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Highlights

- New definition of the local material symmetry group within the nonlinear micromorphic continuum.
- Definitions of micromorphic solids, fluids and subfluids.
- Granular material as amicromorphicsubfluid.

On the material symmetry group for micromorphic media with applications to granular materials

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Abstract

Within the framework of the theory of nonlinear elastic micromorphic continua we introduce the new definition of the local material symmetry group. The group consists of ordered triples of second- and third-order tensors describing such changes of a reference placement that cannot be recognized with any experiment. Using the definition we characterize the micromorphic isotropic media, micromorphic fluids, solids and special intermediate cases called micromorphic subfluids or micromorphic liquid crystals. We demonstrate that some typical behaviour of such complex media as granular materials can be described within the micromorphic subfluids mechanics.

Keywords: material symmetry group, micromorphic continuum, subfluids, granular materials.

1. Introduction

Due to rather complex behaviour the proper description of granular media is a real challenge for continuum mechanics. Indeed, unlike classic media granular materials such as sand or powder demonstrate both fluid- and solid-like behaviour, see [1–5]. In particular, for macroscopic behaviour the interaction forces between particles constituting a granular medium play an important role [1, 5]. Among continual models used for description of granular media let us mention the micromorphic continuum introduced by Mindlin [6] and Eringen and Suhubi [7]. For the actual state of the art in the field of the micromorphic mechanics we refer to [8–10]. In addition to granular media, see e.g. [11–13], the model was used for description of deformations of microstructured media such as foams [14], metamaterials [15, 16], and other micrononhomogeneous media [17–20].

Here we introduce a new definition of the local material symmetry group for a micromorphic medium undergoing finite deformations. The symmetry group relates to a point of the medium and to a chosen reference placement. It consists of all invertible transformations of the reference placement which keep the strain energy density unchanged. Using the definition of the group we characterize the micromorphic solids, fluids and some intermediate cases called micromorphic subfluids or micromorphic liquid crystals. For simple materials subfluids were introduced and extensively discussed by Wang [21], see also [22]. However, let us note that this material model differs from models of liquid crystals such as nematics, cholesterics, smectics and other liquid crystals materials, see, e.g., [23, 24].

The paper is organized as follows. First, in Section 2, we briefly introduce the kinematical descriptors and the objective strain energy density of a nonlinear micromorphic medium. In Section 3 we consider changes of reference placements and the related transformations of the strain mea-

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sures. Then, in Section 4, we introduce the new definition of the local material symmetry group. Finally, with the definition we characterize micromorphic isotropic continua, solids, fluids and micromorphic liquid crystals (micromorphic subfluids). Here we present an example of a model of micromorphic subfluid which can flow at the macroscale whereas at a microlevel it behaves as a solid.

2. Kinematics and objective strain energy density

Kinematics of a micromorphic continuum is described as a mapping from one state called a reference placement into another one called an actual placement. We introduce the placement vectors $\mathbf{x}$ and $\mathbf{X}$ defined in actual and reference placements, respectively. In addition we also introduce two triples of vectors $\mathbf{d}_k$ and $\mathbf{D}_k$, $k = 1, 2, 3$, called directors which are also relate to the actual and reference placements, see [8] for details. So the deformation of a micromorphic continuum is described as mappings

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \quad \mathbf{d}_k = \mathbf{d}(\mathbf{X}). \tag{1}$$

Initial directors $\mathbf{D}_k$ play a role of structural parameters. We introduce the deformation gradient

$$\mathbf{F} = \nabla \mathbf{x}, \tag{2}$$

where ∇ is the nabla-operator in the reference placement defined as in [25, 26]. For example, in Cartesian coordinates X_k with corresponding unit orthogonal vectors $\mathbf{i}_k$, $\mathbf{i}_m \cdot \mathbf{i}_n = \delta_{mn}$,

$$\mathbf{F} = \mathbf{i}_k \otimes \mathbf{x}_{,k}. \tag{3}$$

Hereinafter '$\cdot$' and '$\otimes$' stand for scalar and tensor (dyad) products, respectively. For brevity we use the notation

$$(\ldots)_{,k} = \frac{\partial(\ldots)}{\partial X_k}.$$

Instead of $\mathbf{d}_k$ and $\mathbf{D}_k$ we introduce the microdistortion tensor $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{D}_k \otimes \mathbf{d}_k. \tag{4}$$

Note that $\mathbf{F}$ and $\mathbf{P}$ have a similar form. Indeed, the both tensors are sums of diads where each diad is the tensor product of two vectors such that the first vector is defined in the reference placement whereas the second is defined in the actual placement. Unlike the micropolar elasticity here the directors are not unit and orthogonal to each other, in general. So $\mathbf{P}$ is not orthogonal tensor. We only assume that $\mathbf{P}$ is nonsingular.
For a hyperelastic medium the strain energy density is given by

$$W = W(\mathbf{F}, \mathbf{P}, \nabla\mathbf{P}), \tag{5}$$

The principle of material frame indifference [22, 27] states that W must be invariant under the following transformations

$$\mathbf{x} \to \mathbf{Q} \cdot \mathbf{x}, \quad \mathbf{d}_k \to \mathbf{Q} \cdot \mathbf{d}_k$$

for any orthogonal tensor $\mathbf{Q}$: $\mathbf{Q}^{-1} = \mathbf{Q}^T$. As a result we have that

$$W(\mathbf{F}, \mathbf{P}, \nabla\mathbf{P}) = W(\mathbf{F} \cdot \mathbf{Q}^T, \mathbf{P} \cdot \mathbf{Q}^T, \nabla\mathbf{P} \cdot \mathbf{Q}^T) \tag{6}$$

for all orthogonal tensors $\mathbf{Q}$. In order to satisfy (6) we introduce the polar decomposition of $\mathbf{P}$

$$\mathbf{P} = \mathbf{U} \cdot \mathbf{A},$$

where $\mathbf{U} = (\mathbf{P} \cdot \mathbf{P}^T)^{1/2}$ is a symmetric positive-definite tensor and $\mathbf{A} = \mathbf{U}^{-1} \cdot \mathbf{P}$ is an orthogonal tensor. Substituting into (6) $\mathbf{Q} = \mathbf{A}^{-1} = \mathbf{P}^{-1} \cdot \mathbf{U}$ we get

$$W(\mathbf{F}, \mathbf{P}, \nabla\mathbf{P}) = W(\mathbf{F} \cdot \mathbf{P}^{-1} \cdot \mathbf{U}, \mathbf{U}, \nabla\mathbf{P} \cdot \mathbf{P}^{-1} \cdot \mathbf{U}).$$

As $\mathbf{U}^2 = \mathbf{P} \cdot \mathbf{P}^T$ the objective representation of W take the form

$$W = W(\mathbf{E}, \mathbf{C}, \mathbf{K}) \tag{7}$$

where $\mathbf{E} = \mathbf{F} \cdot \mathbf{P}^{-1}, \mathbf{C} = \mathbf{P} \cdot \mathbf{P}^T, \mathbf{K} = \nabla\mathbf{P} \cdot \mathbf{P}^{-1}$ are the strain measures [8]. It is worth to indicate that $\mathbf{C}$ is a symmetric second-order tensor, $\mathbf{E}$ is a general second-order tensor and $\mathbf{K}$ is a third-order tensor. Let us note that in (5) and (7) we use different sets of arguments, but for simplicity we keep the same notation for W.

3. Changes of reference placement

Let us consider the dependence of the constitutive equation on the choice of the reference placement. Let $\varkappa_1$ and $\varkappa_2$ be two reference placements with placement vectors $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. So $\mathbf{x}$ may depend $\mathbf{X}_1$ or $\mathbf{X}_2$

$$\mathbf{x} = \mathbf{x}(\mathbf{X}_1) = \mathbf{x}(\mathbf{X}_2),$$

and we have two referential nabla-operators ∇_1 and ∇_2 and two deformation gradients

$$\mathbf{F}_1 = \nabla_1 \mathbf{x}, \quad \mathbf{F}_2 = \nabla_2 \mathbf{x}.$$

Introducing the gradient of a mapping from $\varkappa_1$ into $\varkappa_2$ as

$$\mathbf{S} = \nabla_1 \mathbf{X}_2$$

we get the relations

$$\nabla_1 = \mathbf{S} \cdot \nabla_2, \quad \text{and} \quad \mathbf{F}_1 = \mathbf{S} \cdot \mathbf{F}_2. \tag{8}$$

As a reference placement is also characterized by a triple of directors, we introduce in $\varkappa_1$ and $\varkappa_2$ two triples $\{\mathbf{D}_k^{(1)}\}$ and $\{\mathbf{D}_k^{(2)}\}$, $k = 1, 2, 3$. So we have two microdistortion tensors

$$\mathbf{P}_1 = \mathbf{D}_k^{(1)} \otimes \mathbf{d}_k, \quad \mathbf{P}_2 = \mathbf{D}_k^{(2)} \otimes \mathbf{d}_k,$$

related to each other by the formula

$$\mathbf{P}_1 = \mathbf{R} \cdot \mathbf{P}_2, \tag{9}$$

where $\mathbf{R}$ is a nonsingular second-order tensor such that $\mathbf{D}_k^{(1)} = \mathbf{R} \cdot \mathbf{D}_k^{(2)}$.
So the change of reference placement results in the following transformation formulae

$$\mathbf{F} \to \mathbf{S} \cdot \mathbf{F}, \quad \mathbf{P} \to \mathbf{R} \cdot \mathbf{P}, \tag{10}$$

where $\mathbf{S}$ and $\mathbf{P}$ are non-singular second-order tensors describing local transformations of the reference placement.
With (10) we get the transformation rules for strain measures $\mathbf{E}$ and $\mathbf{C}$

$$\mathbf{E} \to \mathbf{S} \cdot \mathbf{E} \cdot \mathbf{R}^{-1}, \quad \mathbf{C} \to \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T. \tag{11}$$

or the derivation of the transformation rule for we consider two strain measures $\mathbf{K}_1$ and $\mathbf{K}_2$ efined in $\varkappa_1$ and $\varkappa_2$, respectively,

$$\mathbf{K}_1 = \nabla_1 \mathbf{P}_1 \cdot \mathbf{P}_1^{-1}, \quad \mathbf{K}_2 = \nabla_2 \mathbf{P}_2 \cdot \mathbf{P}_2^{-1}.$$

Using (8) and (9) we have

$$\begin{aligned}
\mathbf{K}_1 &= [(\mathbf{S} \cdot \nabla_2)(\mathbf{R} \cdot \mathbf{P}_2)] \cdot \mathbf{P}_2^{-1} \cdot \mathbf{R}^{-1} \\
&= (\mathbf{S} \cdot \mathbf{i}_k) \frac{\partial}{\partial X_k^{(2)}} (\mathbf{R} \cdot \mathbf{P}_2) \cdot \mathbf{P}_2^{-1} \cdot \mathbf{R}^{-1} \\
&= (\mathbf{S} \cdot \mathbf{i}_k) \mathbf{R} \cdot \mathbf{P}_{2,k} \cdot \mathbf{P}_2^{-1} \cdot \mathbf{R}^{-1} \\
&\quad + (\mathbf{S} \cdot \mathbf{i}_k) \mathbf{R}_{,k} \cdot \mathbf{P}_2 \cdot \mathbf{P}_2^{-1} \cdot \mathbf{R}^{-1} \\
&= \mathbf{S} \cdot [\mathbf{R} * \mathbf{K}_2] + \mathbf{L},
\end{aligned}$$

where we introduce new operation $*$ between second- and third-order tensors as follows

$$\begin{aligned}
\mathbf{R} * \mathbf{K} &= \mathbf{R} * (K_{mnk} \mathbf{i}_m \otimes \mathbf{i}_n \otimes \mathbf{i}_k) \\
&= K_{mnk} \mathbf{i}_m \otimes (\mathbf{R} \cdot \mathbf{i}_n) \otimes (\mathbf{i}_k \cdot \mathbf{R}^{-1}),
\end{aligned}$$

and $\mathbf{L} = \mathbf{S} \cdot (\nabla_2 \mathbf{R}) \cdot \mathbf{R}^{-1}$.
So, in addition to (11) we have the transformation formula for $\mathbf{K}$

$$\mathbf{K} \to \mathbf{S} \cdot [\mathbf{R} * \mathbf{K}] + \mathbf{L}. \tag{12}$$

In general, the form of constitutive equations depends on the reference configuration. The strain energy densities related to $\varkappa_1$ and $\varkappa_2$ are denoted, respectively, W_1 and W_2. The energy stored in an arbitrary part of the volume occupied by the micromorphic continuum does not depend on the choice of the reference placement, that is

$$\int_{V_1} W_1 \, \mathrm{d}V_1 = \int_{V_2} W_2 \, \mathrm{d}V_2.$$

Using the formula $\mathrm{d}V_2 = |\det \mathbf{S}| \mathrm{d}V_1$ we obtain that

$$W_1(\mathbf{E}_1, \mathbf{C}_1, \mathbf{K}_1) = |\det \mathbf{S}| W_2(\mathbf{E}_2, \mathbf{C}_2, \mathbf{K}_2). \tag{13}$$

The last formula with transformation rules (11) and (12) gives the opportunity to introduce the local material symmetry group $\mathcal{G}_\varkappa$ as a set of such transformations of the reference placement $\varkappa$ which do not affect the form of the strain energy density. In other words, the strain energy density is insensitive to any change of the reference placement describing by the transformations belonging to $\mathcal{G}_\varkappa$.

4. Local material symmetry group

First, let us recall that for simple materials the material symmetry group consists of transformations preserving the material density [22, 27]. Here we will also consider only unimodular tensors $\mathbf{S}$, $|\det \mathbf{S}| = 1$. In a similar way we assume that unrecognizable transformations of the microdistortion tensor should be also unimodular: $|\det \mathbf{R}| = 1$. In what follows we use the following nomenclature for some tensor groups:
Unim is the unimodular group, $Unim = \{\mathbf{S} : \det \mathbf{S} = \pm 1\}$;
Orth is the orthogonal group, $Orth = \{\mathbf{Q} : \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I}\}$, $\mathbf{I}$ is the unit tensor;
Lin_3 is the linear group of third-order tensors without specific constraints concerning index symmetries.
Here *Unim* and *Orth* are groups with respect to multiplication, and Lin_3 is the group with regard to addition.
We introduce
Definition 1. *A set of ordered triples of two second-order unimodular tensors* $\mathbf{S}$ *and* $\mathbf{R}$ *and third-order tensor* $\mathbf{L}$

$$\mathcal{G}_\varkappa(\mathbf{X}) = \{\mathbb{X} = (\mathbf{S}, \mathbf{R}, \mathbf{L})\}$$

is called the local material symmetry group if the following relation is valid

$$W(\mathbf{E}, \mathbf{C}, \mathbf{K}) = W(\mathbf{S} \cdot \mathbf{E} \cdot \mathbf{R}^{-1}, \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T, \mathbf{S} \cdot [\mathbf{R} * \mathbf{K}] + \mathbf{L})$$

for a given point $\mathbf{X}$ *in the reference placement* $\varkappa$ *and for all admissible tensors* $\mathbf{E}$, $\mathbf{C}$, $\mathbf{K}$ *from the domain of the definition of* W. *The set* $\mathcal{G}_\varkappa(\mathbf{X})$ *is a group relative to the operation defined as follows*

$$\mathbb{X}_1 \circ \mathbb{X}_2 \equiv (\mathbf{S}_1, \mathbf{R}_1, \mathbf{L}_1) \circ (\mathbf{S}_2, \mathbf{R}_2, \mathbf{L}_2) = (\mathbf{S}_1 \cdot \mathbf{S}_2, \mathbf{R}_1 \cdot \mathbf{R}_2, \mathbf{L}_1 + \mathbf{S}_1 \cdot [\mathbf{R}_1 * \mathbf{L}_2]). \quad (14)$$

Indeed, let $\mathbb{X}_1 \equiv (\mathbf{S}_1, \mathbf{R}_1, \mathbf{L}_1) \in \mathcal{G}_\varkappa(\mathbf{X})$ and $\mathbb{X}_2 \equiv (\mathbf{S}_2, \mathbf{R}_2, \mathbf{L}_2) \in \mathcal{G}_\varkappa(\mathbf{X})$. This means that

$$\begin{aligned} W(\mathbf{E}, \mathbf{C}, \mathbf{K}) &= W(\mathbf{S}_1 \cdot \mathbf{E} \cdot \mathbf{R}_1^{-1}, \mathbf{R}_1 \cdot \mathbf{C} \cdot \mathbf{R}_1^T, \\ &\quad \mathbf{S}_1 \cdot [\mathbf{R}_1 * \mathbf{K}] + \mathbf{L}_1) \\ &= W(\mathbf{S}_2 \cdot \mathbf{E} \cdot \mathbf{R}_2^{-1}, \mathbf{R}_2 \cdot \mathbf{C} \cdot \mathbf{R}_2^T, \\ &\quad \mathbf{S}_2 \cdot [\mathbf{R}_2 * \mathbf{K}] + \mathbf{L}_2). \end{aligned}$$

Using these relations we prove that

$$\begin{aligned} &W(\mathbf{S}_1 \cdot \mathbf{S}_2 \cdot \mathbf{E} \cdot \mathbf{R}_2^{-1} \cdot \mathbf{R}_2^{-1}, \mathbf{R}_1 \cdot \mathbf{R}_2 \cdot \mathbf{C} \cdot \mathbf{R}_2^T \cdot \mathbf{R}_1^T, \\ &\quad \mathbf{S}_1 \cdot [\mathbf{R}_1 * (\mathbf{S}_2 \cdot [\mathbf{R}_2 * \mathbf{K}] + \mathbf{L}_2)] + \mathbf{L}_1) \\ &= W(\mathbf{S}_2 \cdot \mathbf{E} \cdot \mathbf{R}_2^{-1}, \mathbf{R}_2 \cdot \mathbf{C} \cdot \mathbf{R}_2^T, \\ &\quad \mathbf{S}_2 \cdot [\mathbf{R}_2 * \mathbf{K}] + \mathbf{L}_2) \\ &= W(\mathbf{E}, \mathbf{C}, \mathbf{K}). \end{aligned}$$

Thus, $\mathbb{X}_1 \circ \mathbb{X}_2 \in \mathcal{G}_\varkappa(\mathbf{X})$.
In what follows for brevity we omit dependence on $\mathbf{X}$, so instead of $\mathcal{G}_\varkappa(\mathbf{X})$ we use $\mathcal{G}_\varkappa$ if is not necessary to underline the dependence on $\mathbf{X}$.
The unit element of $\mathcal{G}_\varkappa$ is $(\mathbf{I}, \mathbf{I}, \mathbf{O})$, where $\mathbf{O}$ is the zero third-order tensor. The inverse to $\mathbb{X} = (\mathbf{S}, \mathbf{R}, \mathbf{L})$ is $\mathbb{X}^{-1} = (\mathbf{S}^{-1}, \mathbf{R}^{-1}, -\mathbf{S}^{-1} \cdot [\mathbf{R}^{-1} * \mathbf{L}])$.
Let us note that the definition and the group operation are quite similar to introduced in the case of micropolar continua [28, 29], of material surfaces of various types [30] and shells [31], but for the micropolar media the symmetry groups consists of second-order tensors only and there is another transformation formula of the strain energy density. There is also a similarity with definition of the symmetry group for strain-gradient materials where the material symmetry group consists of ordered couples of tensors [32] or ordered triples [33].

5. Micromorphic isotropic media, fluids, solids and subfluids

Similar to the nonlinear elasticity [22, 27] and to the micropolar elasticity [28] we characterize the typical classes of materials.
Definition 2. *The micromorphic elastic continuum is called isotropic if there exists a reference placement* $\varkappa$ *called undistorted, such that the material symmetry group related to* $\varkappa$ *contains the group* $\mathcal{O}$ *consisting of all orthogonal tensors*

$$\mathcal{O} \subset \mathcal{G}_\varkappa, \quad \mathcal{O} = \{\mathbb{X} = (\mathbf{Q}, \mathbf{Q}, \mathbf{O}), \mathbf{Q} \in Orth\}.$$

From Definition 2 it follows that

$$W(\mathbf{E}, \mathbf{C}, \mathbf{K}) = W(\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot [\mathbf{Q} * \mathbf{K}]).$$

So W is an isotropic function of its arguments. Note that here $\mathbf{Q}\cdot[\mathbf{Q}*\mathbf{K}]$ is the Rayleigh product of $\mathbf{Q}$ and $\mathbf{K}$, see [32, 33]. For further analysis the technique of representation of higher-order tensors may be very useful, see e.g. [34–36].

Definition 3. *We call a micromorphic elastic continuum the micromorphic elastic fluid if the material symmetry $\mathcal{G}_\varkappa$ is given by*

$$\mathcal{G}_\varkappa = \mathcal{U},$$

$$\mathcal{U} \equiv \{(\mathbf{S},\mathbf{R},\mathbf{L}) : \mathbf{S} \in Unim, \mathbf{R} \in Unim, \mathbf{L} \in Lin_3\}.$$

Obviously, $\mathcal{U}$ is the maximal group as it contains all admissible elements. So the micromorphic elastic fluid is isotropic. The symmetry group for the micromorphic fluid does not depend on the choice of the reference configuration like in the case of simple elastic [22] or micropolar fluids [28].

In order to find the representation of W for the micromorphic fluid let us first consider the invariance under transformations based on elements $\{(\mathbf{I},\mathbf{I},\mathbf{L}), \mathbf{L} \in Lin_3\}$. From Definition 1 we have that

$$W(\mathbf{E},\mathbf{C},\mathbf{K}) = W(\mathbf{E},\mathbf{C},\mathbf{K}+\mathbf{L}) \quad \forall \mathbf{L} \in Lin_3.$$

From this it follows that W does not depend on $\mathbf{K}$: $W = W(\mathbf{E},\mathbf{C})$. Then let us take as $\mathbf{S}$ and $\mathbf{R}$ tensors

$$\mathbf{S} = (\det \mathbf{F})\mathbf{F}^{-1}, \quad \mathbf{R} = (\det \mathbf{P})\mathbf{P}^{-1}.$$

Obviously, the both tensors are unimodular. So we get

$$\begin{aligned} W(\mathbf{E},\mathbf{C}) =& W(\mathbf{S}\cdot\mathbf{E}\cdot\mathbf{R}^{-1}, \mathbf{R}\cdot\mathbf{C}\cdot\mathbf{R}^T) \\ =& W\big((\det\mathbf{F})\mathbf{F}^{-1}\cdot\mathbf{E}\cdot(\det\mathbf{P})^{-1}\mathbf{P}, \\ & (\det\mathbf{P})\mathbf{P}^{-1}\cdot\mathbf{C}\cdot(\det\mathbf{P})\mathbf{P}^{-T}\big) \\ =& W\big((\det\mathbf{F})(\det\mathbf{P})^{-1}\mathbf{I}, (\det\mathbf{P})\mathbf{I}\big) \\ =& W(\det\mathbf{F}, \det\mathbf{P}). \end{aligned}$$

As a result, the strain energy density of a micromorphic fluid depends on two scalars describing macro- and micro-volume changes, respectively.

For simple and micropolar elastic materials the material symmetry group is constructed with the help of orthogonal transformations describing rotations and reflections of reference placement, see [22, 27, 28]. Using the same reasoning as in [28] we use the following definition of micromorphic solids.

Definition 4. *A micromorphic elastic continuum is called a micromorphic elastic solid if there exists a reference placement $\varkappa$ called undistorted, such that the material symmetry group related to $\varkappa$, is given by*

$$\mathcal{G}_\varkappa = \mathcal{R}_\varkappa, \quad \mathcal{R}_\varkappa \equiv \{(\mathbf{Q},\mathbf{Q},\mathbf{0}) : \mathbf{Q} \in \mathcal{O}_\varkappa \subset Orth\},$$

where $\mathcal{O}_\varkappa$ is a subgroup of $Orth$. Note that here we assumed that both tensors $\mathbf{S}$ and $\mathbf{R}$ coincide with the same orthogonal tensor.

Characterizing the micromorphic continua within the introduced notion of the material symmetry group we can also introduce various intermediate classes of continua wich are neither solids nor fluids. Following [21] where the simple elastic subfluids were introduced, see also [22], we will call these media the micromorphic subfluids or micromorphic liquid crystals.

Definition 5. *A micromorphic elastic continuum is called a micromorphic elastic subfluid or micromorphic elastic liquid crystal if its material symmetry group $\mathcal{G}_\varkappa$, related to a reference placement $\varkappa$ called undistorted, contains elements which are not members of the orthogonal group $\mathcal{O}$ and $\mathcal{G}_\varkappa$ does not coincide with the maximal group $\mathcal{U}$:*

$$\mathcal{G}_\varkappa \neq \mathcal{U}, \quad \exists \mathbb{X} \in \mathcal{G}_\varkappa : \mathbb{X} \notin \mathcal{O}.$$

Since the structure of the material symmetry group is more complex than in the case of simple materials, the number of possible micromorphic subfluids is greater than the number of simple subfluids [21].

As an example of a micromorphic subfluid let us consider a granular medium consisting of infinitesimal solid deformable particles which can freely move with the respect to each other. In other words, the considered medium behaves like a fluid at the macroscale and as an isotropic solid at the scale of single particle. In order to catch such behaviour let us consider the following material symmetry group

$$\begin{aligned}\mathcal{G}_{\varkappa} = &\{\mathbb{X} = (\mathbf{S}, \mathbf{R}, \mathbf{L}) : \\ &\mathbf{S} \in Unim, \mathbf{R} \in Orth, \mathbf{L} \in Lin_3\}.\end{aligned}$$

In order to derive the corresponding form of W first we observe that W does not depend on $\mathbf{K}$ as in the case of micromorphic fluid: $W = W(\mathbf{E}, \mathbf{C})$. So the invariance requires that

$$W(\mathbf{E}, \mathbf{C}) = W(\mathbf{S} \cdot \mathbf{E} \cdot \mathbf{R}^{-1}, \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T). \quad (15)$$

Substituting in (15) $\mathbf{S} = (\det \mathbf{E})\mathbf{E}^{-1}$ and $\mathbf{R} = \mathbf{I}$ we get

$$W(\mathbf{E}, \mathbf{C}) = W((\det \mathbf{E})\mathbf{I}, \mathbf{C}).$$

Using the identities

$$\det \mathbf{E} = \frac{\det \mathbf{F}}{\det \mathbf{P}} = \frac{\det \mathbf{F}}{\det \mathbf{U}} = \frac{\det \mathbf{F}}{(\det \mathbf{C})^{1/2}}$$

we transform the latter equation into

$$W(\mathbf{E}, \mathbf{C}) = W(\det \mathbf{F}, \mathbf{C}).$$

So Eq. (15) takes the form

$$W(\det \mathbf{F}, \mathbf{C}) = W(\det \mathbf{F}, \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T).$$

Thus, W is an isotropic function of $\mathbf{C}$ whereas dependence on $\mathbf{E}$ reduces to a dependence on $\det \mathbf{F}$. As a result, W is the function of $J = \det \mathbf{F}$ and the principal invariants of $\mathbf{C}$:

$$W = W(J, I_1, I_2, I_3),$$

where $I_1 = \operatorname{tr} \mathbf{C}$, $I_2 = \frac{1}{2}[I_1^2 - \operatorname{tr} \mathbf{C}^2]$, $I_3 = \det \mathbf{C}$.

6. Conclusions

We have formulated the new definition of the local material symmetry group for a micromorphic continuum. The group consists of ordered triples of tensors which make the strain energy density invariant under changes of a reference placement. From a physical point of view, this means that the changes of a reference placement described by members of the material symmetry group are unrecognizable via experiments. In particular, we define micromorphic isotropic media, solids, fluids and subfluids. The micromorphic subfluids present a class of materials which can be useful for description of such microstructured media as granular, soils, suspensions.

Let us note the the introduced here notion of the local material symmetry group can be easily extended for inelastic materials. It can also be very useful for the derivation of consistent representation of nonlinear constitutive equations of enhanced media, see for example the case of micropolar media [28, 29, 31]. Indeed, often we a priori know symmetry properties of a medium or can construct a material with requested optimal symmetries. Knowing in advance material symmetry constraints we can derive the corresponding form of constitutive equations. This is more important for such complex media as briefly considered here granular media. Further extensions of the material symmetry group analysis can also be useful for other generalized continua. In particular, using given here definition one can analyze other classes of materials with microdistortion such as the relaxed micromorphic continuum [37]. In a similar way, using the definition of the material symmetry group in [32] one can characterize new classes of continua such as recently discovered reduced strain-gradient materials [38–40], capillary fluids [41, 42], porous media [43, 44].

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