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On solvability of initial boundary-value problems of micropolar elastic shells with rigid inclusions

Victor A Eremeyev
Department of Civil and Environmental Engineering and Architecture (DICAAR), University of Cagliari, Cagliari, Italy; WILIŚ, Gdańsk University of Technology, Gdańsk, Poland; National Research Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia

Leonid P Lebedev
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

Violetta Konopińska-Zmysłowska
WILIŚ, Gdańsk University of Technology, Gdańsk, Poland

Abstract
The problem of dynamics of a linear micropolar shell with a finite set of rigid inclusions is considered. The equations of motion consist of the system of partial differential equations (PDEs) describing small deformations of an elastic shell and ordinary differential equations (ODEs) describing the motions of inclusions. Few types of the contact of the shell with inclusions are considered. The weak setup of the problem is formulated and studied. It is proved a theorem of existence and uniqueness of a weak solution for the problem under consideration.

Keywords
Micropolar shell, six-parameter shell, dynamics, weak solutions, rigid inclusions, weak setup, uniqueness and existence of weak solution

1. Introduction

Among various models of thin-walled structures, the six-parametric theory of shells, known also as the micropolar shell theory, or the resultant shell theory, has an origin in Reissner's works [1,2]; in more detail, it is presented in previous works [3–7]. Within the model, the kinematics of a shell is described through two surface fields, which are translations and rotations defined on the base surface of the shell. In this way, we get six kinematical degrees of freedom of a shell particle as in the case of rigid body

Corresponding author:
Victor A Eremeyev, Department of Civil and Environmental Engineering and Architecture (DICAAR), University of Cagliari, Via Marengo, 2, 09123 Cagliari, Italy.
Email: victor.eremeyev@gmail.com

dynamics [8] or Cosserat continuum [6]. The counterparts of translations and rotations are stress resultants and surface couple stresses including so-called drilling moment, that is, a moment related to the rotation about the normal to the base surface. The equations of the six-parameter shell theory could be derived using the through-the-thickness integration procedure [3–5] or within the direct approach as in Eremeyev and his colleagues [6,7]; see also [9–11] and the references therein. As a result, on the boundary of a micropolar shell, we have six load boundary conditions that give a possibility to describe the kinematics of multifolded shells or interaction of a shell with rigid bodies. In particular, the kinematics of multifolded (branching) shells was discussed in previous works [5,12]. In the design of spatial structures which include one-dimensional and two-dimensional structures, and three-dimensional solids and their further numerical study [13–17], various joints play a significant role. It is worth to mention here asymptotic techniques in the theory of shells and plates, which could be useful for the further analysis of stress behaviour for high-frequency oscillations in the vicinity of inclusions [18–20]. The interest in the modelling of joint behaviour of elastic shells and rigid inclusions is motivated by some applications such as gyroscopic structures attached to a flexible elastic support, see, for example, the work by Carta et al. and Awrejcewicz et al. [21,22]; for aerospace engineering, see the work by Qatu and Andreev et al. [23,24]; for material processing, Wan et al. [25]; for design of elastic metamaterial plates, Miranda et al., Cai et al., and Ma et al. [26–28]; or for modelling of protein motions in biomembranes, Steigmann [29]. In the models, the inclusions possess their own dynamics which affects the deformation of an elastic support. From the mathematical point of view, the model of an elastic shell with inclusions consists of a system of partial differential equations (PDEs), which corresponds to the shell, and a system of ordinary differential equations (ODEs), which describes the motion of inclusions. An interaction between the shell and inclusions could be perfect or non-perfect. For a non-perfect contact, an inclusion can have certain motions not affecting shell deformations. In other words, for the non-perfect contact, the inclusion could exhibit some rigid body motions non-affecting the shell deformation. So the model of a shell with such inclusions requires further study related to uniqueness and existence of solutions. The mathematical analysis of boundary-value problems (BVPs) in the shell theory was performed in many works [30,31]. Existence of solutions for Cosserat and micropolar shells was studied in a less number of papers [9,32–34] and the reference therein. In particular, in Eremeev and Lebedev [34], static BVPs for shells with rigid inclusions were studied, where only perfect contact conditions were considered. Here, we extend the results [32,34] towards dynamic problems for linear micropolar shells with inclusions considering both perfect and non-perfect interface conditions.

The paper is organized as follows. First, we briefly introduce the basic equations of micropolar shells in Section 2. Here, we also present the motion equations for rigid inclusions and discuss possible types of the contact of the shell with inclusions. In Section 3 we introduce the least action principle and derive the natural boundary conditions along shell-inclusion interfaces. Finally, in Section 4 we define a weak solution using the virtual work principle. To study the weak setup of the dynamic problem, some energy spaces are introduced and characterized through Sobolev's spaces. For a mixed initial BVP, the existence and uniqueness of weak solutions on a finite time interval are proven.

2. Governing equations of dynamics

2.1. Kinematics and dynamics of a micropolar shell

In what follows, we restrict the theory to small deformations. A micropolar shell is considered as a material surface, that is, a smooth enough base surface S with a contour $L = \partial S$, which possesses a mass density, energy, and other physical characteristics. The deformation of the shell is described through two surface fields:

$$\mathbf{u} = \mathbf{u}(s^1, s^2, t), \quad \boldsymbol{\phi} = \boldsymbol{\phi}(s^1, s^2, t), \tag{1}$$

where $\mathbf{u}$ and $\boldsymbol{\phi}$ are vectors of translations and rotations given as vector-valued functions of surface coordinates s^α, $\alpha = 1, 2$, and time t.

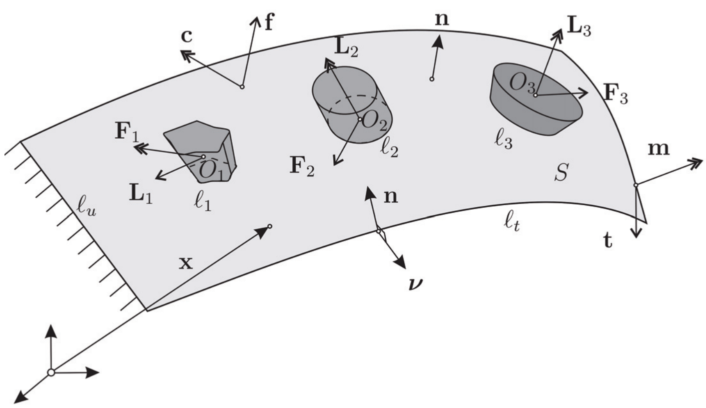


Figure 1. An elastic shell with three rigid inclusions.
From now on, we use types of arrow heads distinguishing geometrical vectors (position, normal, tangent vectors), force vectors, and couples. In particular, for couples we use double arrows.

Let us introduce surface nabla-operator as follows [35]:

$$\nabla = \mathbf{x}^\alpha \frac{\partial}{\partial s^\alpha}, \quad \mathbf{x}^\alpha \cdot \mathbf{x}_\beta = \delta^\alpha_\beta, \quad \mathbf{x}_\alpha = \frac{\partial \mathbf{x}}{\partial s^\alpha}, \quad \alpha, \beta = 1, 2, \tag{2}$$

where $\mathbf{x} = \mathbf{x}(s^1, s^2)$ is a position vector of a point on S and δ^α_β is the Kronecker symbol; Einstein's summation rule is used throughout.

The equations of shell motion [32,35] are

$$\nabla \cdot \mathbf{T} + \mathbf{f} = \rho\ddot{\mathbf{u}} + \rho\mathbf{j}_1 \cdot \ddot{\boldsymbol{\phi}}, \tag{3}$$

$$\nabla \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{c} = \rho\mathbf{j}_1^T \cdot \ddot{\mathbf{u}} + \rho\mathbf{j}_2 \cdot \ddot{\boldsymbol{\phi}}, \tag{4}$$

where $\mathbf{T}$ and $\mathbf{M}$ are stress resultant and surface couple stress tensors, respectively; $\mathbf{f}$ and $\mathbf{c}$ are external forces and couples per unit area, respectively; ρ is a surface mass density; symbol ^{T} stands for the transpose tensor; and $\mathbf{j}_1$ and $\mathbf{j}_2$ are tensors of rotatory inertia. Here, the overdot denotes the derivative with respect to t. Expression $(\ldots)_\times$ denotes the vectorial invariant (Gibbsian cross) of a second-order tensor [35]. For example, for a dyad of two vectors $\mathbf{a}$ and $\mathbf{b}$, we get $(\mathbf{a} \otimes \mathbf{b})_\times = \mathbf{a} \times \mathbf{b}$. Hereafter, "$\cdot$," "$\otimes$," and "$\times$" stand for dot, dyadic, and cross products, respectively.

Equations (3) and (4) correspond to the following expression for the surface density of kinetic energy:

$$K = \frac{1}{2}\rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \rho\dot{\mathbf{u}} \cdot \mathbf{j}_1 \cdot \dot{\boldsymbol{\phi}} + \frac{1}{2}\rho\dot{\boldsymbol{\phi}} \cdot \mathbf{j}_2 \cdot \dot{\boldsymbol{\phi}}, \tag{5}$$

which is assumed to be a positive quadratic form of the linear $\dot{\mathbf{u}}$ and angular $\dot{\boldsymbol{\phi}}$ velocities. So we have the inequality

$$K \geqslant C_K(\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \dot{\boldsymbol{\phi}} \cdot \dot{\boldsymbol{\phi}}),$$

where C_K is a positive constant independent of $\mathbf{u}$ and $\boldsymbol{\phi}$. Let us note that hereafter we suppose all the quantities and equations to be transformed to the dimensionless form.

For a hyperelastic shell, there exists a surface strain energy density W given as a function of two strain measures:

$$W = W(\mathbf{e}, \mathbf{k}), \quad \mathbf{e} = \nabla\mathbf{u} + \mathbf{P} \times \boldsymbol{\phi}, \quad \mathbf{k} = \nabla\boldsymbol{\phi}, \tag{6}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ is the surface unit tensor, $\mathbf{I}$ is the three-dimensional unit tensor, and $\mathbf{n}$ is the unit normal to S (see Figure 1).

The stress measures are expressed through W by the formulae

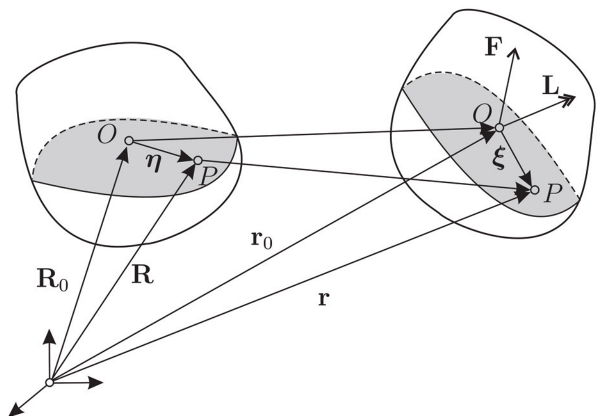


Figure 2. A motion of a rigid body.

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{e}}, \quad \mathbf{M} = \frac{\partial W}{\partial \mathbf{k}}.$$

In what follows, we assume that W is a positive definite quadratic form of $\mathbf{e}$ and $\mathbf{k}$, so

$$W(\mathbf{e}, \mathbf{k}) = \frac{1}{2}\mathbf{e} : \mathbf{C} : \mathbf{e} + \mathbf{e} : \mathbf{B} : \mathbf{k} + \frac{1}{2}\mathbf{k} : \mathbf{D} : \mathbf{k}, \tag{7}$$

$$W(\mathbf{e}, \mathbf{k}) \geqslant C\big(\| \mathbf{e}\|^2 + \| \mathbf{k}\|^2\big), \tag{8}$$

where $\mathbf{C}$, $\mathbf{B}$, and $\mathbf{D}$ are stiffness fourth-order tensors; ":" is the double dot product, $\| (\ldots)\|^2 = (\ldots) : (\ldots)$; and C is a positive constant independent of $\mathbf{e}$ and $\mathbf{k}$. For some material symmetries, the stiffness tensors were presented in Eremeyev and Pietraszkiewicz [36].

The equations of motion should be complemented by boundary conditions along L. Here, L consists of an external contour ℓ and N interfaces ℓ_i between the shell and inclusions, $L = \ell \cup \ell_1 \cup \ldots \cup \ell_N$ (Figure 1). We assume the mixed boundary conditions on ℓ:

$$\mathbf{u}|_{\ell_u} = \mathbf{0}, \quad \boldsymbol{\phi}|_{\ell_u} = \mathbf{0}, \quad \boldsymbol{\nu} \cdot \mathbf{T}|_{\ell_t} = \mathbf{t}, \quad \boldsymbol{\nu} \cdot \mathbf{M}|_{\ell_t} = \mathbf{m}. \tag{9}$$

So the part ℓ_u is fixed, whereas along the rest $\ell_t = \ell \backslash \ell_u$, the line forces $\mathbf{t}$ and couples $\mathbf{m}$ are given. Here, $\boldsymbol{\nu}$ is the outward normal vector to ℓ lying in the tangent plane to S.

The boundary conditions on interfaces ℓ_i, $i = 1, \ldots N$ for various possible types of contact will be discussed in Section 2.3.

The motion equations and the boundary conditions should be complemented by the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}^\circ, \quad \boldsymbol{\phi}|_{t=0} = \boldsymbol{\phi}^\circ, \quad \dot{\mathbf{u}}|_{t=0} = \mathbf{v}^\circ, \quad \dot{\boldsymbol{\phi}}|_{t=0} = \boldsymbol{\omega}^\circ, \tag{10}$$

where $\mathbf{u}^\circ$, $\boldsymbol{\phi}^\circ$, $\mathbf{v}^\circ$, and $\boldsymbol{\omega}^\circ$ are initial data, that is, the initial translations, rotations, linear velocity, and angular velocity, respectively. Note that we assume that $\mathbf{u}^\circ$ and $\boldsymbol{\phi}^\circ$ are consistent with kinematic boundary conditions, that is, $\mathbf{u}^\circ|_{\ell_u} = \mathbf{0}$, $\boldsymbol{\phi}^\circ|_{\ell_u} = \mathbf{0}$, $\mathbf{v}^\circ|_{\ell_u} = \mathbf{0}$, and $\boldsymbol{\omega}^\circ|_{\ell_u} = \mathbf{0}$.

2.2. Rigid body dynamics

Following previous works [6,8], let us briefly recall the equations of dynamics of a rigid body. Let a rigid body $\mathcal{B}$ occupy volumes v and V in a reference and current placements (see Figure 2). The kinematics of $\mathcal{B}$ could be described as a translation of an arbitrary point O of $\mathcal{B}$ called the pole and a rotation about O. For simplicity, we use the centre of mass as the pole. In a reference placement, a position vector of an arbitrary point P of $\mathcal{B}$ is given by

$$\mathbf{R} = \mathbf{R}_0 + \boldsymbol{\xi},$$

where $\mathbf{R}_0$ is the position vector of O and $\boldsymbol{\xi}$ is the vector from O to P. In a current placement, we have a similar representation:

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\eta},$$

where $\mathbf{r}_0$ and $\boldsymbol{\eta}$ are the position vectors of O and from O to P, respectively. Introducing the rotation tensor $\mathbf{Q}$, we get the relation between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$: $\boldsymbol{\eta} = \mathbf{Q} \cdot \boldsymbol{\xi}$. So the displacement $\mathbf{u}$ of P is

$$\mathbf{u} \equiv \mathbf{r} - \mathbf{R} = \mathbf{u}_0 + \mathbf{Q} \cdot \boldsymbol{\xi} - \boldsymbol{\xi}, \tag{11}$$

where $\mathbf{u}_0 = \mathbf{r}_0 - \mathbf{R}_0$ is the translation of pole O. As a result, the velocity of P is given by

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \boldsymbol{\eta}, \tag{12}$$

where $\mathbf{v}_0 = \dot{\mathbf{u}}_0$ is a linear velocity of the pole and $\boldsymbol{\omega}$ is an angular velocity introduced as $\boldsymbol{\omega} = -1/2(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T)_\times$.

The kinetic energy of $\mathcal{B}$ is given by

$$K_\mathcal{B} = \frac{1}{2} M \mathbf{v}_0 \cdot \mathbf{v}_0 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega}, \tag{13}$$

where M and $\mathbf{J}$ are the mass and the inertia tensor of $\mathcal{B}$, respectively. The formulae for $\mathbf{J}$ are

$$\mathbf{J} = \iiint\limits_v \rho \boldsymbol{\eta} \times \mathbf{I} \times \boldsymbol{\eta} \, dv = \mathbf{Q} \cdot \mathbf{J}_0 \cdot \mathbf{Q}^T, \quad \mathbf{J}_0 = \iiint\limits_V \rho \boldsymbol{\xi} \times \mathbf{I} \times \boldsymbol{\xi} \, dV, \tag{14}$$

where ρ is the mass density of $\mathcal{B}$ and $\mathbf{J}_0$ is the referential tensor of inertia.

The motion equations for $\mathcal{B}$ follow from the balance of momentum and moment of momentum; they take the form

$$M \dot{\mathbf{v}}_0 = \mathbf{F}, \quad (\mathbf{J} \cdot \boldsymbol{\omega})^{\cdot} = \mathbf{L}, \tag{15}$$

where $\mathbf{F}$ and $\mathbf{L}$ are the total force and total torque vectors acting on $\mathcal{B}$.

In case of small rotations, we can use the following replacements [6,37]:

$$\mathbf{Q} = \mathbf{I} + \mathbf{I} \times \boldsymbol{\vartheta}, \quad \mathbf{J} = \mathbf{J}_0, \quad \boldsymbol{\omega} = \dot{\boldsymbol{\vartheta}},$$

where $\boldsymbol{\vartheta}$ is the vector of infinitesimal rotations.

In addition to equation (15), we also pose the initial conditions

$$\mathbf{u}_0|_{t=0} = \mathbf{u}_0{}^\circ, \quad \boldsymbol{\vartheta}|_{t=0} = \boldsymbol{\vartheta}^\circ, \quad \dot{\mathbf{u}}_0|_{t=0} = \mathbf{v}_0{}^\circ, \quad \dot{\boldsymbol{\vartheta}}|_{t=0} = \boldsymbol{\omega}^\circ \tag{16}$$

with initial data $\mathbf{u}_0{}^\circ$, $\boldsymbol{\vartheta}^\circ$, $\mathbf{v}_0{}^\circ$, and $\boldsymbol{\omega}^\circ$.

Obviously, for N rigid bodies-inclusions, we have a system of $2N$ equations of motion and $4N$ initial conditions in terms of $\mathbf{u}_i$ and $\boldsymbol{\vartheta}_i$, $i = 1, \ldots N$:

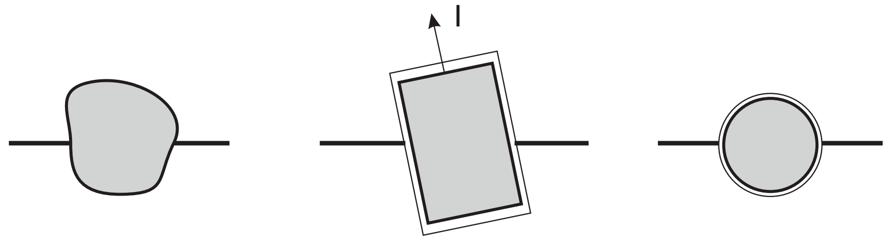
$$M_i \dot{\mathbf{v}}_i = \mathbf{F}_i, \quad \mathbf{J}_i \cdot \ddot{\boldsymbol{\vartheta}}_i = \mathbf{L}_i, \tag{17}$$

$$\mathbf{u}_i|_{t=0} = \mathbf{u}_i{}^\circ, \quad \boldsymbol{\vartheta}_i|_{t=0} = \boldsymbol{\vartheta}_i{}^\circ, \quad \dot{\mathbf{u}}_i|_{t=0} = \mathbf{v}_i{}^\circ, \quad \dot{\boldsymbol{\vartheta}}_i|_{t=0} = \boldsymbol{\omega}_i{}^\circ, \tag{18}$$

where M_i and $\mathbf{J}_i$ are the mass and the moment of inertia of the ith inclusion, respectively.

2.3. Interaction with rigid inclusions

In what follows, we consider the following contact conditions between the shell and rigid inclusions:



(a) (b) (c)

Figure 3. Interactions between a shell and rigid inclusion: (a) perfect contact, (b) constraint rotations, and (c) free rotations.

1. *A perfect contact*. Here the shell is rigidly connected to the body as shown in Figure 3(a). As a result, the rigid body cannot move without deformation of the shell;
2. *A contact with constraint-free rotations*. In this case, there is an axis with an unit director **l** such that rotations about this axis do not produce any deformation of the shell (Figure 3(b));
3. *Free rotations*. It is when the rigid body can rotate freely about any axis without deformations of the shell (Figure 3(c)).

For the perfect contact, there is no sliding between the shell and inclusion, so the translations and rotations have no jump of discontinuity across the interface:

$$\mathbf{u}|_{\ell_i} = \mathbf{u}_i + \boldsymbol{\vartheta}_i \times \boldsymbol{\xi}(s)|_{\ell_i}, \quad \boldsymbol{\phi}|_{\ell_i} = \boldsymbol{\vartheta}_i, \tag{19}$$

where s is the arc-length parameter along ℓ_i. In other words, the points of ℓ_i may exhibit only infinitesimal rigid body motions.

For the other types of contact, the inclusions can possess free rotation. In particular, Case 2 corresponds to the free rotation about $\mathbf{l}-$axis s, whereas Case 3 relates to entirely free rotations. The corresponding kinematic conditions could be derived from the analysis of a velocity distribution in a rigid body and shell; they take the form

$$\mathbf{u}|_{\ell_i} = \mathbf{u}_i + \boldsymbol{\vartheta}_i^{\|} \times \boldsymbol{\xi}(s)|_{\ell_i}, \quad \boldsymbol{\phi}|_{\ell_i} = \boldsymbol{\vartheta}_i^{\|}, \tag{20}$$

$$\mathbf{u}|_{\ell_i} = \mathbf{u}_i. \tag{21}$$

In equation (20), we decompose J_i as follows:

$$\boldsymbol{\vartheta}_i = \boldsymbol{\vartheta}_i^{\perp} + \boldsymbol{\vartheta}_i^{\|}, \quad \boldsymbol{\vartheta}_i^{\perp} = (\boldsymbol{\vartheta}_i \cdot \mathbf{l})\mathbf{l}, \quad \boldsymbol{\vartheta}_i^{\|} = (\mathbf{I} - \mathbf{l} \otimes \mathbf{l}) \cdot \boldsymbol{\vartheta}_i.$$

To distinguish these three cases, we denote the interfaces as ℓ_i', ℓ_i'', and ℓ_i''', respectively. In other words, boundary conditions (19) are given along ℓ_i', whereas equations (20) and (21) are assigned along ℓ_i'' and ℓ_i''', respectively.

The dynamic counterparts of equations (19), (20), and (21), that is, dynamic boundary conditions, can be derived within the variational approach.

3. The least action principle

So for small deformations of an elastic shell with N rigid inclusions, we have the following kinematic descriptors:

- Two vector-valued surface fields $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\boldsymbol{\phi} = \boldsymbol{\phi}(\mathbf{x}, t)$, $\mathbf{x} \in S$;
- $2N$ vector-valued functions $\mathbf{u}_i = \mathbf{u}_i(t)$ and $\boldsymbol{\vartheta}_i = \boldsymbol{\vartheta}_i(t)$, which describe translations and rotations of ith inclusion, $i = 1, 2 \ldots N$.

In what follows, we consider three types of inclusions as described above. So depending on the type of shell-inclusion interaction, $\mathbf{u}_i$ and $\boldsymbol{\vartheta}_i$ are subjected to one of the conditions (19), (20), or (21). As a result,

$\mathbf{u}$, $\boldsymbol{\phi}$, $\mathbf{u}_i$, and $\boldsymbol{\vartheta}_i$ are not kinematically independent, in general. Indeed, from equation (19), we see that $\mathbf{u}_i$ and $\boldsymbol{\phi}_i$ are entirely determined by $\mathbf{u}$ and $\boldsymbol{\phi}$. For equation (20), $\boldsymbol{\vartheta}_i^{\perp}$ is independent, that is, rotations about $\mathbf{l}$ are independent of the shell deformations, whereas for equation (21) all the rotations $\boldsymbol{\vartheta}_i$ are kinematically independent. Nevertheless, we consider the full set $\{\mathbf{u}, \boldsymbol{\phi}, \mathbf{u}_i, \boldsymbol{\vartheta}_i\}$ as primary variable subjected constraints (19), (20), and (21). Obviously, initial data introduced in equations (10) and (18) should be also consistent with equation (19), (20), or (21).

Let us consider the variational statement of the problem under consideration using the least action (Hamilton–Ostrogradski) principle. It has the form

$$\delta\mathcal{H} = 0, \quad \delta\mathcal{H} = \int_{t_0}^{t_1} (\delta\mathcal{K} - \delta\mathcal{W})\, dt, \tag{22}$$

where

$$\mathcal{K} = \iint_S K\, dS + \sum_{i=1}^{N} K_i, \quad K_i = \frac{1}{2} M_i \dot{\mathbf{u}}_i \cdot \dot{\mathbf{u}}_i + \frac{1}{2} \dot{\boldsymbol{\vartheta}}_i \cdot \mathbf{J}_i \cdot \dot{\boldsymbol{\vartheta}}_i, \tag{23}$$

$$\begin{aligned} \delta\mathcal{W} = {} & \delta \iint_S W\, dS - \iint_S (\mathbf{f} \cdot \delta\mathbf{u} + \mathbf{c} \cdot \delta\boldsymbol{\phi})\, dS \\ & - \int_{\ell_t} (\mathbf{t} \cdot \delta\mathbf{u} + \mathbf{m} \cdot \delta\boldsymbol{\phi})\, ds - \sum_{i=1}^{N} (\mathbf{F}_i \cdot \delta\mathbf{u}_i + \mathbf{L}_i \cdot \delta\boldsymbol{\vartheta}_i). \end{aligned} \tag{24}$$

We introduce the kinematically admissible variations of translations and rotations: $\delta\mathbf{u}$, $\delta\boldsymbol{\phi}$, $\delta\mathbf{u}_i$, and $\delta\boldsymbol{\vartheta}_i$. On the external boundary ℓ_u, these new functions satisfy the boundary conditions

$$\delta\mathbf{u}\big|_{\ell_u} = \mathbf{0}, \quad \delta\boldsymbol{\phi}\big|_{\ell_u} = \mathbf{0}. \tag{25}$$

The compatibility conditions along interface ℓ_i (depending on the type of shell-inclusion interaction) are

$$\delta\mathbf{u}\big|_{\ell_i'} = \delta\mathbf{u}_i + \delta\boldsymbol{\vartheta}_i \times \boldsymbol{\xi}(s)\big|_{\ell_i'}, \quad \delta\boldsymbol{\phi}\big|_{\ell_i'} = \delta\boldsymbol{\vartheta}_i, \tag{26}$$

$$\delta\mathbf{u}\big|_{\ell_i''} = \delta\mathbf{u}_i + \delta\boldsymbol{\vartheta}_i^{\parallel} \times \boldsymbol{\xi}(s)\big|_{\ell_i''}, \quad \delta\boldsymbol{\phi}\big|_{\ell_i''} = \delta\boldsymbol{\vartheta}_i^{\parallel}, \tag{27}$$

$$\delta\mathbf{u}\big|_{\ell_i'''} = \delta\mathbf{u}_i. \tag{28}$$

We also assume the standard assumptions for the principle [38]:

$$\delta\mathbf{u}\big|_{t=t_0} = \delta\mathbf{u}\big|_{t=t_1} = \mathbf{0}, \quad \delta\boldsymbol{\phi}\big|_{t=t_0} = \delta\boldsymbol{\phi}\big|_{t=t_1} = \mathbf{0}, \tag{29}$$

$$\delta\mathbf{u}_i\big|_{t=t_0} = \delta\mathbf{u}_i\big|_{t=t_1} = \mathbf{0}, \quad \delta\boldsymbol{\vartheta}_i\big|_{t=t_0} = \delta\boldsymbol{\vartheta}_i\big|_{t=t_1} = \mathbf{0}, \tag{30}$$

where t_0 and t_1 are two time instants, $t_0 < t_1$.

Calculating $\delta\mathcal{H}$, we come to

$$
\begin{aligned}
\delta\mathcal{H} = & \int_{t_0}^{t_1}\iint_S \left(\rho\dot{\mathbf{u}}\cdot\delta\dot{\mathbf{u}} + \delta\dot{\mathbf{u}}\cdot\rho\mathbf{j}_1\cdot\dot{\boldsymbol{\phi}} + \dot{\mathbf{u}}\cdot\rho\mathbf{j}_1\cdot\delta\dot{\boldsymbol{\phi}} + \delta\dot{\boldsymbol{\phi}}\cdot\rho\mathbf{j}_2\cdot\dot{\boldsymbol{\phi}}\right) dS\,dt \\
& + \sum_{i=1}^{N}\int_{t_0}^{t_1}\left(M_i\dot{\mathbf{u}}_i\cdot\delta\dot{\mathbf{u}}_i + \dot{\boldsymbol{\vartheta}}_i\cdot\mathbf{J}_i\cdot\delta\dot{\boldsymbol{\vartheta}}_i\right) dt \\
& - \int_{t_0}^{t_1}\iint_S [\mathbf{T}(\mathbf{e},\mathbf{k}) : \delta\mathbf{e} + \mathbf{M}(\mathbf{e},\mathbf{k}) : \delta\mathbf{k}]\, dS\,dt \\
& + \int_{t_0}^{t_1}\iint_S (\mathbf{f}\cdot\delta\mathbf{u} + \mathbf{c}\cdot\delta\boldsymbol{\phi})\, dS\,dt \\
& + \int_{t_0}^{t_1}\int_{\ell_t} (\mathbf{t}\cdot\delta\mathbf{u} + \mathbf{m}\cdot\delta\boldsymbol{\phi})\, ds\,dt \\
& + \int_{t_0}^{t_1}\sum_{i=1}^{N} (\mathbf{F}_i\cdot\delta\mathbf{u}_i + \mathbf{L}_i\cdot\delta\boldsymbol{\vartheta}_i)\, dt.
\end{aligned}
$$

Here by $\delta\mathbf{e}$ and $\delta\mathbf{k}$ we denote the variations of strain measures:

$$
\delta\mathbf{e} \equiv \mathbf{e}(\delta\mathbf{u},\delta\boldsymbol{\phi}) = \nabla\delta\mathbf{u} + \mathbf{P}\times\delta\boldsymbol{\phi}, \quad \delta\mathbf{k} \equiv \mathbf{k}(\delta\boldsymbol{\phi}) = \nabla\delta\boldsymbol{\phi}.
$$

For brevity, we introduce the bilinear forms

$$
\begin{aligned}
B_K(\dot{\mathbf{u}},\dot{\boldsymbol{\phi}};\delta\dot{\mathbf{u}},\delta\dot{\boldsymbol{\phi}}) = & \iint_S \big(\rho\dot{\mathbf{u}}\cdot\delta\dot{\mathbf{u}} + \delta\dot{\mathbf{u}}\cdot\rho\mathbf{j}_1\cdot\dot{\boldsymbol{\phi}} \\
& + \dot{\mathbf{u}}\cdot\rho\mathbf{j}_1\cdot\delta\dot{\boldsymbol{\phi}} + \delta\dot{\boldsymbol{\phi}}\cdot\rho\mathbf{j}_2\cdot\dot{\boldsymbol{\phi}}\big)\, dS, \\
B_E(\mathbf{u},\boldsymbol{\phi};\delta\mathbf{u},\delta\boldsymbol{\phi}) = & \iint_S [\mathbf{T}(\mathbf{e},\mathbf{k}) : \delta\mathbf{e} + \mathbf{M}(\mathbf{e},\mathbf{k}) : \delta\mathbf{k}]\, dS, \\
B_I(\dot{\mathbf{u}}_i,\dot{\boldsymbol{\vartheta}}_i;\delta\dot{\mathbf{u}}_i,\delta\dot{\boldsymbol{\vartheta}}_i) = & \sum_{i=1}^{N}(M_i\dot{\mathbf{u}}_i\cdot\delta\dot{\mathbf{u}}_i + \dot{\boldsymbol{\vartheta}}_i\cdot\mathbf{J}_i\cdot\delta\dot{\boldsymbol{\vartheta}}_i),
\end{aligned}
$$

and the linear forms

$$
\begin{aligned}
L(\delta\mathbf{u},\delta\boldsymbol{\phi}) = & \iint_S (\mathbf{f}\cdot\delta\mathbf{u} + \mathbf{c}\cdot\delta\boldsymbol{\phi})\, dS + \int_{\ell_t} (\mathbf{t}\cdot\delta\mathbf{u} + \mathbf{m}\cdot\delta\boldsymbol{\phi})\, ds, \\
L_I(\delta\mathbf{u}_i,\delta\boldsymbol{\vartheta}_i) = & \sum_{i=1}^{N} (\mathbf{F}_i\cdot\delta\mathbf{u}_i + \mathbf{L}_i\cdot\delta\boldsymbol{\vartheta}_i),
\end{aligned}
$$

so $\delta\mathcal{H}$ takes the form

$$\delta\mathcal{H} = \int_{t_0}^{t_1} [B_K(\dot{\mathbf{u}}, \dot{\boldsymbol{\phi}}; \delta\dot{\mathbf{u}}, \delta\dot{\boldsymbol{\phi}}) + B_I(\dot{\mathbf{u}}_i, \dot{\boldsymbol{\vartheta}}_i; \delta\dot{\mathbf{u}}_i, \delta\dot{\boldsymbol{\vartheta}}_i) \\ - B_E(\mathbf{u}, \boldsymbol{\phi}; \delta\mathbf{u}, \delta\boldsymbol{\phi}) + L(\delta\mathbf{u}, \delta\boldsymbol{\phi}) + L_I(\delta\mathbf{u}_i, \delta\boldsymbol{\vartheta}_i)]\, dt. \tag{31}$$

Thus, the least action principle (22) takes the form

$$\int_{t_0}^{t_1} B_E(\mathbf{u}, \boldsymbol{\phi}; \delta\mathbf{u}, \delta\boldsymbol{\phi})\, dt \\ = \int_{t_0}^{t_1} [B_K(\dot{\mathbf{u}}, \dot{\boldsymbol{\phi}}; \delta\dot{\mathbf{u}}, \delta\dot{\boldsymbol{\phi}}) + B_I(\dot{\mathbf{u}}_i, \dot{\boldsymbol{\vartheta}}_i; \delta\dot{\mathbf{u}}_i, \delta\dot{\boldsymbol{\vartheta}}_i) \\ + L(\delta\mathbf{u}, \delta\boldsymbol{\phi}) + L_I(\delta\mathbf{u}_i, \delta\boldsymbol{\vartheta}_i)]\, dt \tag{32}$$

for all variations $\delta\mathbf{u}$, $\delta\boldsymbol{\phi}$, $\delta\mathbf{u}_i$, and $\delta\boldsymbol{\vartheta}_i$ satisfying equations (25)–(30). Integrating by parts, we get

$$\delta\mathcal{H} = \int_{t_0}^{t_1} \iint_S \left[\left(\nabla \cdot \mathbf{T} + \mathbf{f} - \rho\ddot{\mathbf{u}} - \rho\mathbf{j}_1 \cdot \ddot{\boldsymbol{\phi}}\right) \cdot \delta\mathbf{u} \right. \\ \left. + \left(\nabla \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{c} - \rho\mathbf{j}_1^T \cdot \ddot{\mathbf{u}} - \rho\mathbf{j}_2 \cdot \ddot{\boldsymbol{\phi}}\right) \cdot \delta\boldsymbol{\phi}\right] dS\, dt \\ + \sum_{i=1}^{N} \int_{t_0}^{t_1} \left[(-M_i\ddot{\mathbf{u}}_i + \mathbf{F}_i) \cdot \delta\mathbf{u}_i + \left(-\mathbf{J}_i \cdot \ddot{\boldsymbol{\vartheta}}_i + \mathbf{L}_i\right) \cdot \delta\boldsymbol{\vartheta}_i\right] dt \\ + \int_{t_0}^{t_1} \int_{\ell_t} (\mathbf{t} \cdot \delta\mathbf{u} + \mathbf{m} \cdot \delta\boldsymbol{\phi})\, ds\, dt \\ - \int_{t_0}^{t_1} \int_{\partial S} (\boldsymbol{\nu} \cdot \mathbf{T} \cdot \delta\mathbf{u} + \boldsymbol{\nu} \cdot \mathbf{M} \cdot \delta\boldsymbol{\phi})\, ds\, dt = 0.$$

From this, by the standard procedure of the calculus of variations, we can obtain the equation of motion (3) and (4), and the static boundary conditions $(9)_3$ and $(9)_4$. Using equations (26), (27), and (28), we get also the dynamic conditions for rigid inclusions:

$$M_i\ddot{\mathbf{u}}_i = \mathbf{F}_i - \int_{\ell_i'} \boldsymbol{\nu} \cdot \mathbf{T}\, ds, \tag{33}$$

$$\mathbf{J}_i \cdot \ddot{\boldsymbol{\vartheta}}_i = \mathbf{L}_i - \int_{\ell_i'} [\boldsymbol{\nu} \cdot \mathbf{M} + \boldsymbol{\xi}(s) \times (\boldsymbol{\nu} \cdot \mathbf{T})]\, ds, \tag{34}$$

$$M_i\ddot{\mathbf{u}}_i = \mathbf{F}_i - \int_{\ell_i''} \boldsymbol{\nu} \cdot \mathbf{T}\, ds, \tag{35}$$

$$\mathbf{J}_i \cdot \ddot{\boldsymbol{\vartheta}}_i = \mathbf{L}_i - \int_{\ell_i''} [\boldsymbol{\nu} \cdot \mathbf{M} + \boldsymbol{\xi}(s) \times (\boldsymbol{\nu} \cdot \mathbf{T})] \cdot (\mathbf{I} - \mathbf{l} \otimes \mathbf{l})\, ds, \tag{36}$$

$$M_i \ddot{\mathbf{u}}_i = \mathbf{F}_i - \int_{\ell_i'''} \boldsymbol{\nu} \cdot \mathbf{T}\, ds, \quad \mathbf{J}_i \cdot \ddot{\boldsymbol{\vartheta}}_i = \mathbf{L}_i, \tag{37}$$

which play a role of boundary conditions for the shell and simultaneously give us the motion equations for the inclusions. Note that here $\mathbf{F}_i$ and $\mathbf{L}_i$ are an external force and torque applied to ith inclusion, whereas in equation (17) the same notation is used for resultant force and moments including interactions with an environment.

4. The principle of virtual work and weak solutions

In order to introduce the weak setup for the dynamic problem under consideration, we introduce the principle of virtual work as in Lebedev et al. [39] Let us note that we consider a Cauchy problem, that is, the problem with initial data given at $t = 0$, whereas equation (32) is formulated for the BVP with conditions given at two time instants, $t = t_0$, $t = t_1$. From now on, for simplicity we will use $t_0 = 0, t_1 = T$. The expression of $\delta\mathcal{H}$ can be used as a basis for the weak setup, but we have to replace equations (29) and (30) by conditions at a time instant $t = T > 0$:

$$\delta\mathbf{u}|_{t=T} = \mathbf{0}, \quad \delta\boldsymbol{\phi}|_{t=T} = \mathbf{0}, \quad \delta\mathbf{u}_i|_{t=T} = \mathbf{0}, \quad \delta\boldsymbol{\vartheta}_i|_{t=T} = \mathbf{0}. \tag{38}$$

For initial BVPs, the principle of virtual work, and so the weak setup of the dynamic problem, for a solution and arbitrary virtual quantities satisfying equation (38) is formulated as follows:

$$\begin{aligned} &\int_0^T [B_K(\dot{\mathbf{u}}, \dot{\boldsymbol{\phi}}; \delta\dot{\mathbf{u}}, \delta\dot{\boldsymbol{\phi}}) + B_I(\dot{\mathbf{u}}_i, \dot{\boldsymbol{\vartheta}}_i; \delta\dot{\mathbf{u}}_i, \delta\dot{\boldsymbol{\vartheta}}_i) \\ &= \int_0^T \iint_S (\mathbf{T} : \delta\mathbf{e} + \mathbf{M} : \delta\mathbf{k}) dS\, dt \\ &+ \int_0^T L(\delta\mathbf{u}, \delta\boldsymbol{\phi}) - L_I(\delta\mathbf{u}_i, \delta\boldsymbol{\vartheta}_i)]\, dt \\ &- B_K\left(\mathbf{v}^\circ, \boldsymbol{\phi}^\circ; \delta\mathbf{u}|_{t=0}, \delta\boldsymbol{\phi}|_{t=0}\right) + B_I\left(\mathbf{v}_i{}^\circ, \boldsymbol{\omega}_i{}^\circ; \delta\mathbf{u}_i|_{t=0}, \delta\boldsymbol{\vartheta}_i|_{t=0}\right), \end{aligned} \tag{39}$$

where $\delta\mathbf{u}$, $\delta\boldsymbol{\phi}$, $\delta\mathbf{u}_i$, and $\delta\boldsymbol{\vartheta}_i$ satisfy equations (25)–(28) and (38). Applying integration by parts similarly to the previous section, from equation (39) we get the equations of motion (3) and (4), the natural boundary conditions $(9)_3$, $(9)_4$, and (33)–(37), and the initial conditions $(10)_3$, $(10)_4$, $(18)_3$, and $(18)_4$.

The virtual work principle is a basis for introduction of weak solutions of the dynamic problem. Let us note that here equation (39) includes a part of the initial conditions for the velocities, whereas the initial data for translations and rotations are treated as the ones which should be formulated explicitly.

Now for shortness, we introduce the notation as in Eremeev and Lebedev [34]: $\mathbf{U} = ((\mathbf{u}, \boldsymbol{\phi}), \mathbf{u}_1, \ldots, \mathbf{u}_n, \boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_N)$ and, respectively, $\delta\mathbf{U} = ((\delta\mathbf{u}, \delta\boldsymbol{\phi}), \delta\mathbf{u}_1, \ldots, \delta\mathbf{u}_n, \delta\boldsymbol{\vartheta}_1, \ldots, \delta\boldsymbol{\vartheta}_N)$. On the set of $\mathbf{U}$ where $(\mathbf{u}, f)$ are smooth functions satisfying boundary conditions $(9)_3$ and $(9)_4$, we introduce an inner product defined by the energy terms for $(\mathbf{u}, \boldsymbol{\phi})$:

$$(\mathbf{U}, \delta\mathbf{U})_V = \iint_S (\mathbf{T} : \delta\mathbf{e} + \mathbf{M} : \delta\mathbf{k})\, dS. \tag{40}$$

Definition 4.1. The completion of the set of elements $\mathbf{U}$, where $(\mathbf{u}, \boldsymbol{\phi})$ are smooth vector functions satisfying geometric conditions $(9)_3$ and $(9)_4$, in the norm $\|\cdot\|_V$ induced by the inner product (40) is called the energy space V.

Note that for the perfect contacts, the deformation of the shell and the contact conditions define uniquely all $\mathbf{u}_i, \boldsymbol{\vartheta}_j$.

In what follows, we use standard Sobolev's $W^{1,2}$ and Lebesgue's L^2 spaces; see the work by Adams and Fournier, and Lions and Magenes [40,41] for more detail. In addition, for vector-valued functions we use the notation $\mathbf{u} \in (W^{1,2}(S))^3$ iff each Cartesian component of $\mathbf{u}$ belongs to $W^{1,2}(S)$.

Clearly V is a Hilbert space. In Eremeyev and Lebedev [32,34], it is shown that if S is smooth, Cartesian components of $(\mathbf{u}, \boldsymbol{\phi})$ pertain to space $W^{1,2}(S)$ and moreover the norm $\|\mathbf{U}\|_V$ is equivalent to the norm of $(\mathbf{u}, \boldsymbol{\phi})$ in Sobolev's space $(W^{1,2}(S))^3 \times W^{1,2}(S))^3$.

We also need the norm for $\mathbf{U}$ which is equivalent to the norm of space $(L^2(S))^3 \times (L^2(S))^3 \times (\mathbb{R}^3)^{2n}$ related to the inertial terms of the problem. This norm is induced by the inner product

$$(\mathbf{U}, \delta\mathbf{U})_H = B_K(\mathbf{u}, \boldsymbol{\phi}; \delta\mathbf{u}, \delta\boldsymbol{\phi}) + B_I(\mathbf{u}_i, \boldsymbol{\vartheta}_i; \delta\mathbf{u}_i, \delta\boldsymbol{\vartheta}_i).$$

By Sobolev's theorem and the conditions on the boundary of inclusions and the shell, there is a constant c_0 independent of $\mathbf{U} \in V$ such that

$$\|\mathbf{U}\|_H \leqslant c_0 \|\mathbf{U}\|_V.$$

In other words, the operator of imbedding of H to V is continuous that is appropriate for the theorems in Lions and Magenes [41] for the equations of second order.

Now we can present the definition of the weak solution.

Definition 4.2. $\mathbf{U} \in L^2([0,T]; V) \bigcap_{W^{1,2}} ([0,T]; H)$ is called a weak solution of the dynamic problem for an elastic micropolar shell with rigid inclusions if it satisfies equation (39) for any $\delta\mathbf{U} \in L^2([0,T]; V)$ such that $\delta\mathbf{U}|_{t=T} = 0$ and it also satisfies the first initial conditions $(10)_1$ and $(10)_2$ in L_2 sense, that is,

$$\iint_S \left(\mathbf{u}(\mathbf{x},t)|_{t=0} - \mathbf{u}^\circ(\mathbf{x})\right)^2 dS = 0,$$

$$\iint_S \left(\boldsymbol{\phi}(\mathbf{x},t)|_{t=0} - \boldsymbol{\phi}^\circ(\mathbf{x})\right)^2 dS = 0.$$

Let us suppose that

1. $\mathbf{u}^\circ(\mathbf{x})$ and $\boldsymbol{\phi}^\circ \in W^{1,2}(S)$ and satisfy $(9)_1$ and $(9)_2$, respectively;
2. $\mathbf{v}^\circ(\mathbf{x})$, $\boldsymbol{\omega}^\circ \in L^2(S)$;
3. $\mathbf{f}(\mathbf{x},t)$, $\mathbf{c}(\mathbf{x},t) \in L^2(Q_T)$, $\mathbf{t}(s,t)$, $\mathbf{m}(s,t) \in L^2(B_T)$, where $Q_T = S \times [0,T]$, $B_T = \ell_t \times [0,T]$.

Under these assumptions and the assumptions on the shell geometry from Eremeyev and Lebedev [32] and applying the results of Lions and Magenes [41], we can prove the following.

Theorem 4.3. There exists a weak solution in the sense of Definition 4.2 which is unique.

Proof. The proof follows the proof in Lions and Magenes [41]; also it almost mimics the one for an clamped elastic membrane given in Lebedev et al. [39]. We omit this for brevity, but we should note that both proofs are based on investigation of Faedo–Galerkin's method for the problem and so simultaneously, as a result, we get a theorem on convergence of Faedo–Galerkin's approximations to the weak solution in space $L^2([0,T]; V) \bigcap_{W^{1,2}} ([0,T]; H)$. In a similar way, we can formulate an existence-uniqueness theorem for non-perfect conditions for inclusions given by equations (20) and (21). For this, similar to the procedure in the paper by Eremeyev and Lebedev [42], we should split solution $\mathbf{U}$ into two parts. The first part describes rigid motions of the inclusions and another part is orthogonal in the sense of space H to all the possible rigid motions. In this case, the rigid motions satisfy a system of ordinary differential equations, whereas the procedure for another part of the solution completely repeats the one for the problem with perfect shell–inclusions contact.

5. Conclusion

Here, we have discussed the well-posedness of dynamic problems for linear micropolar shells with rigid inclusions. We have considered three types of interactions between a shell and inclusions, including perfect contact and two types of non-perfect contact. The latter could be useful for modelling of gyroscopic structures and metamaterial thin-wall structures. The system of governing equations includes PDEs and ODEs that require a particular analysis. We formulated the principle of virtual work. Using it, we introduce the corresponding energy space. Finally, the existence and uniqueness of weak solutions were proven.

Let us note an interesting observation related to the equilibrium conditions for non-perfect contact. It is known that a necessary condition for an equilibrium consists of vanishing of the total force and total torque for any part of a structure under consideration. In other words, external forces and couples should be self-balanced; see, for example, conditions given in Eremeyev and Lebedev [32] for a shell with free boundary. Here, for an equilibrium of inclusions, we have to additionally consider the self-balance conditions for each inclusion. Otherwise an inclusion with non-perfect contact could move freely. As a result, for equilibrium of a shell with free boundary, the self-balance conditions consist of condition for the shell and for each inclusion.

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ORCID iD

Victor A Eremeyev https://orcid.org/0000-0002-8128-3262

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