On Some Zarankiewicz Numbers and Bipartite Ramsey Numbers for Quadrilateral

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Abstract

The Zarankiewicz number $z(m, n; s, t)$ is the maximum number of edges in a subgraph of $K_{m,n}$ that does not contain $K_{s,t}$ as a subgraph. The bipartite Ramsey number $b(n_1, \cdots, n_k)$ is the least positive integer b such that any coloring of the edges of $K_{b,b}$ with k colors will result in a monochromatic copy of K_{n_i,n_i} in the i-th color, for some i, $1 \leq i \leq k$. If $n_i = m$ for all i, then we denote this number by $b_k(m)$. In this paper we obtain the exact values of some Zarankiewicz numbers for quadrilateral ($s = t = 2$), and we derive new bounds for diagonal multicolor bipartite Ramsey numbers avoiding quadrilateral. In particular, we prove that $b_4(2) = 19$, and establish new general lower and upper bounds on $b_k(2)$.

1   Introduction

The Zarankiewicz number $z(m, n; s, t)$ is defined as the maximum number of edges in any subgraph G of the complete bipartite graph $K_{m,n}$, such that G does not contain $K_{s,t}$ as a subgraph. Zarankiewicz numbers and related extremal graphs have been studied by numerous authors, including Kővári, Sós, and Turán [9], Reiman [13], Irving [8], and Goddard, Henning, and Oellermann [6]. A compact summary by Bollobás can be found in [2].

The bipartite Ramsey number $b(n_1, \cdots, n_k)$ is the least positive integer b such that any coloring of the edges of the complete bipartite graph $K_{b,b}$ with k colors will result in a monochromatic copy of K_{n_i, n_i} in the i-th color, for some i, $1 \leq i \leq k$. If $n_i = m$ for all i, then we will denote this number by $b_k(m)$. The study of bipartite Ramsey numbers was initiated by Beineke and Schwenk in 1976, and continued by others, in particular Exoo [4], Hattingh and Henning [7], Goddard, Henning, and Oellermann [6], and Lazebnik and Mubayi [10].

In the remainder of this paper we consider only the case of avoiding quadrilateral C_4, i.e. the case of $s = t = 2$. Thus, for brevity, in the following the Zarankiewicz numbers will be written as $z(m, n)$ or $z(n)$, instead of $z(m, n; 2, 2)$ or $z(n, n; 2, 2)$, respectively. Similarly, the only type of Ramsey numbers we will study is the case of $b_k(2)$.

We derive new bounds for $z(m, n)$ and $z(n)$ for some general cases, and in particular we obtain some exact values of $z(n)$ for $n = q^2 + q - h$ and small $h \geq 0$. This permits to establish the exact values of $z(n)$ for all $n \leq 21$, leaving the first open case for $n = 22$. We establish new lower and upper bounds on multicolor bipartite Ramsey numbers of the form $b_k(2)$, and we compute the exact value for the first previously open case for $k = 4$, namely $b_4(2) = 19$. Now the first open case is for $k = 5$, for which we obtain the bounds $26 \leq b_5(2) \leq 28$.

During the time of reviewing and revising this paper we became aware of some recent independent work by others [3, 5, 14] on related problems, which we summarize in Section 5.

2   Zarankiewicz Numbers for Quadrilateral

In 1951, Kazimierz Zarankiewicz [15] asked what is the minimum number of 1's in a 0-1 matrix of order $n \times n$, which guarantees that it has a 2×2 minor of 1's. In the notation introduced above, it asks for the value of $z(n) + 1$.

The results and methods used to compute or estimate $z(n)$ are similar to those in the widely studied case of $\mathrm{ex}(n, C_4)$, where one seeks the maximum number of edges in any C_4-free n-vertex graph. The latter ones may have triangles (though not many since no two triangles can share an edge), which seems to cause that computing $\mathrm{ex}(n, C_4)$ is harder than $z(m)$, when the number of potential edges is about the same at $n \approx m\sqrt{2}$.

The main results to date on $z(m, n)$ or $z(n)$ were obtained in early papers by Kővári, Sós, and Turán (1954, [9]) and Reiman (1958, [13]). A nice compact summary of what is known was presented by Bollobás [2] in 1995.

Theorem 1 ([9], [13], [2])

(a) $z(m, n) \leq m/2 + \sqrt{m^2 + 4mn(n-1)}/2$ *for all $m, n \geq 1$,*

(b) $z(m) \leq (m + m\sqrt{4m-3})/2$,   *for all $m \geq 1$,*

(c) $z(p^2 + p, p^2) = p^2(p+1)$ *for primes p,*

(d) $z(q^2 + q + 1) = (q+1)(q^2 + q + 1)$ *for prime powers q, and*

(e) $\lim_{n \to \infty} z(n)/n^{3/2} = 1$.

In Theorem 1, statement (a) with $m = n$ gives statement (b), (c) is an equality in (a) for $m = p^2 + p, n = p^2$ and primes p, and (d) is an equality in (b) for $m = q^2 + q + 1$ for prime powers q. Statements (b) and (d) are widely cited in contrast to somewhat forgotten (a) and (c). The equality in statement (d) is realized by the point-line bipartite graph of any projective plane of order q. We note that the statement of Theorem 1.3.3. in [2] has a typo in (ii), where instead of $(q-1)$ it should be $(q+1)$. In the remainder of this section we will derive more cases similar to statements (c) and (d). We will be listing explicitly all coefficients in the polynomials involved, hence for easier comparison we restate (d) as

$$z(k^2 + k + 1) = k^3 + 2k^2 + 2k + 1 \tag{1}$$

for prime powers k. The results for new cases which we will consider include both lower and upper bounds on $z(n)$ for $n = k^2 + k + 1 - h$ with small h, $1 \leq h \leq 4$.

Theorem 2 *For prime powers k, for $0 \leq h \leq 4$, and for $n = k^2+k+1-h$, there exist C_4-free subgraphs of $K_{n,n}$ of sizes establishing lower bounds for $z(n)$ as follows:*

$$z(k^2 + k + 1 - h) \geq \begin{cases} k^3 + 2k^2 + 2k + 1 & for \ \ h = 0, \\ k^3 + 2k^2 & for \ \ h = 1, \\ k^3 + 2k^2 - 2k & for \ \ h = 2, \\ k^3 + 2k^2 - 4k + 1 & for \ \ h = 3, \ \ and \\ k^3 + 2k^2 - 6k + 2 & for \ \ h = 4. \end{cases} \tag{2}$$

Proof. For each prime power k, consider the bipartite graph $G_k = (P_k \cup B_k, E_k)$ of a projective plane of order k, on the partite sets P_k (points) and B_k (lines). We have $|P_k| = |B_k| = k^2 + k + 1$, $|E_k| = k^3 + 2k^2 + 2k + 1$, and for $p \in P_k$ and $l \in B_k$, $\{p, l\} \in E_k$ if and only if point p is on line l. One can easily see that G_k is $(k+1)$-regular and C_4-free. We will construct the induced subgraphs $H(k, h)$ of G_k by removing h points from P_k and h lines from B_k, where the removed vertices $\{p_1, \cdots, p_h\} \cup \{l_1, \cdots, l_h\}$ induce $s(h)$ edges in G_k. Then, the number of edges in $H(k, h)$ is equal to

$$|E_k| - 2(k+1)h + s(h). \tag{3}$$

It is easy to choose the removed vertices so that $s(h) = 0, 1, 3, 6, 9$ for $h = 0, 1, 2, 3, 4$, respectively. The case $h = 0$ is trivial, for $h = 1$ we take a point on a line, and for $h = 2$ we take points p_1, p_2, the line l_1 containing them, and a second line l_2 containing p_2, so that $p_1 l_1 p_2 l_2$ forms a path P_3. Consider three points not on a line and three lines defined by them for $h = 3$, then such removed parts induce a C_6. Finally, for $h = 4$, we take three collinear points $\{p_1, p_2, p_3\}$ on line l_1, p_4 not on l_1, and three lines passing through p_4 and the first three points. It is easy to see that these vertices induce a subgraph of $K_{4,4}$ with 9 edges. To complete the proof observe that the right hand sides of (2) are equal to the values of (3) for corresponding h. □

Next, for $1 \leq h \leq 3$, we obtain the upper bound on $z(k^2 + k + 1 - h)$ equal to the lower bound in Theorem 2. We observe that now we do not require k to be a prime power, and that obviously the equality holds in (2) for $h = 0$ by (1).

Theorem 3 *For all $k \geq 2$,*

$$z(k^2 + k + 1 - h) \leq \begin{cases} k^3 + 2k^2 & for \quad h = 1, \\ k^3 + 2k^2 - 2k & for \quad h = 2, \ and \\ k^3 + 2k^2 - 4k + 1 & for \quad h = 3. \end{cases} \qquad (4)$$

Proof. We will proceed with the steps A through G below in a similar way for $h = 1, 2$ and 3, and we will label an item by (X.hi) if it is a part of step X for $h = i$. For $h = 3$ and $k = 2, 3$, it is known that $z(4) = 9$ and $z(10) = 34$ [6] (see also Table 1 below), and these values satisfy (4). Hence, in the rest of the proof we will assume that $k \geq 4$ for $h = 3$. First we prove that

(A.$h1$) $z(k^2 + 1, k^2 + k) < (k + 1)(k^2 + 1)$,

(A.$h2$) $z(k^2 - k + 1, k^2 + k - 1) < (k + 1)(k^2 - k + 1)$, and

(A.$h3$) $z(k^2 - 2k + 2, k^2 + k - 2) < (k + 1)(k^2 - 2k + 2)$.

In (A.hi) we aim at the smallest m, so that $z(m, n) < (k+1)m$ can still be proven by our method for $n = k^2 + k + 1 - h$. Suppose for contradiction that a bipartite graph H, with the partite sets L and R of suitable orders, attains any right hand side in (A). We will count the number of paths P_3 of type LRL in H. Since

(B.$h1$) $\frac{(k+1)(k^2+1)}{k^2+k} = k + \frac{k+1}{k^2+k} = k + \frac{1}{k}$,

(B.$h2$) $\frac{(k+1)(k^2-k+1)}{k^2+k-1} = (k - 1) + \frac{2k}{k^2+k-1}$, and

(B.$h3$) $\frac{(k+1)(k^2-2k+2)}{k^2+k-2} = (k - 2) + \frac{4k-2}{k^2+k-2}$,

we conclude that the minimum number of such paths is achieved in H when R has the degree sequence of

(C.$h1$) $(k + 1)$ vertices of degree $(k + 1)$ and $(k^2 - 1)$ vertices of degree k,

(C.$h2$) $2k$ vertices of degree k and $(k^2 - k - 1)$ vertices of degree $(k - 1)$, or

(C.$h3$) $4k - 2$ vertices of degree $(k - 1)$ and $(k^2 - 3k)$ vertices of degree $(k - 2)$,

respectively. Hence, the number of LRL paths in H is at least

(D.h1) $(k+1)\binom{k+1}{2} + (k^2-1)\binom{k}{2} = \frac{1}{2}k(k+1)(k^2-k+2)$,

(D.h2) $2k\binom{k}{2} + (k^2-k-1)\binom{k-1}{2} = \frac{1}{2}(k-1)(k^3-k^2+k+2)$, or

(D.h3) $(4k-2)\binom{k-1}{2} + (k^2-3k)\binom{k-2}{2} = \frac{1}{2}(k-2)(k^3-2k^2+3k+2)$.

On the other hand

(E.h1) $\binom{k^2+1}{2} = \frac{1}{2}k^2(k^2+1)$,

(E.h2) $\binom{k^2-k+1}{2} = \frac{1}{2}(k-1)(k^3-k^2+k)$, and

(E.h3) $\binom{k^2-2k+2}{2} = \frac{1}{2}(k^4-4k^3+7k^2-6k+2)$.

Observe that the following hold:

(F.h1) $k(k+1)(k^2-k+2) > k^2(k^2+1)$ for $k \geq 1$,

(F.h2) $(k-1)(k^3-k^2+k+2) > (k-1)(k^3-k^2+k)$ for $k \geq 2$, and

(F.h3) $(k-2)(k^3-2k^2+3k+2) > (k^4-4k^3+7k^2-6k+2)$ for $k \geq 4$,

which imply that (D.hi) > (E.hi) for the three cases and for k as specified in (F). Consequently, in all these cases there exist two LRL paths in H which share both of their endpoints. This creates C_4 which is a contradiction, and thus (A) holds. Further, we can see that any C_4-free bipartite graph with partite sets L and R of orders as in H must have the minimum degree on part L at most k (otherwise (A) would not be true).

Finally, consider any C_4-free bipartite graph G with both partite sets of order $n = k^2 + k + 1 - h$. Any of its subgraphs of partite orders of H must have at least one vertex of degree at most k in L, and together with (A) this implies that G has at most

(G.h1) $(k+1)k^2 + kk = k^3 + 2k^2$,

(G.h2) $(k+1)(k^2-k) + k(2k-1) = k^3 + 2k^2 - 2k$, or

(G.h3) $(k+1)(k^2-2k+1) + k(3k-3) = k^3 + 2k^2 - 4k + 1$

edges for $h = 1, 2, 3$, respectively. These values are the same as the upper bounds claimed in (4), which completes the proof of Theorem 3. □

Theorem 4 *For any prime power k, and also for $k = 1$,*

$$z(k^2 + k + 1 - h) = \begin{cases} k^3 + 2k^2 + 2k + 1 & \text{for } h = 0, \\ k^3 + 2k^2 & \text{for } h = 1, \\ k^3 + 2k^2 - 2k & \text{for } h = 2, \text{ and} \\ k^3 + 2k^2 - 4k + 1 & \text{for } h = 3. \end{cases}$$

Proof. Theorems 1(d), 2 and 3 imply the equality for all prime powers k. The easy cases for $k = 1$ hold as well, as can be checked in Table 1. $\quad\square$

Goddard, Henning and Oellermann obtained the value $z(18) = 81$, and their proof is a special case of our Theorems 2 and 3 for $k = 4$ and $h = 3$. We were not able to prove the general upper bound of $k^3 + 2k^2 - 6k + 2$ for $h = 4$, but we expect that it is true. We could only obtain one special case for $k = 4$, namely $z(17) = 74$, which is established later in this section in Lemma 6. Thus, we consider that Theorem 2 and the known values of $z(n)$ for $n = k^2 + k - 3$, $k = 2, 3, 4$ (see Table 1), provide strong evidence for the following conjecture.

Conjecture 5 *For any prime power k,*

$$z(k^2 + k - 3) = k^3 + 2k^2 - 6k + 2.$$

Previous work by others [6], Theorem 4, computations using `nauty`, the special case in Lemma 6 below, and the comments above, give all the values of $z(n)$ for $n \le 21$. They are listed in Table 1, together with the parameters k and h when applicable. This leaves $z(22)$ as the first open case. Note that in Table 1 the only cases not covered by Theorem 4 or Conjecture 5 are those for $n = 8, 14, 15$ and 16.

n	k	h	$z(n)$	n	k	h	$z(n)$	n	k	h	$z(n)$
1	1	2	1	8			24	15			61
2	1	1	3	9	3	4	29	16			67
3	1	0	6	10	3	3	34	17	4	4	74
4	2	3	9	11	3	2	39	18	4	3	81
5	2	2	12	12	3	1	45	19	4	2	88
6	2	1	16	13	3	0	52	20	4	1	96
7	2	0	21	14			56	21	4	0	105

Table 1: $z(n)$ for $1 \le n \le 21$ with k, h for $n = k^2 + k + 1 - h$, $h \le 4$.

With the help of the package `nauty` developed by Brendan McKay [12], one can easily obtain the values of $z(n)$ for $n \leq 16$ and confirm the values of related numbers and extremal graphs presented in [6]. However, `nauty` cannot complete this task for $n \geq 17$. The cases $18 \leq n \leq 21$ are settled by Theorem 4, hence we fill in the only missing case of $n = 17$ with the following lemma.

Lemma 6 $z(17) = 74$.

Proof. For the upper bound, suppose that there exists a C_4-free bipartite graph $H = (L \cup R, E)$ with $|L| = |R| = 17$, which has 75 edges. Since $z(16) = 67$, then for every edge $\{u, v\} \in E$ we must have $\deg(u) + \deg(v) \geq 9$. Let u be a vertex of minimum degree δ in L. Clearly $\delta \leq 4$. Removing u from L gives a subgraph H' of $K_{16,17}$ with $75 - \delta$ edges and minimum degree δ' in the part R of H'. Now, $z(16) = 67$ implies that $\delta + \delta' \geq 8$, which in turn leaves the only possibility $\delta = \delta' = 4$. Hence, all four neighbors of u in H, $\{v_1, v_2, v_3, v_4\}$, must have degree at least 5. Furthermore, in order to avoid C_4, the neighborhoods N_i of v_i cannot have any other intersection than $\{u\}$. Thus, $L \setminus \{u\}$ is partitioned into four 4-sets $N_i \setminus \{u\}$ and $\deg(v_i) = 5$, for $1 \leq i \leq 4$. All of the other 13 vertices in R are not connected to u, they can have at most one adjacent vertex in each of the N_i's, and hence they have degree at most 4. This implies that H has at most 72 edges, which yields a contradiction.

The lower bound construction with 74 edges is provided by Theorem 2 with $k = h = 4$. $\square$

Finally, we note that the method of the proof of Lemma 6 cannot be applied to the first open case of Conjecture 5, $z(27)$, since it would require a good bound for an open case of $z(26)$.

3 Bipartite Ramsey Numbers $b_k(2)$

The determination of values of $b_k(2)$ appears to be difficult. The only known exact results are: Beineke and Schwenk proved that $b_2(2) = 5$ [1], Exoo found the second value $b_3(2) = 11$ [4], and in the next section we show that $b_4(2) = 19$.

A construction by Lazebnik and Woldar [11] yields $r_k(C_4) \geq k^2 + 2$ for prime powers k, where $r_k(G)$ is the classical Ramsey number defined as the least n such that there is a monochromatic copy of G in any k-coloring of

the edges of K_n. We use a slight modification of a similar construction from [10], furthermore only for the special case of graphs avoiding C_4 (versus r-uniform hypergraphs avoiding $K_{2,t+1}^{(r)}$). In addition, in our case we color the edges of K_{k^2,k^2}, while for the graph case in [11, 10] the edges of K_{k^2+1} are colored. This gives us a new lower bound on $b_k(2)$, which almost doubles an easy bound $b_k(2) \geq r_k(C_4)/2$, as follows:

Theorem 7 *For any prime power k, we have*

$$b_k(2) \geq k^2 + 1.$$

Proof. Let k be any prime power and let $n = k^2$. We will define a k-coloring of the edges of $K_{n,n}$ without monochromatic C_4's. Let F be a k-element field, and consider the partite sets $L = \{(a,b) \in F \times F\}$ and $R = \{(a',b') \in F \times F\}$. Color an edge between two vertices in L and R with color $\alpha \in F$ if and only if

$$a \cdot a' - b - b' = \alpha.$$

Denote by G_α the graph consisting of the edges in color α. We claim that G_α contains no monochromatic copy of C_4. First we argue that for $(p_1, s_1), (p_2, s_2) \in L$, $(p_1, s_1) \neq (p_2, s_2)$, the system

$$
\begin{aligned}
p_1 x - s_1 - y &= \beta \\
p_2 x - s_2 - y &= \beta
\end{aligned}
$$

has at most one solution $(x,y) \in R$ for every $\beta \in F$. Suppose that:

$$
\begin{aligned}
p_1 x - s_1 - y &= \beta & (5) \\
p_2 x - s_2 - y &= \beta & (6) \\
p_1 x' - s_1 - y' &= \beta & (7) \\
p_2 x' - s_2 - y' &= \beta & (8)
\end{aligned}
$$

Adding (6) and (7) and subtracting (5) and (8) yields $(p_2 - p_1)(x - x') = 0$, which implies that $p_1 = p_2$ or $x = x'$. If $p_1 = p_2$, then (5) and (6) imply that $s_1 = s_2$, yielding a contradiction $(p_1, s_1) = (p_2, s_2)$. On the other hand, if $x = x'$, then (6) and (8) imply $y = y'$, which gives $(x,y) = (x',y')$. $\square$

Our next theorem improves by one the upper bound on $b_k(2)$ established by Hattingh and Henning in 1998 [7], for all $k \geq 5$.

Theorem 8 *For all $k \geq 5$,*

$$b_k(2) \leq k^2 + k - 2.$$

Proof. For $n = k^2 + k - 2$, suppose that there exists a k-coloring $\mathcal{C}$ of the edges of $K_{n,n}$ without monochromatic C_4's. Theorem 3 with $h = 3$ implies that $\mathcal{C}$ has at most $k^3 + 2k^2 - 4k + 1$ edges in any of the colors, and thus at most $m = k(k^3 + 2k^2 - 4k + 1)$ edges in $\mathcal{C}$ are colored. One can easily check that $m < n^2$ for $k \geq 5$, which completes the proof. $\square$

We note that the bound of Theorem 8 is better than one which could be obtained by the same method using Theorem 1(b) instead of Theorem 3. Observe also that in the proof of Theorem 8 with $k = 4$ there is no contradiction, since using $z(18) = 81$ one obtains $n^2 = m = 324$, and hence a 4-coloring $\mathcal{C}$ of $K_{18,18}$ is not ruled out. Indeed, we have constructed a few of them, and one is presented in the next section.

4   The Ramsey Number $b_4(2)$

Theorem 9
$$b_4(2) = 19.$$

Proof. The same reasoning as in Theorem 8, but now for $k = 4$ and $n = k^2 + k - 1$, gives $m = 4z(19) = 352 < 361 = n^2$, which implies the upper bound. The lower bound follows from a 4-coloring $\mathcal{D}$ of $K_{18,18}$ without monochromatic C_4's presented in Figures 1 and 2. This completes the proof, though we will still give an additional description and comments on the coloring $\mathcal{D}$ in the following. $\square$

Goddard et al. [6] showed that any extremal graph for $z(18)$ must have the degree sequence $n_4 = n_5 = 9$ on both partite sets. By using a computer algorithm, we have found that such graph is unique up to isomorphism, and thus it also must be the same as one described in the proof of Theorem 2 for $k = 4$ and $h = 3$. Let us denote it by G_{18}, and consider its labeling as in Figure 1. Note that four 9×9 quarters of G_{18} have the structure

$$G_{18} = \begin{bmatrix} 3C_6 & S^T \\ S & 9K_2 \end{bmatrix},$$

$$
\begin{array}{ccc|ccc|ccc||ccc|ccc|ccc}
0&1&1 & 0&0&0 & 0&0&0 & 1&0&0 & 1&0&0 & 1&0&0\\
1&0&1 & 0&0&0 & 0&0&0 & 0&1&0 & 0&1&0 & 0&1&0\\
1&1&0 & 0&0&0 & 0&0&0 & 0&0&1 & 0&0&1 & 0&0&1\\
\hline
0&0&0 & 0&1&1 & 0&0&0 & 1&0&0 & 0&0&1 & 0&1&0\\
0&0&0 & 1&0&1 & 0&0&0 & 0&1&0 & 1&0&0 & 0&0&1\\
0&0&0 & 1&1&0 & 0&0&0 & 0&0&1 & 0&1&0 & 1&0&0\\
\hline
0&0&0 & 0&0&0 & 0&1&1 & 0&0&1 & 1&0&0 & 0&1&0\\
0&0&0 & 0&0&0 & 1&0&1 & 1&0&0 & 0&1&0 & 0&0&1\\
0&0&0 & 0&0&0 & 1&1&0 & 0&1&0 & 0&0&1 & 1&0&0\\
\hline\hline
1&0&0 & 1&0&0 & 0&1&0 & 1&0&0 & 0&0&0 & 0&0&0\\
0&1&0 & 0&1&0 & 0&0&1 & 0&1&0 & 0&0&0 & 0&0&0\\
0&0&1 & 0&0&1 & 1&0&0 & 0&0&1 & 0&0&0 & 0&0&0\\
\hline
1&0&0 & 0&1&0 & 1&0&0 & 0&0&0 & 1&0&0 & 0&0&0\\
0&1&0 & 0&0&1 & 0&1&0 & 0&0&0 & 0&1&0 & 0&0&0\\
0&0&1 & 1&0&0 & 0&0&1 & 0&0&0 & 0&0&1 & 0&0&0\\
\hline
1&0&0 & 0&0&1 & 0&0&1 & 0&0&0 & 0&0&0 & 1&0&0\\
0&1&0 & 1&0&0 & 1&0&0 & 0&0&0 & 0&0&0 & 0&1&0\\
0&0&1 & 0&1&0 & 0&1&0 & 0&0&0 & 0&0&0 & 0&0&1\\
\end{array}
$$

Figure 1: G_{18}, the unique extremal graph for $z(18)$.

$$
\begin{array}{ccc|ccc|ccc||ccc|ccc|ccc}
2&1&1 & 3&3&4 & 4&4&3 & 1&3&2 & 1&4&2 & 1&2&3\\
1&2&1 & 4&3&3 & 3&4&4 & 2&1&3 & 2&1&4 & 3&1&2\\
1&1&2 & 3&4&3 & 4&3&4 & 3&2&1 & 4&2&1 & 2&3&1\\
\hline
4&4&3 & 2&1&1 & 3&3&4 & 1&3&2 & 2&3&1 & 2&1&4\\
3&4&4 & 1&2&1 & 4&3&3 & 2&1&3 & 1&2&3 & 4&2&1\\
4&3&4 & 1&1&2 & 3&4&3 & 3&2&1 & 3&1&2 & 1&4&2\\
\hline
3&3&4 & 4&4&3 & 2&1&1 & 2&4&1 & 1&3&2 & 2&1&3\\
4&3&3 & 3&4&4 & 1&2&1 & 1&2&4 & 2&1&3 & 3&2&1\\
3&4&3 & 4&3&4 & 1&1&2 & 4&1&2 & 3&2&1 & 1&3&2\\
\hline\hline
1&2&4 & 1&2&4 & 2&1&3 & 1&2&2 & 4&3&4 & 3&3&4\\
4&1&2 & 4&1&2 & 3&2&1 & 2&1&2 & 4&4&3 & 4&3&3\\
2&4&1 & 2&4&1 & 1&3&2 & 2&2&1 & 3&4&4 & 3&4&3\\
\hline
1&2&3 & 2&1&4 & 1&2&4 & 3&4&3 & 1&2&2 & 4&4&3\\
3&1&2 & 4&2&1 & 4&1&2 & 3&3&4 & 2&1&2 & 3&4&4\\
2&3&1 & 1&4&2 & 2&4&1 & 4&3&3 & 2&2&1 & 4&3&4\\
\hline
1&4&2 & 2&3&1 & 2&4&1 & 3&4&4 & 4&3&3 & 1&2&2\\
2&1&4 & 1&2&3 & 1&2&4 & 4&3&4 & 3&4&3 & 2&1&2\\
4&2&1 & 3&1&2 & 4&1&2 & 4&4&3 & 3&3&4 & 2&2&1\\
\end{array}
$$

Figure 2: $\mathcal{D}$, a 4-coloring of the edges of $K_{18,18}$ without monochromatic C_4's.

where S is the point-block bipartite subgraph of $K_{9,9}$ obtained from the unique Steiner triple system on 9 points with any three parallel blocks removed (out of the total of 12 blocks).

Coloring $\mathcal{D}$ has 81 edges in each of the four colors, and each of them induces a graph isomorphic to G_{18}. Note that colors 1 and 2 swap and overlay their corresponding quarters $3C_6$ and $9K_2$, so that $6K_{3,3}$ is formed. The colors 3 and 4 have the same structure. We have constructed 8 non-isomorphic colorings with the same properties as those listed for $\mathcal{D}$, but

there may be more of them. They were constructed as follows: First, we overlayed two quarters $3C_6$ and $9K_2$ of the two first colors as in $\mathcal{D}$, and then we applied some heuristics to complete the overlay of the first two colors. Finally, the bipartite complement of this overlay was split into colors 3 and 4 by standard SAT-solvers. These were applied to a naturally constructed Boolean formula, whose variables decide which of the colors 3 or 4 is used for still uncolored edges, so that no monochromatic C_4 is created. Many successful splits were made, but only 8 of them were nonisomorphic (20 if the colors are fixed under isomorphisms), and all of them have the same structure as $\mathcal{D}$.

The first open case of $b_k(2)$ is now for 5 colors, for which we know that $26 \leq b_5(2) \leq 28$. The lower bound is implied by Theorem 7, while the upper bound by Theorem 8. We believe that the correct value is 28.

Theorem 10
$$26 \leq b_5(2) \leq 28.$$

Conjecture 11
$$b_5(2) = 28.$$

5 Addendum

We would like to add some notes on other independent work about which we became aware while our paper was in review. This includes the work by Steinbach and Posthoff [14], who achieved the lower bound construction for $b_4(2)$, but by very different means. Interestingly, their construction is isomorphic to ours. The essence of our Theorem 3 is subsumed by results in the paper by Damásdi, Héger and Szőnyi [3], but our proofs are much simpler. Finally, Fenner, Gasarch, Glover and Purewal [5] wrote a very extensive survey of the area of grid colorings, which are essentially equivalent to edge colorings of complete bipartite graphs.

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