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On rotary inertia of microstuctured beams and variations thereof

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ABSTRACT

We discuss the classic rotary inertia notion and extend it for microstructured beams introducing new microinertia parameters as an additional dynamic response to microstructure changes. Slender structures made of beam- or platelet-lattice metamaterials may exhibit not only large translations and rotations but also general deformations of inner structure. Here we considered a few examples of beam-like structures and derive their inertia properties which include effective mass density, rotary inertia and microinertia. Extended dynamic characteristics related to enhanced kinematics may be crucial for description of origami-like structures or other beam-lattice metamaterials.

0. Introduction

After works by Bresse [1], Rayleigh [2], Timoshenko [3], and Mindlin [4] the notion of rotary inertia became common in structural mechanics, see also [5–7] for more detail. By definition, the rotary inertia determines a part of kinetic energy related to rotational motions. From the mathematical point of view rotary inertia can be defined as the second moment of mass density distribution in a cross-sectional area. Restricting to beams with solid cross-sectional area one can see that rotary inertia has an order of h^3 where h is a thickness. As a result, in some cases it could be neglected. On the other hand, new microstructured material such as beam-lattice composites and metamaterials [8–10], origami/kirigami structures [11–14] may essentially extend dynamic properties of slender structures including mass density distribution moments of higher order as well as dynamic response to additional kinematic variables such as warping or microdeformations.

In addition to classic and nonclassic beam models it is worth to mention other one-dimensional and quasi one-dimensional discrete and continuum models such as chains and lattice-like structures with short- and long-range interactions, see e.g. [15–24] and the references therein. They may exhibit a rather complex dynamics. For example, mass-in-mass chains [25] possesses to tune band gaps due to relative motions of additional masses. In [26] it was shown that the presence of torsional springs in an origami-like structure may essentially affect dispersion relations. Discussing rotary inertia it worth to mention here gyroscopic systems, where dynamics of spinners plays a crucial role [27–30].

The main aim of the paper is to bring attention to non-classical kinetic constitutive relations, that is to possible forms of kinetic energy of microstructured materials. The paper is organized as follows. First, we briefly recall equations of dynamics and possible forms of kinetic energy for systems with finite number of degrees of freedom, beams, plates and shells in Section 1. Then in Section 2 we give an example of a system with one degree of freedom with essential non-linearity in inertia. Section 3 presents an example of a beam-like structure with dominant rotary inertia proportional to h^2. Both discrete and continuum models were presented. In Section 4 we combine models introduced previously and discuss a motion of a pantographic beam using again both discrete and continuum models. The last example shows an importance of microinertia. Finally, in conclusions we briefly discussed other models of continua which involves rotary inertia and their extensions with their similarities to beam models.

1. Overview of rotary inertia

1.1. Discrete systems

Dynamics of a discrete conservative system can be described using the Euler–Lagrange equation, see [31,32],

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, n, \tag{1}$$

where $L = L(q_i, \dot{q}_i)$ is a Lagrangian given as a function of generalized coordinates $q_i = q_i(t)$ and its velocities $\dot{q}_i = \dot{q}_i(t)$, and the overdot stands for the derivative with respect to time t.

At this level of generality it is difficult to extract a part related to rotations or to other non-translational motions. Moreover, even kinetic

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energy is not defined yet. For many mechanical systems L can be represented as

$$L = K(q_i, \dot{q}_i) - W(q_i),$$

where K and W are kinetic and potential energies, respectively.

In particular, in rigid body dynamics [31] we first meet tensors of inertia as a measure of rotary inertia. Indeed, here a kinetic energy is given by

$$K = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + \mathbf{v} \cdot \mathbf{J}_1 \cdot \boldsymbol{\omega}, \tag{2}$$

where $\mathbf{v}$ and $\boldsymbol{\omega}$ are linear and angular velocities, respectively, M is the total mass, $\mathbf{J}$ and $\mathbf{J}_1$ are second-order tensors of inertia, and $\cdot$ is the dot product. By definition $\mathbf{J}$ is positive definite, whereas $\mathbf{J}_1$ vanishes if we choose the center of mass as an origin of the frame. Both inertia tensors are responsible for the rotational part of the kinetic energy and play a crucial role in rigid body dynamics. In fact, with M they characterize a mass distribution in the considered rigid body. In the case of finite rotations $\mathbf{J}$ and $\mathbf{J}_1$ depend on rotations as follows

$$\mathbf{J} = \mathbf{Q} \cdot \bar{\mathbf{J}} \cdot \mathbf{Q}^T, \quad \mathbf{J}_1 = \mathbf{Q} \cdot \bar{\mathbf{J}}_1 \cdot \mathbf{Q}^T, \tag{3}$$

where $\mathbf{Q} = \mathbf{Q}(t)$ is a tensor of rotations, and $\bar{\mathbf{J}}$ and $\bar{\mathbf{J}}_1$ are constant referential tensors of inertia. Eq. (3) brings an essential nonlinearity in the equations of motion of a rigid body.

1.2. Beams, plates and shells

The classical Bernoulli–Euler beam model is based on the equation

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \ddot{w} = 0, \tag{4}$$

where $w = w(x,t)$ is the deflection, x is the axial coordinate, E is Young's modulus, I is the moment of inertia, A is the cross-sectional area, and ρ is the mass density. The Bernoulli–Euler model does not take into account rotary inertia.

In order to take it into account, Bresse and Rayleigh modified this model as follows

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \ddot{w} - \rho I \frac{\partial^2 \ddot{w}}{\partial x^2} = 0. \tag{5}$$

Discrete flexural structure considered in [21] results in this model under the long-wave regime, i.e. a Rayleigh beam resting on a Winkler foundation.

Finally, Timoshenko and Ehrenfest [7] extended the previous model taking into account shear deformations. Vibrations of the Timoshenko beam are described through two equations

$$EI \frac{\partial^2 \psi}{\partial x^2} + \kappa \mu A \left(\frac{\partial w}{\partial x} - \psi \right) = \rho I \ddot{\psi}, \tag{6}$$

$$\kappa \mu A \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) = \rho A \ddot{w}, \tag{7}$$

where ψ is the angle of rotation of the beam cross-section, μ is the shear modulus and κ is the shear correction factor. Excluding ψ from (6) and (7) we get the following equation

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \ddot{w} - \rho \left(I + \frac{EI}{\kappa \mu} \right) \frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\rho^2 I}{\kappa \mu} \ddddot{w} = 0. \tag{8}$$

Note that in all differential equations presented above we assume that external loads are absent.

Kinetic energies for these models have the form

$$K_{BE} = \frac{1}{2} \int_0^{\ell} \rho A \dot{w}^2 \, dx, \tag{9}$$

$$K_R = \frac{1}{2} \int_0^{\ell} \rho \left[A \dot{w}^2 + I \left(\frac{\partial \dot{w}}{\partial x} \right)^2 \right] dx, \tag{10}$$

$$K_T = \frac{1}{2} \int_0^{\ell} \rho \left[A \dot{w}^2 + I \dot{\psi}^2 \right] dx, \tag{11}$$

where ℓ is the beam length.

Considering homogeneous beams we see that the rotary inertia has an order h^3 where h is the thickness of the beam. For example, for a beam with a rectangular cross-section of width b and thickness h we have that $A = bh$ and $I = bh^3/12$. This observation may lead to the conclusion that rotary inertia may be negligible at least in some cases. For example, discussions of these cases can be found in [5–7,33], see also [34]. On the other hand, this is not the case for thin-walled beams, see e.g [35]. For example, in [36] it was shown that under some conditions the neglecting of rotary inertia leads to about 170% of the relative error in natural frequencies.

Considering microstructured materials it is worth to mention here space-fractional non-local models of beams based on fractional derivatives [37,38], the foundations of the theory could be found in [39, 40]. Here the classic form of kinetic energy as (10) or(11) was used, see e.g. [41,42]. It is interesting that here instead of (8) we have a similar equation but with mixed time- and space-fractional derivatives, see [42] for more details.

For spatial motions of beams instead of scalar measure of rotary inertia I we have to consider one or a few tensors of inertia, see e.g. [34,43–46]. Let briefly recall the Cosserat curve model (directed curve) which is often used for modeling of beams, see e.g. [43,44,47]. Within the model two independent kinematical fields are introduced that are the vector of displacements $\mathbf{u} = \mathbf{u}(x,t)$ and orthogonal tensor $\mathbf{Q} = \mathbf{Q}(x,t)$, where x is the Lagrangian axial coordinate. $\mathbf{Q}$ describes rotations of beam cross-section. As a result, kinetic energy is a quadratic form which depends on linear $\mathbf{v} = \dot{\mathbf{u}}$ and angular $\boldsymbol{\omega}$ velocities

$$K_C = \frac{1}{2} \int_0^{\ell} \left(\rho \mathbf{v} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + 2 \mathbf{v} \cdot \mathbf{J}_1 \cdot \boldsymbol{\omega} \right) dx. \tag{12}$$

Here ρ is the referential linear mass density (mass per unit length in a reference placement), $\boldsymbol{\omega}$ follows from the formula $\dot{\mathbf{Q}} = \mathbf{Q} \times \boldsymbol{\omega}$ where $\times$ is the cross product. Inertia tensors $\mathbf{J}$ and $\mathbf{J}_1$ depend on $\mathbf{Q}$ as in Eq. (3). For identification of inertia tensors we refer to [48–50].

It is worth also to mention the higher-order theories of beams such as discussed in [51] where the three-dimensional field of displacements was represented in series with respect to cross-sectional coordinates. Here equations of motions includes of higher moments such as

$$I^{(m,n)} = \iint_A \rho y^m z^n \, dA,$$

where y and z are Lagrangian coordinates in the cross-sectional area A, m and n are integers. Note that with these notations $A = I^{(0,0)}$ and $I = I^{(0,1)}$.

In the case of plates and shells we have situation with rotary inertia similar to beams. For example, equation of motion of a Kirchhoff plate with rotary inertia takes the form [34]

$$D\Delta^2 w + \rho h \ddot{w} - \rho \frac{h^3}{12} \Delta \ddot{w} = 0, \tag{13}$$

where $w = w(x,y,t)$ is the deflection, D is the bending stiffness, Δ is the two-dimensional Laplace operator, and h is the plate thickness. Eq. (13) is 2D counterpart of (5), whereas Mindlin's equations of motion [4,34] are analogous to Timoshenko and Ehrenfest model (6), (7). For higher order theories of plates one should also consider higher-order moments, see [52].

Within nonlinear resultant six-parametric theory of shells the rotary inertia were discussed in [53], see also [54,55]. The most general form of a kinetic energy density was proposed in [56]. It is introduced as a positive quadratic form of linear $\mathbf{v}$ and angular $\boldsymbol{\omega}$ velocities

$$K_s = \frac{1}{2} \iint_A (\mathbf{v} \cdot \mathbf{J}_0 \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{J}_1 \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathbf{J}_2 \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega}) \, dA, \tag{14}$$

$$\mathbf{J}_0 = \mathbf{J}_0^T, \quad \mathbf{J}_1^T = \mathbf{J}_2, \quad \mathbf{J}^T = \mathbf{J},$$

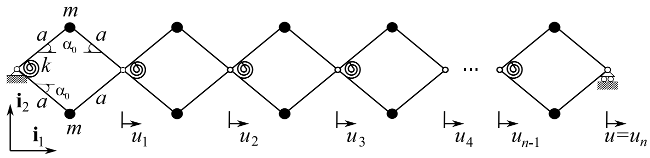

Fig. 1. Pantographic bar.

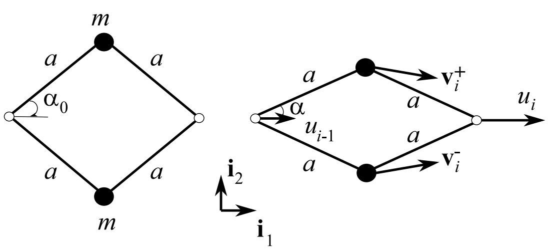

Fig. 2. Deformation of a pantographic cell: initial and current placements.

where A is a base shell surface in a reference placement, $\mathbf{J}_0$, $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}$ are tensors of inertia which maybe a rather complex functions of rotations, strains, gradient of strains, and strain rates. Obviously, any enhancement of shell kinematics requires additional terms in the kinetic energy, in general.

2. Dynamics of pantographic bar

First, let us consider motions of a pantographic bar shown in Fig. 1. The bar consists of n cells, whereas each cell consists of two rigid bars of length $2a$ connected to each other through a hinge with a rotational spring of stiffness k. At the ends of the bars masses m are attached. Without loss of generality we assume that mass of bars is negligible. Obviously, as bars are rigid the pantographic bar has one degree of freedom. For deformable pantographic materials we refer to [9] where also some applications are considered. As a kinematical descriptor we use the current angle $\alpha = \alpha(t)$ with initial angle α_0, see Fig. 2. So we have formulae

$$u_i = 2ai(\cos\alpha - \cos\alpha_0), \quad i = 1, 2, \dots, n, \tag{15}$$

$$\begin{aligned}\mathbf{v}_i^{\pm} =& (2i-1)a(\cos\alpha - \cos\alpha_0)\mathbf{i}_1\\ &\pm a(\sin\alpha - \sin\alpha_0)\mathbf{i}_2,\end{aligned} \tag{16}$$

where u_i is a displacement of ith hinge, $\mathbf{v}_i^{\pm}$ are vectors of displacements of upper and lower masses, respectively, and $\mathbf{i}_1$ and $\mathbf{i}_2$ are Cartesian unit base vectors. The corresponding velocities are given by the formulae

$$\dot{u}_i = -2ai\sin\alpha\dot{\alpha}, \tag{17}$$

$$\dot{\mathbf{v}}_i^{\pm} = -(2i-1)a\sin\alpha\dot{\alpha}\mathbf{i}_1 \pm a\cos\alpha\dot{\alpha}\mathbf{i}_2. \tag{18}$$

Elastic energy is given by the equation

$$W = \frac{1}{2}kn(\alpha - \alpha_0)^2, \tag{19}$$

which corresponds to the energy of n rotational springs.

Kinetic energy has more complex form

$$\begin{aligned}K =& \sum_{i=1}^{n} m|\mathbf{v}_i^{\pm}|^2\\ =& \sum_{i=1}^{n} ma^2\left[(2i-1)^2\sin^2\alpha + \cos^2\alpha\right]\dot{\alpha}^2\\ =& mna^2\left[\frac{4}{3}(n^2-1)\sin^2\alpha + 1\right]\omega^2\\ =& \frac{M}{2}a^2\left[\frac{4}{3}(n^2-1)\sin^2\alpha + 1\right]\omega^2,\end{aligned} \tag{20}$$

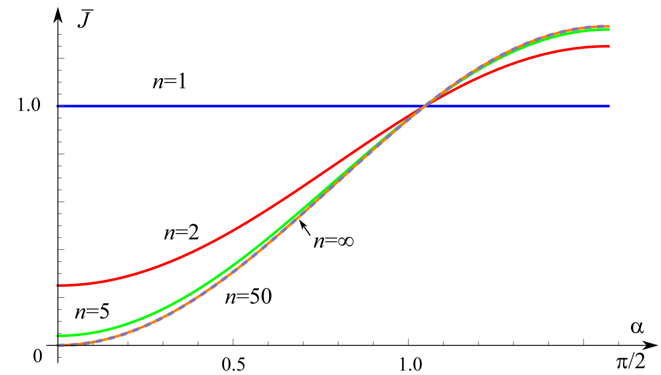

Fig. 3. Inertia parameter vs. α. Here $\bar{J} \equiv J/Ma^2n^2$ and dashed curve relates to $n \to \infty$.

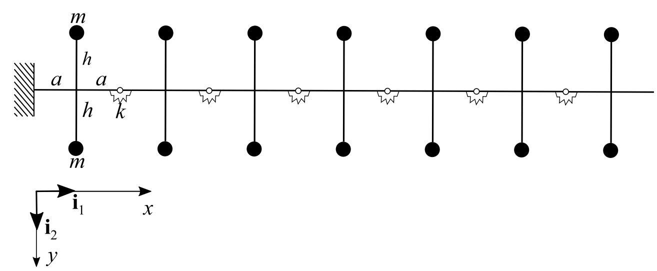

Fig. 4. Cross-beam.

where $M = 2mn$ is the total mass of the bar and $\omega = \dot{\alpha}$. It is natural to call the factor

$$J = Ma^2\left[\frac{4}{3}(n^2-1)\sin^2\alpha + 1\right] \tag{21}$$

the rotary inertia as it relates to rotations of the bars. Obviously, as J depends nonlinearly on α we have a dependence of the rotary inertia on deformations.

Equation of motion with respect to α follows from (1) with $L = K - W$.

In the case of large number of cells, i.e. at $n \to \infty$, we have an asymptotic formula

$$J_\infty = \frac{4}{3}Ma^2n^2\sin^2\alpha. \tag{22}$$

Graphs of normalized inertia $\bar{J} \equiv J/Ma^2n^2$ are given in Fig. 3 for various values of n. Here the dashed curve relates to $\bar{J}_\infty$, it almost coincides with $\bar{J}$ at $n = 50$ (orange line).

So one can see that inertia properties may essentially depend on microstructure, i.e. on the number of cells, and on kinematical descriptor itself. Let us note that we cannot say that the pantographic bar possesses rotary inertia as it is one-dimensional system which does not exhibit any rotation at the macroscale. Nevertheless the rather complex form of inertia corresponds to relative rotations of masses about the cell center of mass. We can call J microinertia parameter.

3. Cross-beam structure

In order to demonstrate a beam-like elastic system with dominant rotary inertia, let us consider a structure as shown in Fig. 4. It again consists of n cells connected through rotational springs of stiffness k. Each cell has a shape of a cross made of two rigid and rigidly connected bars of length $2a$ and $2h$, respectively. In addition, there are two masses attached to the ends of the vertical bar.

As kinematical descriptors we again choose angles $\phi_i = \phi_i(t)$, $i = 1, 2, \dots, n+1$, see Fig. 5. Motion of the ith cell can be described as a

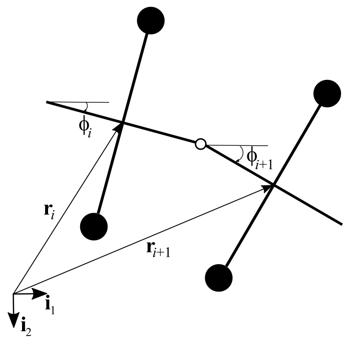

Fig. 5. Deformation of the cross-beam.

motion of its center of mass and relative rotation about it. We have the following relations

$$\mathbf{R}_i = (2i-1)a\mathbf{i}_1, \tag{23}$$

$$\mathbf{r}_i = \left(\sum_{j=1}^{i-1} 2a\cos\phi_j + a\cos\phi_i\right)\mathbf{i}_1 + \left(\sum_{j=1}^{i-1} 2a\sin\phi_j + a\sin\phi_i\right)\mathbf{i}_2, \tag{24}$$

where $\mathbf{R}_i$ and $\mathbf{r}_i$ are the position vectors of center of mass in initial and current placements, respectively. As a result, the ith cell has the translational velocity

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = -\left(\sum_{j=1}^{i-1} 2a\sin\dot{\phi}_j + a\sin\phi_i\dot{\phi}_i\right)\mathbf{i}_1 + \left(\sum_{j=1}^{i-1} 2a\cos\phi_j\dot{\phi}_j + a\cos\phi_i\dot{\phi}_i\right)\mathbf{i}_2, \tag{25}$$

and the angular velocity $\omega_i \equiv \dot{\phi}_i$.

Elastic energy is a sum of energies stored in springs. So it is given by the formula

$$W = \frac{1}{2}\sum_{i=2}^{n-1} k(\phi_i - \phi_{i-1})^2. \tag{26}$$

Kinetic energy has the form

$$K = \frac{1}{2}\sum_{i=2}^{n-1}(M_c\mathbf{v}_i\cdot\mathbf{v}_i + J\omega_i^2), \tag{27}$$

where M_c is the total mass of the cell and J is the moment of inertia given by the formulae

$$M_c = m_v + m_h + 2\,m,$$
$$J = \frac{1}{3}m_v h^2 + \frac{1}{3}m_h a^2 + 2mh^2.$$

Here m_v and m_h are masses of vertical and horizontal bars, respectively. Obviously, under assumptions that $a \ll h$ and $m_v, m_h \ll m$ the influence of attached masses becomes dominant. In what follows for simplicity let as consider the case $m_v = m_h = 0$. In other words, we assume that the mass is concentrated in attached masses.

Discrete model discussed above has a continuum limit at $n\to\infty$ (or at $a\to 0$). Let M be the total mass of the beam and ℓ be its length. In order to keep the total mass and length constant we assume that

$$M_c = \frac{M}{n}, \quad a = \frac{L}{2n}.$$

We introduce the linear mass density $\bar{\rho} = M/L$ and the angle ϕ as a differentiable function of x and t. So ϕ_i can be identified as follows

$$\phi_i = \phi(x_i, t), \quad x_0 = a, \quad x_{i+1} = x_i + 2a, \quad i = 0,\dots,n-1.$$

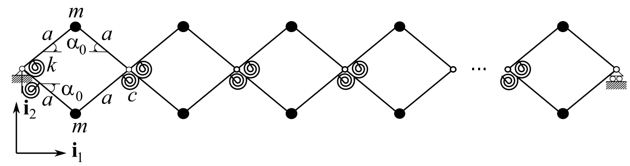

Fig. 6. Pantographic beam.

Instead of Eqs. (23), (24), and (25) we have their continuum counterparts

$$\mathbf{R} = x\mathbf{i}_1, \quad \mathbf{r} = \int_0^x \cos\phi\,dx\,\mathbf{i}_1 + \int_0^x \sin\phi\,dx\,\mathbf{i}_2 \tag{28}$$

$$\mathbf{v} = \dot{\mathbf{r}} = -\int_0^x \omega\sin\phi\,dx\,\mathbf{i}_1 + \int_0^x \omega\cos\phi\,dx\,\mathbf{i}_2, \tag{29}$$

where $\omega = \dot{\phi}(x,t)$. Potential and kinetic energies take the form

$$W = \frac{1}{2}\int_0^{\ell} k\left(\frac{\partial\phi}{\partial x}\right)^2 dx, \tag{30}$$

$$K = \frac{1}{2}\int_0^{\ell}\left(\bar{\rho}\mathbf{v}\cdot\mathbf{v} + \bar{J}\omega^2\right) dx, \tag{31}$$

where we keep the same notation for the bending stiffness k and the rotary inertia $\bar{J}$ is given by $\bar{J} = \bar{\rho}h^2$. For small rotations (30) has the same up to notations form as (10), i.e coincides with the Bresse–Rayleigh model. The difference in values of inertia parameters as here mass is concentrated far for the center of the beam. So the cross-beam returns us to rotary inertia as in was introduced by Bresse and Rayleigh.

4. Pantographic-beam

In order to demonstrate a beam-like structure having both rotary and microinertia or higher order inertia let us consider a pantographic beam as shown in Fig. 6. In the following we use the models introduced in the previous sections and modify a previously considered pantographic bar as follows. Now the beam consists of n cells, whereas each cell consists of four rigid bars of length a connected to each other with perfect hinges and one rotational spring of stiffness k. Between cells we introduce a rotational springs of stiffness c, so cells can exhibit relative rotations. In fact, this model can be treated as a relaxed pantographic bar which can be bended or as a cross-beam with additional mass dynamics. Dynamics of more complex pantographic beams were studied in [57–60]. Here for our purposes we essentially simplified them to extract exactly essential inertia terms. In [61] dynamics of another pantographic beam-like structure was analyzed where terms higher than third order have been neglected.

Deformation of a pantographic beam can be modeled through two angles, ϕ_i and α_i, $i = 1,\dots,n$, see Fig. 7. ϕ_i describes a rotation of the ith cell whereas α_i relates to the change of shape of the same cell. Initial positions of the center of mass of the ith cell and the attached masses can be described using vectors $\mathbf{R}_i$ and $\mathbf{X}_i^{\pm}$, respectively. These vectors are given by the formulae

$$\mathbf{R}_i = (2i-1)a\cos\alpha_0\mathbf{i}_1, \quad \mathbf{X}_i^{\pm} = \mathbf{R}_i \pm a\sin\alpha_0\mathbf{i}_2. \tag{32}$$

In the current placement at time t they become

$$\mathbf{r}_i = \left(\sum_{j=1}^{i-1} 2a\cos\alpha_j\cos\phi_j + a\cos\alpha_i\cos\phi_i\right)\mathbf{i}_1 + \left(\sum_{j=1}^{i-1} 2a\sin\alpha_j\sin\phi_j + a\sin\alpha_i\sin\phi_i\right)\mathbf{i}_2, \tag{33}$$

$$\mathbf{x}_i^{\pm} = \mathbf{r}_i \pm \mathbf{y}_i, \quad \mathbf{y}_i = a\sin\alpha_i\left[\sin\phi_i\mathbf{i}_1 + \cos\phi_i\mathbf{i}_2\right]. \tag{34}$$

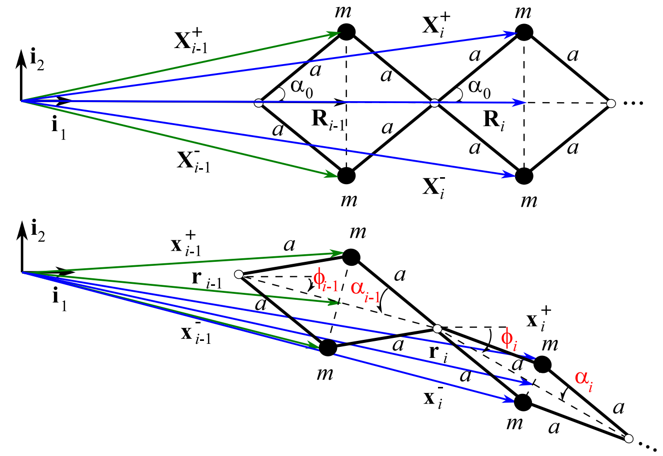

Fig. 7. Deformation of two pantographic cells: initial (above) and current (below) placements.

With this kinematics the potential and kinetic energies of the beam take the form

$$W = \frac{1}{2}\sum_{i=1}^{n} k(\alpha_i-\alpha_0)^2 + \frac{1}{2}c(\alpha_1+\phi_1-\alpha_0)^2 + \frac{1}{2}\sum_{i=2}^{n} c(\alpha_i+\alpha_{i-1}+\phi_i-\phi_{i-1}-2\alpha_0)^2, \tag{35}$$

$$\begin{aligned} K &= \frac{1}{2}\sum_{i=1}^{n} m\left(\dot{\mathbf{x}}_i{}^+\cdot\dot{\mathbf{x}}_i{}^+ + \dot{\mathbf{x}}_i{}^-\cdot\dot{\mathbf{x}}_i{}^-\right)\\ &= \sum_{i=1}^{n} m\left(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i + \dot{\mathbf{y}}_i\cdot\dot{\mathbf{y}}_i\right)\\ &= \sum_{i=1}^{n} m\mathbf{v}_i\cdot\mathbf{v}_i\\ &\quad + \sum_{i=1}^{n} m\left(\sin^2\alpha_i\dot{\phi}_i^2 + \cos^2\alpha_i\dot{\alpha}_i^2\right). \end{aligned} \tag{36}$$

For derivation of a continuum model of the pantographic beam at $n\to\infty$ we again introduce differentiable functions, $\mathbf{r}=\mathbf{r}(x,t)$, $\alpha=\alpha(x,t)$, and $\phi=\phi(x,t)$, the total mass $M=2mn$, and the initial length $\ell=2an\cos\alpha_0$. So we get that $m=M/2n$ and $a=\ell/(2n\cos\alpha_0)$. The continuum limit of the kinetic energy is given by the formula

$$K = \frac{1}{2}\int_0^\ell \bar{\rho}\mathbf{v}\cdot\mathbf{v}\,dx + \frac{1}{2}\int_0^\ell \underline{\bar{\rho}\left(\sin^2\alpha\,\dot{\phi}^2+\cos^2\alpha\,\dot{\alpha}^2\right)}, \tag{37}$$

where $\bar{\rho}=M/\ell=m/(a\cos\alpha_0)$ is the linear mass density. Underlined terms in (37) can be interpreted as parts of the kinetic energy related to rotations and microdeformations, respectively. So we can call factors $\bar{\rho}\sin^2\alpha$ and $\bar{\rho}\cos^2\alpha$ rotary inertia and microinertia, respectively. Both inertia parameters essentially depend on α, i.e on deformations.

Conclusions and discussion

In our paper we discuss rotary inertia and more general microinertia considering one-dimensional models of microstructured bars and beams. Three examples are considered. The first one about a pantographic bar demonstrated a possible non-linearity in inertia. The second shows a beam-like structure with dominant rotary inertia. Finally, considering pantographic beam we see that the structure has two inertia parameters, that are rotary inertia and microinertia. Additional microinertia relates to changes of cross-sectional area and also bases on relative rotations of masses about a center of mass.

Let us note that similar extension towards more complex kinetic constitutive equations can be expected in the case of microstructured continua. As the first example we should mention the Cosserat continuum model (micropolar medium) [62–64]. Indeed, introduced by Cosserat brothers more than one hundred years ago this model describes translational and rotational (orientational) interactions. As a result, rotations play a role of an additional kinematical descriptor independent on translations. So a kinetic energy in the Cosserat continuum naturally contains a part related to rotations. Moreover, the Cosserat curve and six-parameter shell models can be treated as an one- and two-dimensional Cosserat continua embedded into the three-dimensional space. The simplest form of the kinetic energy function is given by the formula

$$K_m = \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\rho j\boldsymbol{\omega}\cdot\boldsymbol{\omega},$$

where again $\mathbf{v}$ and $\boldsymbol{\omega}$ are the linear and angular velocities, respectively, ρ is the current mass density, and j is a scalar measure of rotary inertia. In general, this form relates to material particles of spherical shape that is obviously a serious simplification. Eringen [65] introduce the tensor of microinertia $\mathbf{j}$ for micropolar fluids as an additional constitutive parameter with own balance law. Moreover, he used this approach to model liquid crystals [66] and suspensions [67]. Recently, some generalizations were proposed in [68] and in [69] where two microinertia tensors were identified through a homogenization of an elastic network, see also [70,71] and the references therein. The most general form of the kinetic energy within micropolar theory is almost coincides with (14), see [64].

Another enhanced model of continua with non-trivial kinetic constitutive relation called micromorphic continuum was introduced in [72, 73]. Instead of rotation tensor here we faced a second-order tensor of microdeformation $\mathbf{P}$ as an additional kinematical variable. So a simple form of a kinetic energy density has the form

$$K_{mm} = \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\rho j\,\dot{\mathbf{P}}:\dot{\mathbf{P}},$$

with a scalar microinertia j. Hereinafter $:$ is the double dot product. Balance law for the microinertia tensor was later discussed by Eringen [74]. In general, for a micromorphic medium one can consider a general quadratic form of $\mathbf{v}$ and $\dot{\mathbf{P}}$ as a kinetic equation. In this case we get

$$K_{mm} = \frac{1}{2}\mathbf{v}\cdot\mathbf{I}_0\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{I}_1:\dot{\mathbf{P}} + \frac{1}{2}\dot{\mathbf{P}}:\mathbf{I}_2:\dot{\mathbf{P}},$$

where $\mathbf{I}_0$, $\mathbf{I}_1$, and $\mathbf{I}_2$ are tensors of microinertia of second-, third-, and fourth-order, respectively. Obviously, the latter formula requires a lot of microinertia parameters which should be identified. Relaxed micromorphic model was proposed in [75] that requires less number of parameters.

Considering media with additional degrees of freedom we also mention nematic liquid crystals [76,77] and their solid counterparts [78]. Here we have an unit vector $\mathbf{n}$ called director as an additional kinematic variable. As a result, the kinetic energy density takes the form

$$K_n = \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}j\dot{\mathbf{n}}\cdot\dot{\mathbf{n}},$$

where j is an orientational inertia. Liquid crystals, solid liquid crystals, micropolar and micromorphic media and their variations can be treated as further generalizations of the Timoshenko beam model. Indeed, for both models we have vector equations of motions that are analogous to (6) and (7).

Finally, we conclude our brief review of continua with non-trivial kinetic constitutive equations considering strain gradient elasticity [72, 79–81]. In addition to extension of a deformation energy density to a function of strains and its gradients, here we have an extension of a kinetic energy. For example, in the case of the Toupin–Mindlin strain gradient elasticity the kinetic energy takes the form

$$K_{sg} = \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\nabla\mathbf{v}:\mathbf{J}:\nabla\mathbf{v},$$

where **J** is a fourth-order tensor of microinertia and ∇ is the nabla-operator. For isotropic solids **J** contains three independent inertia parameters. The corresponding equation of motion is similar to (5) as it contain mixed spatial–temporal derivatives of displacements. For analysis of dynamic properties within strain gradient elasticity we also refer to [82].

Analysis provided in this paper shows that microstructured materials may demonstrate an essentially extension of rotary inertia and microinertia properties also for continua models. In particular, one may expect a rather complex dependence of microinertia on deformations as well as a dominant role of inertia related to other micromotions.

CRediT authorship contribution statement

Victor A. Eremeyev: Conceptualization, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Isaac Elishakoff:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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