if*

$$\gamma(G - x) < \gamma(G).$$

Proof. Let G be a graph belonging to the family $\mathcal{F}$. We use the induction on the number k of operations performed to construct the graph G. If $k = 0$, then $G \in \mathcal{F}_0$, and then by Lemma 4.4 we obtain that $\gamma(G - x) < \gamma(G)$ if and only if $x \in A(G)$.

Now assume that the result is true for every graph $G' = G_{k-1}$ of the family $\mathcal{F}$ constructed by $k-1$ operations, e.g. $\gamma(G'-x) < \gamma(G')$ if and only if $x \in A(G')$. Let $G = G_k$ be a graph of the family $\mathcal{F}$ constructed by k operations.

First assume that G is obtained from G' by Operation $\mathcal{T}_1$. That is G is obtained from G' by adding a path (w, y, z) and the edge vw to a vertex $v \in V(G')$ with $sta(v) = A$. Then $sta(w) = sta(y) = B$, and $sta(z) = A$. Thus, $A(G) = A(G') \cup \{z\}$.

By Lemma 4.6, $\gamma(G-x) = \gamma(G'-x)+1$. Let $x \in A(G)$. If $x \in A(G')$, then clearly any minimum dominating set of $G'-x$ may be extended to a dominating set of $G-x$ by adding to it y. Thus,

$$\gamma(G-x) \leq \gamma(G'-x) + 1 < \gamma(G') + 1 = \gamma(G)$$

and therefore $\gamma(G-x) < \gamma(G)$. If $x \in A(G) - A(G')$, then $x = z$. Since $\gamma(G'-v) < \gamma(G')$, there exists a dominating set of $G-z$ containing w and of cardinality $\gamma(G')$. Thus, by the induction hypothesis

$$\gamma(G-z) \leq \gamma(G') = \gamma(G) - 1$$

and therefore $\gamma(G-z) < \gamma(G)$.

Now let $x \notin A(G)$. Then $x \in B(G)$. If $x \in B(G')$, then by Lemma 4.6, $\gamma(G-x) = \gamma(G'-x)+1$. Thus, by the induction hypothesis

$$\gamma(G-x) = \gamma(G'-x) + 1 \geq \gamma(G') + 1 = \gamma(G)$$

and therefore $\gamma(G-x) \geq \gamma(G)$. If $x \in B(G) - B(G')$, then $x \in \{w, y\}$. Assume first $x = w$. Then $\gamma(G-w) = \gamma(G')+1 = \gamma(G)$. If $x = y$, then the situation is similar. Hence, if $x \in B(G)$ then removing x from G will not decrease the domination number.

Assume now G is obtained from G' by Operation $\mathcal{T}_2$. That is G is obtained from G' by adding a path (w, y) and the edge vw to a vertex $v \in V(G')$ with $sta(v) = B$. Then $sta(w) = B$, and $sta(y) = A$. Thus, $A(G) = A(G') \cup \{y\}$.

Any minimum dominating set of G' may be extended to a dominating set of G by adding to it w. Thus, $\gamma(G) \leq \gamma(G')+1$. On the other hand, suppose $\gamma(G) = \gamma(G')$. Without loss of generality we assume $w \in D$ and $y, v \notin D$, where D is a minimum dominating set of G. Then $D - \{w\}$ would be a dominating set of $G'-v$ of cardinality smaller that $|D|$, implying that $\gamma(G'-v) < \gamma(G')$, which is impossible, since by the induction hypothesis, $v \in B(G')$. Thus, we conclude that $\gamma(G) = \gamma(G')+1$.

Let $x \in A(G)$. If $x \in A(G')$, then by a similar reasoning as above we conclude that $\gamma(G-x) = \gamma(G'-x)+1$. Hence

$$\gamma(G-x) = \gamma(G'-x) + 1 < \gamma(G') + 1 = \gamma(G)$$

and therefore $\gamma(G-x) < \gamma(G)$. If $x \in A(G) - A(G')$, then $x = y$. Since $v \in B(G')$, there exists a vertex $u \in A(G')$ such that $uv \in E(G')$. Moreover, since $\gamma(G'-u) < \gamma(G')$, there exists a minimum dominating set of G' containing v, say D'. Then D' is a dominating set of $G-y$ of cardinality $\gamma(G')$, which implies that $\gamma(G-y) < \gamma(G)$.

Now let $x \notin A(G)$. Then $x \in B(G)$. If $x \in B(G')$, then by a similar reasoning as above we conclude that $\gamma(G-x) = \gamma(G'-x)+1$. Hence

$$\gamma(G-x) = \gamma(G'-x) + 1 \geq \gamma(G') + 1 = \gamma(G)$$

and therefore $\gamma(G-x) \geq \gamma(G)$. If $x \in B(G) - B(G')$, then $x = w$. Then clearly $\gamma(G-w) = \gamma(G') + 1 = \gamma(G)$. Hence, if $x \in B(G)$ then removing x from G does not decrease the domination number. $\square$

Lemma 4.8. *If G is a graph belonging to the family $\mathcal{F}$, then*

$$\operatorname{msd}(G) = 3.$$

Proof. Let G be a graph belonging to the family $\mathcal{F}$. We use the induction on the number k of operations performed to construct the graph G. If $k = 0$, then $G \in \mathcal{F}_0$, and by Lemma 4.5 we obtain that $\operatorname{msd}(G) = 3$.

Now assume that the result is true for every graph $G' = G_{k-1}$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let $G = G_k$ be a graph of the family $\mathcal{F}$ constructed by k operations. It suffices to show that $\operatorname{msd}(G) = 3$, or equivalently, $\gamma(G_{e,2}) \leq \gamma(G)$ for any edge $e \in E(G)$.

Let us start assumming that G is obtained from G' by Operation $\mathcal{T}_1$. That is G is obtained from G' by adding a path (w, y, z) and the edge vw to a vertex $v \in V(G')$ with $sta(v) = A$. Then $sta(w) = sta(y) = B$, and $sta(z) = A$.

Let $e \in E(G)$. Assume additionally $e \in E(G')$. Then Lemma 4.6 implies that $\gamma(G') = \gamma(G) - 1$ and $\gamma(G'_{e,2}) = \gamma(G_{e,2}) - 1$. Since $\gamma(G'_{e,2}) \leq \gamma(G')$, the inequality $\gamma(G_{e,2}) \leq \gamma(G)$ easily follows. Assume now $e \in E(G) - E(G')$. Then $e \in \{vw, wy, yz\}$. Observe that graphs $G_{vw,2}$, $G_{wy,2}$ and $G_{yz,2}$ are isomorphic, so it suffices to consider the graph $G_{wy,2}$. Since $v \in A(G)$, Lemma 4.7 implies that $\gamma(G-v) < \gamma(G)$. Let D_0 be a minimum dominating set of $G - v$. Since y is a strong support vertex in $G - v$, $y \in D_0$. Then $D = D_0 \cup \{w\}$ is a minimum dominating set of G containing w and y. Moreover, D is a dominating set of $G_{wy,2}$, which implies that $\gamma(G_{e,2}) \leq \gamma(G)$ for each edge $e \in E(G)$.

Now assume G is obtained from G' by Operation $\mathcal{T}_2$. That is G is obtained from G' by adding a path (w, y) and the edge vw to a vertex $v \in V(G')$ with $sta(v) = B$. Then $sta(w) = B$, and $sta(y) = A$.

Let $e \in E(G)$. Assume additionally $e \in E(G')$. Then any minimum dominating set of $G'_{e,2}$ may be extended to a dominating set of $G_{e,2}$ by adding to it y. Hence, $\gamma(G_{e,2}) - 1 \leq \gamma(G'_{e,2})$. Since $\operatorname{msd}(G') = 3$, $\gamma(G'_{e,2}) = \gamma(G')$ and since $v \in B(G')$, $\gamma(G') \leq \gamma(G' - v)$. Moreover, there exists a minimum dominating set of G, say D, containing w and not containing v, y. Then $D - \{w\}$ is a dominating set of $G' - v$, impying that $\gamma(G' - v) \leq \gamma(G) - 1$. Summing up,

$$\gamma(G_{e,2}) - 1 \leq \gamma(G'_{e,2}) = \gamma(G') \leq \gamma(G' - v) \leq \gamma(G) - 1.$$

Assume now $e \in E(G) - E(G')$. Then $e \in \{vy, yz\}$. Observe that graphs $G_{vw,2}$ and $G_{wy,2}$ are isomorphic, so it suffices to consider the graph $G_{vw,2}$. Since $v \in B(G)$ there exists a vertex $z \in A(G)$ such that $vz \in E(G)$. Then Lemma 4.7 implies that $\gamma(G - z) < \gamma(G)$, so there exists a minimum dominating set of G, say D, containing v and w. Then D is a dominating set of $G_{vw,2}$, which implies that $\gamma(G_{e,2}) \leq \gamma(G)$ for each edge $e \in E(G)$. $\square$

Lemma 4.9. *If G is unicyclic and* $\operatorname{msd}(G) = 3$*, then G belongs to the family $\mathcal{F}$.*

Proof. Let G be a graph with $\operatorname{msd}(G) = 3$ and let $v \in V(G)$ be a leaf adjacent to $u \in V(G)$. Then $\gamma(G_{uv,2}) = \gamma(G)$ and thus by Lemma 4.6, $\gamma(G - v) = \gamma(G) - 1$. Therefore v is $\gamma(G)$-critical. Moreover, no vertex of G is adjacent to more than one leaf.

Let G be a unicyclic graph with $\operatorname{msd}(G) = 3$ and denote by $C_k = (c_1, c_2, \ldots, c_k)$ the unique cycle of G. For each vertex c_i, where $i = 1, 2, \ldots, k$, denote by $T(c_i)$ the tree rooted in c_i, that is the connected subgraph containing c_i and obtained from G by deleting edges $c_{i-1}c_i$ and c_ic_{i+1} (the indices are taken modulo k and added 1, if needed). Additionally, let $h(T(c_i))$ be the height of $T(c_i)$. We proceed by induction on the number vertices of G.

Case 1. If $|V(T(c_i))| = 1$ for each $i = 1, 2, \ldots, k$, then G is a cycle and by [4], G has $n = 4, 7, 10, \ldots$ vertices and thus, $G \in \mathcal{C} \subseteq \mathcal{F}$.
Case 2. If $|V(T(c_i))| \leq 2$ for each $i = 1, 2, \ldots, k$ and at least one tree, say $T(c_1)$, has exactly two vertices. If additionally $|V(T(c_i))| = 1$ for each $i = 2, 3, \ldots, k$, then it is easy to check that no such a graph have the multisubdivision number equal to three. Thus, $|V(T(c_1))| = 2$ and $|V(T(c_i))| = 2$ for at least one $i = 2, 3, \ldots, n$. Then G is a unicyclic graph such that each vertex belonging to the cycle is of degree 2 or is a support vertex of degree 3 and at least two vertices of the cycle are of degree 3.

Denote by w_1 the leaf adjacent to c_1. Let T be the tree obtained from G by removing vertices c_1, w_1 and attaching paths $(u_1, u_2), (v_1, v_2)$ and edges u_2c_2, v_2c_k. Since w_1 is a leaf, there exists a minimum dominating set of G, say D_G containing c_1 and not containing w_1. Thus, $(D_G - \{w_1\}) \cup \{u_2, v_2\}$ is a dominating set of T of cardinality $\gamma(G) + 1$. On the other hand, since v_1 and u_1 are of degree 1, there exists a minimum dominating set of T, say D_T, not containing u_1, v_1 and containing u_2, v_2. Then $(D_T - \{u_2, v_2\}) \cup \{c_1\}$ is a dominating set of G of cardinality $\gamma(T) - 1$. We conclude $\gamma(T) = \gamma(G) + 1$.

Now let $e \in E(T) \cap E(G)$ and let D_2 be a minimum dominating set of $G_{e,2}$ containing c_1. Then $(D_2 - \{c_1\}) \cup \{u_2, v_2\}$ is a dominating set of $T_{e,2}$. Therefore

$$\gamma(T_{e,2}) \leq \gamma(G_{e,2}) + 1 = \gamma(G) + 1 = \gamma(T) \tag{4.1}$$

and since subdividing an egde can not decrease the domination number of a graph, $\gamma(T_{e,2}) = \gamma(T)$. Assume $e \in \{u_1u_2, u_2c_2\}$. Observe that graphs $T_{u_1u_2,2}$ and $T_{u_2c_2,2}$ are isomorphic, so it suffices to consider the graph $T_{u_2c_2,2}$. Let D_3 be a minimum dominating set of $G_{c_1c_2,2}$ containing c_1. Then $(D_3 - \{c_1\}) \cup \{u_2, v_2\}$ is a dominating set of $T_{u_2c_2,2}$. Therefore

$$\gamma(T_{u_2c_2,2}) \leq \gamma(G_{c_1c_2,2}) + 1 \leq \gamma(G) + 1 = \gamma(T)$$

and again $\gamma(T_{e,2}) = \gamma(T)$. The situation when $e \in \{v_1v_2, v_2c_k\}$ is similar and thus is ommited. We conclude that $\operatorname{msd}(T) = 3$. Then Corollary 3.3 implies that $T \in \mathcal{T}$. Since G is a unicyclic graph such that each vertex belonging to the cycle is of degree 2 or is a support vertex of degree 3 and at least two vertices of the cycle are of degree 3, Observation 3.1 implies that on the (u_1, v_1)-path in T exists a vertex of degree 3 labelled B. Therefore G may be obtained from T and so G belongs the Subclass $\mathcal{B} \subseteq \mathcal{F}_0$.

Therefore we conclue that if G is a unicyclic graph with $\mathrm{msd}(G) = 3$ and $|V(T(c_1))| = 2$, then G belongs to the family $\mathcal{F}$.
Case 3. If $|V(T(c_i))| \leq 3$ for each $i = 1, 2, \ldots, k$ and at least one tree, say $T(c_1)$, has exactly three vertices. If $h(T(c_1)) = 1$, then c_1 is adjacent to more than one leaf and it is easy to check that no such a graph have the multisubdivision number equal to three. Thus $h(T(c_1)) = 2$.

Denote by w_1 the leaf adjacent to w_2 and let w_2 be adjacent to c_1. Let T be the tree obtained from G by removing from G vertices c_1, w_2, w_1 and attaching paths $(u_1, u_2, u_3), (v_1, v_2, v_3)$ and edges u_3c_2, v_3c_k. Since $\gamma(G - w_1) < \gamma(G)$, there exists a minimum dominating set of G, say D_G, containing w_2 and not containing c_1, w_1. Thus, $(D_G - \{w_2\}) \cup \{u_2, v_2\}$ is a dominating set of T of cardinality $\gamma(G) + 1$. On the other hand, there exists a minimum dominating set of T, say D_T, not containing u_1, v_1, u_3, v_3 and containing u_2, v_2. Then $(D_T - \{u_2, v_2\}) \cup \{w_2\}$ is a dominating set of G of cardinality $\gamma(T) - 1$. We conclude $\gamma(T) = \gamma(G) + 1$.

Now let $e \in E(T) \cap E(G)$ and let D_2 be a minimum dominating set of $G_{e,2}$ containing w_2. Then $(D_2 - \{w_2\}) \cup \{u_2, v_2\}$ is a dominating set of $T_{e,2}$. Therefore

$$\gamma(T_{e,2}) \leq \gamma(G_{e,2}) + 1 = \gamma(G) + 1 = \gamma(T) \tag{4.2}$$

and since subdividing an egde can not decrease the domination number of a graph, $\gamma(T_{e,2}) = \gamma(T)$. Assume $e \in \{u_1u_2, u_2u_3, u_3c_2\}$. Observe that graphs $T_{u_1u_2,2}$, $T_{u_2u_3,2}$ and $T_{u_3c_2,2}$ are isomorphic, so it suffices to consider the graph $T_{u_2u_3,2}$. Let D_3 be a minimum dominating set of $G_{c_1w_2,2}$ containing w_2. Then $c_1 \in D_3$ and $(D_3 - \{w_2, c_1\}) \cup \{u_2, u_3, v_2\}$ is a dominating set of $T_{u_2u_3,2}$. Therefore

$$\gamma(T_{u_2u_3,2}) \leq \gamma(G_{c_1w_2,2}) + 1 \leq \gamma(G) + 1 = \gamma(T)$$

and again $\gamma(T_{e,2}) = \gamma(T)$. The situation when $e \in \{v_1v_2, v_2v_3, v_3c_k\}$ is similar and thus is ommited. We conclude that $\mathrm{msd}(T) = 3$. Then Corollary 3.3 implies that $T \in \mathcal{T}$. Therefore G may be obtained from T and so G belongs the Subclass $\mathcal{A} \subseteq \mathcal{F}_0$.

We conclude that if G is a unicyclic graph with $\mathrm{msd}(G) = 3$ and $|V(T(c_1))| = 3$, then G belongs to the family $\mathcal{F}$.
Case 4. If $|V(T(c_i))| \geq 4$ for some $i = 1, 2, \ldots, k$. Without loss of generality let $T(c_1)$ has at least four vertices. Then $h(T(c_1)) \geq 2$. Assume the result is true for every unicyclic graph G' with $\mathrm{msd}(G') = 3$ and with $|V(G')| < |V(G)|$. Denote by $P = (w_1, w_2, \ldots, c_1)$ a longest path in $T(c_1)$ starting at c_1. Since G does not have a strong support vertex, $d(w_2) = 2$.

If $d(w_3) = 2$, then Lemma 5.1 implies, that for $G' = G - \{w_1, w_2, w_3\}$ we have $\mathrm{msd}(G') = 3$. Since G' is unicyclic and $|V(G')| < |V(G)|$, the induction hypothesis implies that $G' \in \mathcal{F}$. Consider the graph $G_{w_2w_3,2}$. Since $\mathrm{msd}(G) = 3$, $\gamma(G) = \gamma(G_{w_2w_3,2})$. Since w_2 is a support vertex in $G_{w_2w_3,2}$, w_2 belongs to a minimum dominating set of $G_{w_2w_3,2}$. Without loss of generality w_3 also belongs to a minimum dominating set of $G_{w_2w_3,2}$. For these reasons there exists a minimum dominating set of $G_{w_2w_3,2}$, say D, containing w_2 and w_3. Therefore $D - \{w_2, w_3\}$ is a dominating set of $G - \{w_1, w_2, w_3, w_4\} = G' - w_4$ of cardinality $\gamma(G) - 2$. Thus, $\gamma(G' - w_4) \leq \gamma(G) - 2 = \gamma(G') - 1$. Hence w_4 is a critical vetex, so $w_4 \in A(G')$ and therefore G may be obtained from $G' \in \mathcal{F}$ by Operation $\mathcal{T}_1$. For this reason $G \in \mathcal{F}$.

Consider the situation when $d(w_3) > 2$ and w_3 is a support vertex adjacent to the leaf z_1. Denote $G' = G - \{w_1, w_2\}$. Then Lemma 5.6 implies that $\mathrm{msd}(G') = 3$. Thus by the induction hypothesis, $G' \in \mathcal{F}$. Let D' be a minimum dominating set of $G' - w_3$. Then $z_1 \in D'$ and for this reason D' is also a dominating set in G'. Hence $\gamma(G') \leq \gamma(G' - w_3)$ and thus $w_3 \in B(G')$. Therefore G may be obtained from $G' \in \mathcal{F}$ by Operation $\mathcal{T}_2$. For this reason $G \in \mathcal{F}$.

Now consider the situation when $d(w_3) > 2$ and w_3 is not a support vertex. Then there exists in $T(c_1)$ a longest path $P' = (z_1, z_2, w_3, \ldots, c_1)$ such that $z_2 \neq w_2$ and $z_1 \neq w_1$. Denote $G' = G - \{w_1, w_2\}$. Then Lemma 5.4 implies that $\mathrm{msd}(G') = 3$. Thus by the induction hypothesis, $G' \in \mathcal{F}$. Let D' be a minimum dominating set of $G' - w_3$. Then we may assume that $z_2 \in D'$ and for this reason D' is also a dominating set in G'. Hence $\gamma(G') \leq \gamma(G' - w_3)$ and thus $w_3 \in B(G')$. Therefore G may be obtained from $G' \in \mathcal{F}$ by Operation $\mathcal{T}_2$. For this reason $G \in \mathcal{F}$. $\square$

Here we present the main result of this paper which is an immediate consequence of Lemma 4.8 and Lemma 4.9.

Theorem 4.10. *Let G be a unicyclic graph. Then* $\mathrm{msd}(G) = 3$ *if and only if G belongs to the family $\mathcal{F}$.*

5. LEMMAS USED IN THE PROOF OF THE LEMMA 4.9

In this section we present lemmas used in the proof of the Lemma 4.9.

Lemma 5.1. *Let G be a graph with* $\mathrm{msd}(G) = 3$ *and let $w_1, w_2, w_3 \in V(G)$ induce a path $P_3 = (w_1, w_2, w_3)$ in G, where $d_G(w_1) = 1$, $d_G(w_2) = d_G(w_3) = 2$. Denote $G' = G - \{w_1, w_2, w_3\}$. Then*

$$\mathrm{msd}(G') = 3.$$

Proof. Let G, w_1, w_2, w_3 be as in the assumptions of the lemma. Denote $G' = G - \{w_1, w_2, w_3\}$. Let e be any edge of G'. Then e also belongs to $E(G)$.

Clearly, $\gamma(G') \leq \gamma(G'_{e,2})$. Moreover, Lemma 4.6 implies that $\gamma(G'_{e,2}) + 1 = \gamma(G_{e,2})$ and $\gamma(G) = \gamma(G') + 1$. Thus

$$\gamma(G) = \gamma(G') + 1 \leq \gamma(G'_{e,2}) + 1 = \gamma(G_{e,2}). \tag{5.1}$$

Since $\mathrm{msd}(G) = 3$, $\gamma(G) = \gamma(G_{e,2})$ and we have equalities in the chain (5.1). Therefore $\gamma(G') = \gamma(G'_{e,2})$ and for this reason $\mathrm{msd}(G') = 3$. $\square$

Lemma 5.2. *Let G be a graph such that $w_1, w_2, w_3, z_1, z_2 \in V(G)$ induce a path $P_5 = (w_1, w_2, w_3, z_2, z_1)$ in G, where $d_G(w_1) = d_G(z_1) = 1$, $d_G(w_2) = d_G(z_2) = 2$ and $d_G(w_3) \geq 3$. Then*

$$\gamma(G - \{w_1, w_2\}) = \gamma(G) - 1.$$

Proof. Let $G, w_1, w_2, w_3, z_1, z_2$ be as in the assumption of the lemma. Denote $G' = G - \{w_1, w_2\}$.

Any $\gamma(G')$-set may be expanded to a dominating set of G by adding to it w_2. Thus, $\gamma(G) \leq \gamma(G') + 1$.

On the other hand, there exists a $\gamma(G)$-set D containing w_2, z_2 and not containing w_1, w_3, z_1. Then $D - \{w_2\}$ is a dominating set of G' and hence $\gamma(G') \leq \gamma(G) - 1$.

Therefore $\gamma(G') = \gamma(G) - 1$. □

Lemma 5.3. *Let G be a graph such that $u, v, w, x, y, z_1, z_2 \in V(G)$ induce a path $P_7 = (w_1, w_2, w_3, x, y, z_2, z_1)$ in G, where $d_G(w_1) = d_G(z_1) = 1$, $d_G(w_2) = d_G(x) = d_G(y) = d_G(z_2) = 2$ and $d_G(w_3) \geq 3$. Then*

$$\gamma(G - \{w_1, w_2\}) = \gamma(G) - 1.$$

Proof. Let $G, w_1, w_2, w_3, x, y, z_2, z_1$ be as in the assumption of the lemma. Denote $G' = G - \{w_1, w_2\}$.

Any $\gamma(G')$-set may be expanded to a dominating set of G by adding to it w_2. Thus, $\gamma(G) \leq \gamma(G') + 1$.

On the other hand, there exists a $\gamma(G)$-set D containing w_2, z_2 and not containing w_1, z_1. To dominate x, we may assume that $w_3 \in D$. Then $D - \{w_2\}$ is a dominating set of G' and hence $\gamma(G') \leq \gamma(G) - 1$.

Therefore $\gamma(G') = \gamma(G) - 1$. □

Lemma 5.4. *Let G be a graph with* $\mathrm{msd}(G) = 3$ *and let $w_1, w_2, w_3, z_1, z_2 \in V(G)$ induce a path $P_5 = (w_1, w_2, w_3, z_2, z_1)$ in G, where $d_G(w_1) = d_G(z_1) = 1$, $d_G(w_2) = d_G(z_2) = 2$ and $d_G(w_3) \geq 3$. Then*

$$\mathrm{msd}(G - \{w_1, w_2\}) = 3.$$

Proof. Let $G, w_1, w_2, w_3, z_1, z_2$ be as in the assumptions of the lemma. Denote $G' = G - \{w_1, w_2\}$. Let e be any edge of G'. Then e also belongs to $E(G)$.

Clearly, $\gamma(G') \leq \gamma(G'_{e,2})$. If $e \in \{w_3z_2, z_1z_2\}$, then by Lemma 5.3, $\gamma(G'_{e,2}) = \gamma(G_{e,2}) - 1$. Similarly, if $e \notin \{w_3z_2, z_1z_2\}$, then Lemma 5.2 implies the same equality. Therefore,

$$\gamma(G) = \gamma(G') + 1 \leq \gamma(G'_{e,2}) + 1 = \gamma(G_{e,2}). \tag{5.2}$$

Since $\mathrm{msd}(G) = 3$, $\gamma(G) = \gamma(G_{e,2})$ and we have equalities in the chain (5.2). For this reason $\gamma(G') = \gamma(G'_{e,2})$ and hence $\mathrm{msd}(G') = 3$. □

Lemma 5.5. *Let G be a graph such that $w_1, w_2, w_3, z_1 \in V(G)$ induce a path $P_4 = (w_1, w_2, w_3, z_1)$ in G, where $d_G(w_1) = d_G(z_1) = 1$, $d_G(w_2) = 2$ and $d_G(w_3) \geq 3$. Then*

$$\gamma(G - \{w_1, w_2\}) = \gamma(G) - 1.$$

Proof. Let G, w_1, w_2, w_3, z_1 be as in the assumption of the lemma. Denote $G' = G - \{w_1, w_2\}$.

Any $\gamma(G')$-set may be expanded to a dominating set of G by adding to it w_2. Thus, $\gamma(G) \leq \gamma(G') + 1$.

On the other hand, there exists a $\gamma(G)$-set D containing w_2, w_3 and not containing w_1, z_1. Then $D - \{w_2\}$ is a dominating set of G' and hence $\gamma(G') \leq \gamma(G) - 1$.

Therefore $\gamma(G') = \gamma(G) - 1$. □

Lemma 5.6. *Let G be a graph with* $\mathrm{msd}(G) = 3$ *and let* $w_1, w_2, w_3, z_1 \in V(G)$ *induce a path* $P_4 = (w_1, w_2, w_3, z_1)$ *in* G, *where* $d_G(w_1) = d_G(z_1) = 1$, $d_G(w_2) = 2$ *and* $d_G(w_3) \geq 3$. *Then*

$$\mathrm{msd}(G - \{w_1, w_2\}) = 3.$$

Proof. Let G, w_1, w_2, w_3, z_1 be as in the assumptions of the lemma. Denote $G' = G - \{w_1, w_2\}$ and denote by w' a neighbour of w_3 different from w_2 and z_1.

Now, let e be any edge of G'. Then e also belongs to $E(G)$. Clearly, $\gamma(G') \leq \gamma(G'_{e,2})$.

Lemma 5.5 implies that $\gamma(G') = \gamma(G) - 1$ and if $e \neq w_3 z_1$, then $\gamma(G'_{e,2}) = \gamma(G_{e,2}) - 1$.

If $e = w_3 z_1$, then any $\gamma(G'_{w_3 z_1,2})$-set may be expanded to a dominating set of $G_{w_3 z_1,2}$ by adding to it w_2. Thus, $\gamma(G'_{w_3 z_1,2}) \leq \gamma(G_{w_3 z_1,2}) - 1$. On the other hand, since $\gamma(G) = \gamma(G_{w_w w',2})$, there exists a $\gamma(G)$-set D containing w_2, w_3, w' and not containing w_1, z_1. Thus $(D - \{w_2, w_3\}) \cup \{z'\}$, where z' is the neighbour of z_1 in $G'_{w_3 z_1,2}$, is a dominating set of $G'_{w_3 z_1,2}$. Hence $\gamma(G'_{w_3 z_1,2}) \leq \gamma(G_{w_3 z_1,2}) - 1$. Therefore, $\gamma(G'_{e,2}) = \gamma(G_{e,2}) - 1$ for any edge $e \in E(G')$.

In summary we obain the following chain

$$\gamma(G) = \gamma(G') + 1 \leq \gamma(G'_{e,2}) + 1 = \gamma(G_{e,2}). \tag{5.3}$$

Since $\mathrm{msd}(G) = 3$, $\gamma(G) = \gamma(G_{e,2})$ and we have equalities in the chain (5.3). In particular, $\gamma(G') = \gamma(G'_{e,2})$ implying that $\mathrm{msd}(G') = 3$. □

6. DISCUSSION

In this paper, we constructively characterized all connected unicyclic graphs with the domination multisubdivision number equal to 3. The results we obtained using the characterization of trees with the domination subdivision number equal to 3 described by Aram, Sheikholeslami and Favaron [1].

Future research on our topic could involve the following questions. Recall that for each tree T with at least three vertices, $\mathrm{sd}(T) = \mathrm{msd}(T)$. It would be interesting to know if this is true also for unicyclic graphs.

Another question is whether the descibed operation applied to unicyclic graphs described in this paper would produce classes of graphs with more than one cycle and with the domination multisubdivision number equal to 3.

REFERENCES

[1] H. Aram, S.M. Sheikholeslami, O. Favaron, *Domination subdivision number of trees*, Discrete Math. **309** (2009), 622–628.

[2] D. Avella-Alaminos, M. Dettlaff, M. Lemańska, R. Zuazua, *Total domination multisubdivision number of a graph*, Discuss. Math. Graph Theory **35** (2015), 315–327.

[3] S. Benecke, C. M. Mynhardt, *Trees with domination subdivision number one*, Australasian J. Combin. **42** (2008), 201–209.

[4] M. Dettlaff, J. Raczek, J. Topp, *Domination subdivision and domination multisubdivision numbers of graphs*, to appear in Discuss. Math. Graph Theory.

[5] O. Favaron, H. Karami, R. Khoeilar, S.M. Sheikholeslami, *On the total domination subdivision number in some classes of graphs*, Journal od Combinatorial Optimization **20** (2010), 76–84.

[6] O. Favaron, H. Karami, S.M. Sheikholeslami, *Disproof of a conjecture on the subdivision domination number of a graph,* Graphs and Combinatorics **24** (2008), 309–312.

[7] J.H. Hattingh, E.J. Joubert, M. Loizeaux, A.R. Plummer, L.C. van der Merwe, *Restrained domination in unicyclic graphs*, Discuss. Math. Graph Theory **29** (2009), 71–86.

[8] T.W. Haynes, M.A. Henning, L. Hopkins, *Total domination subdivision numbers of trees*, Discrete Math. **286** (2004), 195–202.

[9] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[10] N.J. Rad, *Characterization of total restrained domination edge critical unicyclic graphs*, Australas. J. Combin. **47** (2010), 77–82.

[11] P. Roushini Leely Pushpama, T.N.M. Malini Mai, *Roman domination in unicyclic graphs*, Journal of Discrete Mathematical Sciences and Cryptography **15** (2012), 237–257.

[12] E. Sampathkumar, P.S. Neeralagi, *Domination and neighbourhood critical, fixed, free, and totally free points*, Sankhyä **54** (1992), 403–407.

[13] S. Velammal, *Studies in graph theory: covering, independence, domination and related topics*, Ph.D. Thesis, Manonmaniam Sundaranar University, Tirunelveli, 1997.

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Received: January 1, 2017.
Revised: December 23, 2017.
Accepted: January 9, 2018.