

## ON THE HAT PROBLEM ON A GRAPH

Marcin Krzywkowski

**Abstract.** The topic of this paper is the hat problem in which each of  $n$  players is uniformly and independently fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The solution of the hat problem on a graph is known for trees and for cycles on four or at least nine vertices. In this paper first we give an upper bound on the maximum chance of success for graphs with neighborhood-dominated vertices. Next we solve the problem on unicyclic graphs containing a cycle on at least nine vertices. We prove that the maximum chance of success is one by two. Then we consider the hat problem on a graph with a universal vertex. We prove that there always exists an optimal strategy such that in every case some vertex guesses its color. Moreover, we prove that there exists a graph with a universal vertex for which there exists an optimal strategy such that in some case no vertex guesses its color. We also give some Nordhaus-Gaddum type inequalities.

**Keywords:** hat problem, graph, degree, neighborhood, neighborhood-dominated, unicyclic, universal vertex, Nordhaus-Gaddum.

**Mathematics Subject Classification:** 05C07, 05C38, 05C99, 91A12.

### 1. INTRODUCTION

In the hat problem, a team of  $n$  players enters a room and a blue or red hat is uniformly and independently placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the “seven prisoners puzzle”, was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [26], Die Zeit [6], and abcNews [25]. It is also one of the Berkeley Riddles [4].

The hat problem with  $2^k - 1$  players was solved in [14], and for  $2^k$  players in [11]. The problem with  $n$  players was investigated in [7]. The hat problem and Hamming codes were the subject of [8]. The generalized hat problem with  $n$  people and  $q$  colors was investigated in [24].

There are many known variations of the hat problem (for a comprehensive list, see [22]). For example in [19] there was considered a variation in which players do not have to guess their hat colors simultaneously. In the papers [1, 10, 18] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [16] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [17] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are  $n$  players which have random bits on foreheads, and they have to vote on the parity of the  $n$  bits.

The hat problem and its variations have many applications and connections to different areas of science (for a survey on this topic, see [22]), for example: information technology [5], linear programming [16], genetic programming [9], economics [1, 18], biology [17], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15]. Therefore, it is hoped that the hat problem on a graph considered in this paper is worth exploring as a natural generalization, and may also have many applications.

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [20]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [21] the problem was solved on the cycle  $C_4$ . The problem on cycles on at least nine vertices was solved in [23].

In this paper first we give an upper bound on the maximum chance of success for graphs with neighborhood-dominated vertices. We use this bound to solve the hat problem on the graph obtained from  $K_4$  by the subdivision of one edge. We also prove that there exists a graph having two vertices with the same open neighborhood for which there exists an optimal strategy such that in some situation both those vertices guess their colors. Next we solve the problem on unicyclic graphs containing a cycle on at least nine vertices. We prove that the maximum chance of success is one by two. Then we consider the hat problem on a graph with a universal vertex. We prove that there always exists an optimal strategy such that in every case some vertex guesses its color. Moreover, we prove that there exists a graph with a universal vertex for which there exists an optimal strategy such that in some case no vertex guesses its color. We also give some Nordhaus-Gaddum type inequalities.

## 2. PRELIMINARIES

For a graph  $G$ , the set of vertices and the set of edges we denote by  $V(G)$  and  $E(G)$ , respectively. By complement of  $G$ , denoted by  $\overline{G}$ , we mean a graph which has the same vertices as  $G$ , and two distinct vertices of  $\overline{G}$  are adjacent if and only if they are not adjacent in  $G$ . If  $H$  is a subgraph of  $G$ , then we write  $H \subseteq G$ . Let  $v \in V(G)$ . The open neighborhood of  $v$ , that is  $\{x \in V(G) : vx \in E(G)\}$ , we denote by  $N_G(v)$ . We say that a vertex of  $G$  is universal if it is adjacent to every one of the remaining vertices. By a leaf we mean a vertex having exactly one neighbor. We say that a vertex  $v$  of a graph  $G$  is neighborhood-dominated in  $G$  if there is some other vertex  $w \in V(G)$  such that  $N_G(v) \subseteq N_G(w)$ . We say that a graph is unicyclic if it contains exactly one cycle as a subgraph.

The degree of vertex  $v$ , that is, the number of its neighbors, we denote by  $d_G(v)$ . Thus  $d_G(v) = |N_G(v)|$ . The path (cycle, complete graph, respectively) on  $n$  vertices we denote by  $P_n$  ( $C_n$ ,  $K_n$ , respectively).

Let  $f: X \rightarrow Y$  be a function. If for every  $x \in X$  we have  $f(x) = y$ , then we write  $f \equiv y$ .

Let  $V(G) = \{v_1, v_2, \dots, v_n\}$ . By  $Sc = \{1, 2\}$  we denote the set of colors, where 1 corresponds to the blue color, and 2 corresponds to the red color.

By a case for a graph  $G$  we mean a function  $c: V(G) \rightarrow \{1, 2\}$ , where  $c(v_i)$  means color of vertex  $v_i$ . The set of all cases for the graph  $G$  we denote by  $C(G)$ , of course  $|C(G)| = 2^{|V(G)|}$ .

By a situation of a vertex  $v_i$  we mean a function  $s_i: V(G) \rightarrow Sc \cup \{0\} = \{0, 1, 2\}$ , where  $s_i(v_j) \in Sc$  if  $v_i$  and  $v_j$  are adjacent, and 0 otherwise. The set of all possible situations of  $v_i$  in the graph  $G$  we denote by  $St_i(G)$ , of course  $|St_i(G)| = 2^{d_G(v_i)}$ .

We say that a case  $c$  for the graph  $G$  corresponds to a situation  $s_i$  of vertex  $v_i$  if  $c(v_j) = s_i(v_j)$ , for every  $v_j$  adjacent to  $v_i$ . This implies that a case corresponds to a situation of  $v_i$  if every vertex adjacent to  $v_i$  in the case has the same color as in the situation. Of course, to every situation of the vertex  $v_i$  correspond exactly  $2^{|V(G)| - d_G(v_i)}$  cases.

By a guessing instruction of a vertex  $v_i \in V(G)$  we mean a function  $g_i: St_i(G) \rightarrow Sc \cup \{*\} = \{1, 2, *\}$ , which for a given situation gives the color  $v_i$  guesses if  $g_i(s_i) \neq *$ , otherwise  $v_i$  passes. Thus a guessing instruction is a rule determining the behavior of a vertex in every situation. We say that  $v_i$  never guesses its color if  $v_i$  passes in every situation, that is,  $g_i \equiv *$ .

Let  $c$  be a case, and let  $s_i$  be the situation (of vertex  $v_i$ ) corresponding to that case. The guess of  $v_i$  in the case  $c$  is correct (wrong, respectively) if  $g_i(s_i) = c(v_i)$  ( $* \neq g_i(s_i) \neq c(v_i)$ , respectively). By result of the case  $c$  we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is,  $g_i(s_i) = c(v_i)$  (for some  $i$ ) and there is no  $j$  such that  $* \neq g_j(s_j) \neq c(v_j)$ . Otherwise the result of the case  $c$  is a loss.

By a strategy for the graph  $G$  we mean a sequence  $(g_1, g_2, \dots, g_n)$ , where  $g_i$  is the guessing instruction of vertex  $v_i$ . The family of all strategies for a graph  $G$  we denote by  $\mathcal{F}(G)$ .

If  $S \in \mathcal{F}(G)$ , then the set of cases for the graph  $G$  for which the team wins (loses,

respectively) using the strategy  $S$  we denote by  $W(S)$  ( $L(S)$ , respectively). The set of cases for which the team loses, and no vertex guesses its color we denote by  $Ln(S)$ . By the chance of success of the strategy  $S$  we mean the number  $p(S) = |W(S)|/|C(G)|$ . By the hat number of the graph  $G$  we mean the number  $h(G) = \max\{p(S) : S \in \mathcal{F}(G)\}$ . We say that a strategy  $S$  is optimal for the graph  $G$  if  $p(S) = h(G)$ . The family of all optimal strategies for the graph  $G$  we denote by  $\mathcal{F}^0(G)$ .

By solving the hat problem on a graph  $G$  we mean finding the number  $h(G)$ .

Let  $G$  and  $H$  be graphs. Assume that  $H \subseteq G$ . Since every vertex from the set  $V(G) \setminus V(H)$  can always pass, and every vertex  $v_i \in V(H)$  can ignore the colors of vertices from the set  $N_G(v_i) \setminus N_H(v_i)$ , we get the following relation between numbers  $h(H)$  and  $h(G)$ .

**Fact 2.1.** *If  $H$  is a subgraph of  $G$ , then  $h(H) \leq h(G)$ .*

Since the one-vertex graph is a subgraph of every graph, we get the following corollary.

**Corollary 2.2.** *For every graph  $G$  we have  $h(G) \geq 1/2$ .*

Using the definition of an optimal strategy, we immediately get the following corollary.

**Corollary 2.3.** *Let  $G$  be a graph. If  $S \in \mathcal{F}^0(G)$ , then  $p(S) \geq 1/2$ .*

The following four results are from [20]. The first of them states that there does not exist any graph such that the team can always win.

**Fact 2.4.** *For every graph  $G$  we have  $h(G) < 1$ .*

Now we state that a guess of any other vertex is unnecessary in a case in which some vertex already guesses its color.

**Fact 2.5.** *Let  $G$  be a graph, and let  $S$  be a strategy for this graph. Let  $c$  be a case in which some vertex guesses its color. Then a guess of any other vertex cannot improve the result of the case  $c$ .*

Now there is a sufficient condition for deleting a vertex of a graph without changing its hat number.

**Theorem 2.6.** *Let  $G$  be a graph, and let  $v$  be a vertex of  $G$ . If there exists a strategy  $S \in \mathcal{F}^0(G)$  such that  $v$  never guesses its color, then  $h(G) = h(G - v)$ .*

The next theorem is the solution of the hat problem on trees.

**Theorem 2.7.** *For every tree  $T$  we have  $h(T) = 1/2$ .*

The following solution of the hat problem on cycles on at least nine vertices is a result from [23]. It was obtained by proving that even if every vertex guesses its color in exactly one situation, then in at least half of all cases some vertex guesses its color wrong, causing the loss of the team.

**Theorem 2.8.** *For every integer  $n \geq 9$  we have  $h(C_n) = 1/2$ .*

### 3. HAT PROBLEM ON A GRAPH WITH NEIGHBORHOOD – DOMINATED VERTICES

In this section we consider the hat problem on graphs with neighborhood-dominated vertices.

First, we investigate optimal strategies for such graphs.

**Theorem 3.1.** *Let  $G$  be a graph, and let  $v_1$  and  $v_2$  be vertices of  $G$ . If  $N_G(v_1) \subseteq N_G(v_2)$ , then there exists an optimal strategy for the graph  $G$  such that there is no case in which both vertices  $v_1$  and  $v_2$  guess their colors.*

*Proof.* Suppose that for every optimal strategy for the graph  $G$  there exists a case in which both  $v_1$  and  $v_2$  guess their colors. Let  $S$  be any optimal strategy for  $G$ . Let  $c_1, c_2, \dots, c_k$  be the cases in which both vertices  $v_1$  and  $v_2$  guess their colors. These cases correspond to the situations  $s_2^1, s_2^2, \dots, s_2^k$  of  $v_2$  ( $s_2^i \neq s_2^j$  for  $i \neq j$ ). Let the strategy  $S'$  for the graph  $G$  differ from  $S$  only in that  $v_2$  does not guess its color in the situations  $s_2^1, s_2^2, \dots, s_2^k$ . Since in every one of the cases corresponding to these situations  $v_1$  guesses its color, by Fact 2.5 the guess of  $v_2$  cannot improve the result of any one of these cases. Therefore  $p(S) \leq p(S')$ . Since  $S \in \mathcal{F}^0(G)$ , the strategy  $S'$  is also optimal. In this strategy there is no case in which both  $v_1$  and  $v_2$  guess their colors.  $\square$

**Corollary 3.2.** *Let  $G$  be a graph, and let  $v_1, v_2, \dots, v_k$  be vertices of  $G$  such that  $N_G(v_1) = N_G(v_2) = \dots = N_G(v_k)$ . Then there exists an optimal strategy for the graph  $G$  such that in each situation at most one of the vertices  $v_1, v_2, \dots, v_k$  guesses its color.*

In the next fact we state that there exists a graph having two vertices with the same open neighborhood for which there exists an optimal strategy such that in some situation both those vertices guess their colors.

**Fact 3.3.** *There exists an optimal strategy for the path  $P_3$  such that in some situation the two vertices having the same open neighborhood guess their colors.*

*Proof.* Let  $E(P_3) = \{v_1v_2, v_2v_3\}$ , and let  $S = (g_1, g_2, g_3) \in \mathcal{F}(P_3)$  be the strategy as follows.

$$\begin{aligned} g_1(s_1) &= \begin{cases} 1 & \text{if } s_1(v_2) = 2, \\ 0 & \text{otherwise,} \end{cases} \\ g_2(s_2) &\equiv 1, \\ g_3(s_3) &= \begin{cases} 1 & \text{if } s_3(v_2) = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It means that the vertices proceed as follows.

- The vertex  $v_1$ . If  $v_2$  has the second color, then  $v_1$  guesses it has the first color, otherwise it passes.
- The vertex  $v_2$  always guesses it has the first color.

- The vertex  $v_3$ . If  $v_2$  has the second color, then  $v_3$  guesses it has the first color, otherwise it passes.

It is not difficult to verify that  $|W(S)| = 4$ . Since  $|C(P_3)| = 8$ , we get  $p(S) = 4/8 = 1/2$ . By Theorem 2.7 we have  $h(P_3) = 1/2$ , therefore the strategy  $S$  is optimal. We have  $N_{P_3}(v_1) = N_{P_3}(v_3)$ , and in the strategy  $S$  both vertices  $v_1$  and  $v_3$  guess their colors in the situation when  $v_2$  has the second color.  $\square$

Let  $G$  be a graph, and let  $A_1, A_2, \dots, A_k$  be a partition of the set of vertices of  $G$  such that the open neighborhoods of the vertices of each set  $A_i$  can be linearly ordered by inclusion.

Now we give an upper bound on the chance of success for any strategy for the hat problem on a graph with neighborhood-dominated vertices.

**Theorem 3.4.** *Let  $G$  be a graph, and let  $k$  mean the minimum number of sets to which  $V(G)$  can be partitioned in a way described above. Then  $h(G) \leq k/(k+1)$ .*

*Proof.* Theorem 3.1 implies that there exists a strategy  $S \in \mathcal{F}^0(G)$  such that in every case at most one vertex from each set  $A_i$  guesses its color. The number of cases in which the vertices of  $A_i$  guess their colors in the strategy  $S$  is at most  $2(2^{|V(G)|} - |W(S)|)$ , otherwise the number of cases in which some of these vertices guesses its color wrong is greater than  $2^{|V(G)|} - |W(S)|$ . This implies that the team loses for more than  $2^{|V(G)|} - |W(S)|$  cases, and therefore the number of cases for which the team wins is less than  $|W(S)|$ . This is a contradiction as  $|W(S)|$  is the number of cases for which the team wins. In half of all cases the guesses of the vertices of  $A_i$  are correct, thus their guesses are correct in at most  $2^{|V(G)|} - |W(S)|$  cases. Therefore the number of cases for which the team wins using the strategy  $S$  is less than or equal to  $k(2^{|V(G)|} - |W(S)|)$ . This implies that  $p(S) = |W(S)|/2^{|V(G)|} \leq k/(k+1)$ .  $\square$

Now we use the previous theorem to solve the hat problem on the graph  $H$  (given in Figure 1). This graph is obtained from  $K_4$  by the subdivision of one edge.

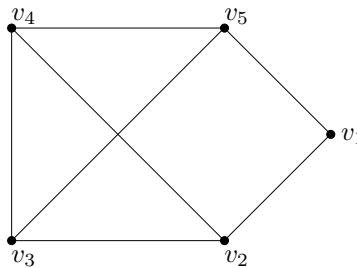


Fig. 1. The graph  $H$

**Fact 3.5.**  $h(H) = 3/4$ .

*Proof.* It is easy to observe that  $N_H(v_1) \subseteq N_H(v_3)$  and  $N_H(v_2) = N_H(v_5)$ . This implies that we can partition the set of vertices of  $H$  into three sets the open neighborhoods of which can be linearly ordered. By Theorem 3.4 we have  $h(G) \leq 3/4$ . On the other hand, by Fact 2.1 we get  $3/4 = h(K_3) \leq h(H)$  as  $K_3 \subseteq H$ .  $\square$

## 4. HAT PROBLEM ON A UNICYCLIC GRAPH

In this section we solve the hat problem on unicyclic graphs containing a cycle on at least nine vertices.

**Theorem 4.1.** *If  $G$  is a unicyclic graph containing the cycle  $C_k$  for some  $k \geq 9$ , then  $h(G) = 1/2$ .*

*Proof.* The result we prove by induction on the number  $n$  of vertices of  $G$ . For  $n = k$  the Theorem holds by Theorem 2.8. Now assume that  $n > k$ . Assume that for every unicyclic graph  $G'$  with  $n - 1$  vertices containing  $C_k$  we have  $h(G') = 1/2$ . Let  $S$  be an optimal strategy for  $G$ . If some vertex, say  $v_i$ , never guesses its color, then by Theorem 2.6 we have  $h(G) = h(G - v_i)$ . If  $v_i$  is a vertex of the cycle, then the graph  $G - v_i$  is a subgraph of a tree. Using Theorem 2.7 we get  $h(G - v_i) \leq 1/2$ , and therefore  $h(G) \leq 1/2$ . On the other hand, by Fact 2.2 we have  $h(G) \geq 1/2$ . If  $v_i$  is a leaf (obviously,  $G$  has at least one leaf), then the graph  $G - v_i$  is a unicyclic graph with  $n - 1$  vertices containing  $C_k$ . By the inductive hypothesis we have  $h(G - v_i) = 1/2$ , and therefore  $h(G) = 1/2$ . Now assume that every vertex of the cycle, and every leaf guesses its color, that is, every one of these vertices guesses its color in at least one situation. We are interested in the possibility when the number of cases for which the team loses is as small as possible. We assume that every one of those vertices guesses its color in exactly one situation, and we prove that these guesses suffice to cause the loss of the team in more than a half of all cases. The vertices of the cycle we denote by  $v_1, v_2, \dots, v_k$ . Let  $v_i$  and  $v_{i+1}$  be adjacent.

First assume that at least three vertices of the cycle have degree at least three. This implies that  $G$  has at least three leaves having different neighbors. Observe that each one of the leaves guesses its color wrong in a quarter of all cases. Since the closed neighborhoods of the leaves are pairwise disjoint, the team wins for at most  $(3/4)^3 = 27/64 < 1/2$  of all cases. This is a contradiction to Corollary 2.3.

Now assume that exactly two vertices of the cycle have degree at least three. Thus  $G$  has at least two leaves, say  $x$  and  $y$ , which have different neighbors. The neighbor of  $x$  ( $y$ , respectively) we denote by  $x'$  ( $y'$ , respectively). Let  $v_i$  mean a vertex of the cycle such that  $x, y, x', y' \notin N_G[v_i]$ . Let us observe that the vertex  $v_i$  guesses its color wrong in  $1/8$  of all cases as it has two neighbors. Each one of the leaves  $x$  and  $y$  guesses its color wrong in a quarter of all cases. Since the closed neighborhoods of the vertices  $x$ ,  $y$ , and  $v_i$  are pairwise disjoint, the team wins for at most  $(3/4)^2 \cdot 7/8 = 63/128 < 1/2$  of all cases. This is a contradiction to Corollary 2.3.

Now assume that exactly one vertex of the cycle, say  $v_1$ , has degree at least three. Let  $x$  mean a leaf of  $T_k$  which is joined with  $v_1$  by a path which does not go through any other vertex of the cycle. The vertex  $x$  guesses its color wrong in a quarter of all cases. Each one of the vertices  $v_3$  and  $v_4$  guesses its color wrong in  $1/8$  of all cases. Let us observe that both these vertices at the same time guess their colors wrong in at most  $1/16$  of all cases. Thus they guess their colors wrong in at least  $1/8 + 1/8 - 1/16 = 3/16$  of all cases. Similarly we conclude that the vertices  $v_7$  and  $v_8$  guess their colors wrong in at least  $3/16$  of all cases. Disjointness of proper



neighborhoods implies that the team wins for at most  $(13/16)^2 \cdot 3/4 = 507/1024 < 1/2$  of all cases. This is a contradiction to Corollary 2.3.  $\square$

## 5. HAT PROBLEM ON A GRAPH WITH A UNIVERSAL VERTEX

Now we consider the hat problem on graphs with a universal vertex.

We have the following property of optimal strategies for such graphs.

**Fact 5.1.** *Let  $G$  be a graph, and let  $v$  be a universal vertex of  $G$ . If  $S$  is an optimal strategy for the graph  $G$ , then for every situation of  $v$ , in at least one of two cases corresponding to this situation some vertex guesses its color.*

*Proof.* Let  $s$  be a situation of  $v$ . Suppose that the strategy  $S$  for the graph  $G$  is optimal, and in the cases  $c$  and  $d$  corresponding to the situation  $s$  no vertex guesses its color. Of course, for both these cases the team loses. Let the strategy  $S'$  for the graph  $G$  differ from  $S$  only in that in the situation  $s$  the vertex  $v$  guesses it has the color which it has in the case  $c$ . In the strategy  $S'$  the result of the case  $c$  is a win, and  $d$  is a loss. This implies that  $|W(S')| = |W(S)| + 1$ , and consequently,

$$p(S') = \frac{|W(S')|}{|C(G)|} = \frac{|W(S)| + 1}{|C(G)|} > \frac{|W(S)|}{|C(G)|} = p(S),$$

a contradiction to the optimality of  $S$ .  $\square$

Now, let us consider a strategy for a graph with a universal vertex such that there are two cases corresponding to the same situation of a universal vertex, and in one of them no vertex guesses its color, while in the second some vertex guesses its color. In the following lemma we give a method of creating a strategy which is not worse than that.

**Lemma 5.2.** *Let  $G$  be a graph and let  $v$  be a universal vertex of  $G$ . Let  $c$  and  $d$  be any cases corresponding to the same situation of  $v$ . Let  $S$  be a strategy for the graph  $G$  such that in the case  $c$  no vertex guesses its color, and in the case  $d$  some vertex guesses its color. Let the strategy  $S'$  for the graph  $G$  differ from  $S$  only in that  $v$ , in the situation to which correspond cases  $c$  and  $d$ , guesses it has the color which it has in the case  $c$ . Then  $p(S') \geq p(S)$ .*

*Proof.* The result of the case  $c$  in the strategy  $S'$  is a win, and in the strategy  $S$  is a loss. The result of the case  $d$  in the strategy  $S'$  is a loss. If the result of the case  $d$  in the strategy  $S$  is also a loss, then  $|W(S')| = |W(S)| + 1$ . If the result of the case  $d$  in the strategy  $S$  is a win, then  $|W(S')| = |W(S)|$ . This implies that  $|W(S')| \geq |W(S)|$ . Therefore  $p(S') \geq p(S)$ .  $\square$

It is possible to prove that if a graph has a universal vertex, then there exists an optimal strategy such that in every case some vertex guesses its color. This implies that to solve the hat problem on a graph with a universal vertex it suffices to examine only strategies such that in every case some vertex guesses its color. Thus if in some

case of a strategy no vertex guesses its color, then we can cease further examining this strategy.

**Theorem 5.3.** *If  $G$  is a graph with a universal vertex, then there exists a strategy  $S \in \mathcal{F}^0(G)$  such that  $|Ln(S)| = 0$ .*

*Proof.* Suppose that for every optimal strategy  $S$  for the graph  $G$  we have  $|Ln(S)| > 0$ . Let  $S'$  be an optimal strategy for  $G$ , and let  $c_1, c_2, \dots, c_n$  be the cases in which no vertex guesses its color. By Fact 5.1, any two of them do not correspond to the same situation of  $v$ . Let the strategy  $S_1$  for the graph  $G$  differ from  $S'$  only in that  $v$ , in the situation to which corresponds the case  $c_1$ , guesses it has the color which it has in the case  $c_1$ . By Lemma 5.2 we have  $p(S_1) \geq p(S')$ . Let the strategy  $S_2$  for the graph  $G$  differ from  $S_1$  only in that  $v$ , in the situation to which corresponds the case  $c_2$ , guesses it has the color which it has in the case  $c_2$ . By Lemma 5.2 we have  $p(S_2) \geq p(S_1)$ . After  $n - 2$  further analogical steps we get the strategy  $S = S_n$  for the graph  $G$  such that  $p(S) \geq p(S_{n-1}) \geq \dots \geq p(S_2) \geq p(S_1) \geq p(S')$ , and there is no case in which no vertex guesses its color. Since the strategy  $S'$  for the graph  $G$  is optimal, and  $p(S) \geq p(S')$ , the strategy  $S$  is also optimal. In every case in the strategy  $S_n$  some vertex guesses its color, thus  $|Ln(S)| = 0$ .  $\square$

In the next fact we state that there exists a graph with a universal vertex for which there exists an optimal strategy such that in some case no vertex guesses its color.

**Fact 5.4.** *There exists a strategy  $S \in \mathcal{F}^0(K_2)$  such that  $|Ln(S)| > 0$ .*

*Proof.* Let  $S = (g_1, g_2) \in \mathcal{F}(K_2)$  be the strategy as follows.

$$g_1(s_1) = \begin{cases} 1 & \text{if } s_1(v_2) = 1, \\ * & \text{otherwise,} \end{cases}$$

$$g_2(s_2) = \begin{cases} 2 & \text{if } s_2(v_1) = 2, \\ * & \text{otherwise.} \end{cases}$$

All cases we present in Table 1.

Table 1.

No	The color of		The guess of		Result
	$v_1$	$v_2$	$v_1$	$v_2$	
1	1	1	+		+
2	1	2			–
3	2	1	–	–	–
4	2	2		+	+

From Table 1 we know that  $|W(S)| = 2$  and  $|Ln(S)| = 1$ . We have  $|C(K_2)| = 4$ , thus  $p(S) = 2/4 = 1/2$ . The graph  $K_2$  is a tree, therefore by Theorem 2.7 we have  $h(K_2) = 1/2$ . Since  $h(K_2) = 1/2$ , the strategy  $S$  is optimal for  $K_2$ . Both vertices  $v_1$  and  $v_2$  are universal, and  $|Ln(S)| = 1$  as in the case in which  $v_1$  has the first color, and  $v_2$  has the second color no vertex guesses its color.  $\square$

## 6. A NORDHAUS-GADDUM TYPE INEQUALITIES

In this section we give some Nordhaus-Gaddum type inequalities.

In the following two theorems we give a lower and upper bounds on the product (sum, respectively) of the hat number of a graph and the hat number of its complement.

**Theorem 6.1.** *For every graph  $G$  we have  $1/4 \leq h(G)h(\overline{G}) < 1$ .*

*Proof.* By Corollary 2.2 we have  $h(G) \geq 1/2$  and  $h(\overline{G}) \geq 1/2$ , so  $h(G)h(\overline{G}) \geq 1/4$ . Since by Fact 2.4 we have  $h(G) < 1$  and  $h(\overline{G}) < 1$ , we get  $h(G)h(\overline{G}) < 1$ .  $\square$

**Theorem 6.2.** *For every graph  $G$  we have  $1 \leq h(G) + h(\overline{G}) < 2$ .*

The proof is similar to that of Theorem 6.1.

Now we prove that for every number greater than or equal to quarter, and smaller than one, there exists a graph for which the product of its hat number and the hat number of its complement is greater than that number.

**Theorem 6.3.** *For every  $\alpha \in [1/4; 1)$  there is a graph  $G$  such that  $h(G)h(\overline{G}) > \alpha$ .*

*Proof.* Let  $G$  be a graph with  $2n$  vertices such that  $V(G) = \{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$  and  $E(G) = \{v_i v_j : i, j \in \{1, 2, \dots, n\}, i \neq j\}$ . It is easy to see that  $E(\overline{G}) = \{v_i v'_j : i, j \in \{1, 2, \dots, n\}\} \cup \{v'_i v'_j : i, j \in \{1, 2, \dots, n\}, i \neq j\}$ . Since  $K_n$  is a subgraph of both graphs  $G$  and  $\overline{G}$ , by Fact 2.1 we have  $h(G) \geq h(K_n)$  and  $h(\overline{G}) \geq h(K_n)$ . To prove that  $h(G)h(\overline{G}) > \alpha$ , it suffices to prove that  $(h(K_n))^2 > \alpha$ , that is  $h(K_n) > \sqrt{\alpha}$ . The authors of [14] have proven that for the hat problem with  $n = 2^k - 1$  players there exists a strategy giving the chance of success  $(2^k - 1)/2^k$ . Since  $\lim_{k \rightarrow \infty} (2^k - 1)/2^k = 1$ , for every  $\alpha \in [1/4; 1)$  there exists a positive integer  $k$  such that for the hat problem with  $n = 2^k - 1$  players there exists a strategy  $S$  such that  $p(S) \geq 1 - 1/2^k = 1 - 1/(n + 1) > \sqrt{\alpha}$ . By definition we have  $h(K_n) \geq p(S)$ , thus  $h(K_n) > \sqrt{\alpha}$ .  $\square$

The following theorem says that for every number greater than or equal to one, and smaller than two, there exists a graph for which the sum of its hat number and the hat number of its complement is greater than that number.

**Theorem 6.4.** *For every  $\alpha \in [1; 2)$  there is a graph  $G$  such that  $h(G) + h(\overline{G}) > \alpha$ .*

The proof is similar to that of Theorem 6.3.

### Acknowledgements

*Thanks are due to the anonymous referee for comments that helped to improve the presentation of the results.*

### REFERENCES

- [1] G. Aggarwal, A. Fiat, A. Goldberg, J. Hartline, N. Immorlica, M. Sudan, *Derandomization of auctions*, Proc. of the 37th Annual ACM Symposium on Theory of Computing, ACM, New York, 2005, 619–625.

- [2] J. Aspnes, R. Beigel, M. Furst, S. Rudich, *The expressive power of voting polynomials*, *Combinatorica* **14** (1994), 135–148.
- [3] R. Beigel, L. Fortnow, F. Stephan, *Infinitely-often autoreducible sets*, *SIAM Journal on Computing* **36** (2006), 595–608.
- [4] *Berkeley Riddles*, [www.ocf.berkeley.edu/~www/riddles/hard.shtml](http://www.ocf.berkeley.edu/~www/riddles/hard.shtml).
- [5] M. Bernstein, *The hat problem and Hamming codes*, *MAA Focus*, November, 2001, 4–6.
- [6] W. Blum, *Denksport für Hutträger*, *Die Zeit*, May 3, 2001.
- [7] M. Breit, D. Deshommes, A. Falden, *Hats required: perfect and imperfect strategies for the hat problem*, *The Journal of the Summer Undergraduate Mathematical Science Research Institute*, Miami University, Summer 2002.
- [8] E. Brown, K. Mellinger, *Kirkman's schoolgirls wearing hats and walking through fields of numbers*, *Mathematics Magazine* **82** (2009), 3–15.
- [9] E. Burke, S. Gustafson, G. Kendall, *A puzzle to challenge genetic programming*, *Genetic Programming*, 136–147, *Lecture Notes in Computer Science*, Springer, 2002.
- [10] S. Butler, M. Hajianghayi, R. Kleinberg, T. Leighton, *Hat guessing games*, *SIAM Journal on Discrete Mathematics* **22** (2008), 592–605.
- [11] G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, *Covering Codes*, North Holland, 1997.
- [12] T. Ebert, *Applications of recursive operators to randomness and complexity*, Ph.D. Thesis, University of California at Santa Barbara, 1998.
- [13] T. Ebert, W. Merkle, *Autoreducibility of random sets: a sharp bound on the density of guessed bits*, *Mathematical Foundations of Computer Science 2002*, 221–233, *Lecture Notes in Computer Science*, 2420, Springer, Berlin, 2002.
- [14] T. Ebert, W. Merkle, H. Vollmer, *On the autoreducibility of random sequences*, *SIAM Journal on Computing* **32** (2003), 1542–1569.
- [15] T. Ebert, H. Vollmer, *On the autoreducibility of random sequences*, *Mathematical Foundations of Computer Science 2000 (Bratislava)*, 333–342, *Lecture Notes in Computer Science*, 1893, Springer, Berlin, 2000.
- [16] U. Feige, *You can leave your hat on (if you guess its color)*, Technical Report MCS04-03, Computer Science and Applied Mathematics, The Weizmann Institute of Science, 2004, 10 pp.
- [17] W. Guo, S. Kasala, M. Rao, B. Tucker, *The hat problem and some variations*, *Advances in Distribution Theory, Order Statistics, and Inference, Statistics for Industry and Technology*, Birkhäuser, Boston, 2007, 459–479.
- [18] N. Immerlica, *Computing with strategic agents*, Ph.D. Thesis, Massachusetts Institute of Technology, 2005.
- [19] M. Krzywkowski, *A modified hat problem*, *Commentationes Mathematicae* **50** (2010), 121–126.
- [20] M. Krzywkowski, *Hat problem on a graph*, *Mathematica Pannonica* **21** (2010), 3–21.

- [21] M. Krzywkowski, *Hat problem on the cycle  $C_4$* , International Mathematical Forum **5** (2010), 205–212.
- [22] M. Krzywkowski, *On the hat problem, its variations, and their applications*, Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica **9** (2010), 55–67.
- [23] M. Krzywkowski, *The hat problem on cycles on at least nine vertices*, Ars Combinatoria **101** (2011), 3–13.
- [24] H. Lenstra, G. Seroussi, *On hats and other covers*, IEEE International Symposium on Information Theory, Lausanne, 2002.
- [25] J. Poulos, *Could you solve this \$1 million hat trick?*, abcNews, November 29, 2001.
- [26] S. Robinson, *Why mathematicians now care about their hat color*, The New York Times, Science Times Section, page D5, April 10, 2001.

Marcin Krzywkowski  
marcin.krzywkowski@gmail.com

Gdańsk University of Technology  
Faculty of Electronics, Telecommunications and Informatics  
ul. Narutowicza 11/12, 80–233 Gdańsk, Poland

*Received: April 9, 2010.*

*Revised: March 31, 2011.*

*Accepted: April 8, 2011.*