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On the mixing property and the ergodic principle for nonhomogeneous Markov chains

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ABSTRACT

We study different types of limit behavior of infinite dimension discrete time nonhomogeneous Markov chains. We show that the geometric structure of the set of those Markov chains which have asymptotically stationary density depends on the considered topologies. We generalize and correct some results from Ganikhodjaev et al. (2006) [3].

1. Introduction

The theory of Markov chains is a well-developed field of mathematics whose applications arise in many different areas of science and technology. However, there are some biological and physical models which cannot be described by homogeneous chains. One of them is a model related to population genetics. To examine the problem of the evolution of biologic system, the notion of a quadratic stochastic process was introduced (see [8] for review). The fundamental issue is the study of the limit behavior of such processes. In [3] the authors considered the concept of the ergodic principle (originally this notion was introduced by Kolmogorov in [7]) for both quadratic stochastic processes and Markov chains and discussed the relationship between them. Unfortunately, some parts of the results obtained in [3] are false, namely Theorem 2.2 and those subsequent theorems which are partly based on it.

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In this paper we study different types of limit behavior, e.g. mixing and ergodicity, of infinite dimension nonhomogeneous Markov chains. We also examine the geometric structure of the set of all discrete time nonhomogeneous Markov chains. We shall see that the set of Markov chains which are mixing is not dense in norm operator topology, but the weaker property, i.e. norm almost mixing, is generic for both norm and strong operator topologies. Finally, we improve on and generalize some results presented in [3].

Throughout the paper we consider

$$\ell^1 = \left\{ \underline{x} = (x_n) : \|\underline{x}\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty, x_n \in \mathbb{R} \right\},$$
$$\mathcal{D} = \{\underline{x} \in \ell^1 : x_n \geqslant 0, \|\underline{x}\|_1 = 1\}.$$

A matrix $[Q_{ij}]_{i,j\in\mathbb{N}}$ is called *stochastic* if

$$Q_{ij} \geqslant 0; \qquad \sum_{j=1}^{\infty} Q_{ij} = 1.$$

The convex set of all stochastic matrices is denoted by $\mathbf{S}$.

Definition 1.1. A family of stochastic matrices

$$\mathbf{Q} = \{[Q_{ij}^{m,n}]_{i,j\in\mathbb{N}} : m, n \in \mathbb{N}, n - m \geqslant 1\}$$

is called a discrete time (nonhomogeneous) Markov chain if for any natural numbers m, l, n such that $m < l < n$ the following condition, known as the Chapman–Kolmogorov equation, is satisfied:

$$Q_{ij}^{m,n} = \sum_{k=1}^{\infty} Q_{ik}^{m,l} Q_{kj}^{l,n}.$$

Every stochastic matrix defines a linear operator $Q^{m,n} : \ell^1 \to \ell^1$ as follows:

$$(Q^{m,n}(\underline{x}))_j = \sum_{i=1}^{\infty} Q_{ij}^{m,n} x_i, \qquad \underline{x} = (x_n) \in \ell^1.$$

The norm of this operator is given by

$$\| |Q^{m,n}| \| = \sup_{\underline{x}\in\mathcal{D}} \|Q^{m,n}\underline{x}\|_1 = 1.$$

Stochasticity of $(Q_{ij}^{m,n})_{i,j\in\mathbb{N}}$ implies that

$$\| |Q^{m,n}| \| = 1 \text{ and } Q^{m,n}(\mathcal{D}) \subset \mathcal{D}.$$

Notice that the Chapman–Kolmogorov equation can be presented in the form

$$\forall_{m<l<n\in\mathbb{N}} \qquad Q^{m,n} = Q^{l,n} \circ Q^{m,l},$$

where $\circ$ stands for the composition of linear operators (multiplication of matrices).

Remark 1.2. Applying a Chapman–Kolmogorov property, a Markov chain $\mathbf{Q}$ may be considered as a mapping

$$\mathbb{N} \ni n \mapsto Q^{n,n+1} \in \mathbf{S}.$$

In fact,

$$Q^{m,n} = Q^{n-1,n} \circ \ldots \circ Q^{m+1,m+2} \circ Q^{m,m+1}.$$

The set of all Markov chains will be denoted by $\mathscr{S}$, i.e.

$$\mathscr{S} = \left\{ (Q^{n,n+1})_{n\geqslant 1} : Q^{n,n+1} \text{ are linear operators defined by } \left\{ [Q^{n,n+1}_{ij}]_{i,j\in\mathbb{N}} \right\} \right\}.$$

To simplify the notation, elements of the set $\mathscr{S}$ will be written in bold, i.e. instead of writing $(Q^{n,n+1})_{n\geqslant 1} \in \mathscr{S}$ we will write $\mathbf{Q} \in \mathscr{S}$.

Definition 1.3. Given $t \in [0, 1]$, a convex combination $\mathbf{T}(t)$ of two nonhomogeneous Markov chains $\mathbf{Q}$ and $\mathbf{R} \in \mathscr{S}$ is defined as

$$T^{n,n+1}(t) = tQ^{n,n+1} + (1-t)R^{n,n+1}.$$

Moreover,

$$T^{m,n}(t) = T^{n-1,n}(t) \circ \ldots \circ T^{m+1,m+2}(t) \circ T^{m,m+1}(t) \text{ for } t \in [0, 1].$$

It follows that $\mathbf{T}(t) \in \mathscr{S}$ for every $t \in [0, 1]$ and, moreover, $[0, 1] \ni t \mapsto \mathbf{T}(t) \in \mathscr{S}$ is continuous (when $\mathscr{S}$ is endowed with a suitable topology), and $\mathbf{T}(0) = \mathbf{R}$, $\mathbf{T}(1) = \mathbf{Q}$. In particular, the set $\mathscr{S}$ has an affine structure and therefore is arcwise connected.

There are several topologies considered in studying the geometric structure of the set $\mathscr{S}$. We have:

(1) The sup norm operator topology induced by metric $\rho_{n.\sup} : \mathscr{S} \times \mathscr{S} \to \mathbb{R}_+ \cup \{0\}$ defined by

$$\rho_{n.\sup}(\mathbf{Q}, \mathbf{T}) = \sup_m |||Q^{m,m+1} - T^{m,m+1}|||.$$

(2) The $\sum$ norm operator topology induced by metric $\rho_{n.\sum} : \mathscr{S} \times \mathscr{S} \to \mathbb{R}_+ \cup \{0\}$ defined by

$$\rho_{n.\sum}(\mathbf{Q}, \mathbf{T}) = \sum_{m=1}^{\infty} \frac{1}{2^m} |||Q^{m,m+1} - T^{m,m+1}|||.$$

(3) The $\sum$ sup strong operator topology induced by the metric $\rho_{so.\sup} : \mathscr{S} \times \mathscr{S} \to \mathbb{R}_+ \cup \{0\}$ defined by

$$\rho_{so.\sup}(\mathbf{Q}, \mathbf{T}) = \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_m \|Q^{m,m+1}e^{(l)} - T^{m,m+1}e^{(l)}\|_1,$$

where $\{e^{(m)}\}$ is a standard basis in ℓ^1, i.e. $e^{(m)} = (\underbrace{0, 0, \ldots, 1}_{m}, 0, \ldots)$, $m \in \mathbb{N}$.

(4) The $\sum\sum$ strong operator topology induced by the metric $\rho_{so.\sum} : \mathscr{S} \times \mathscr{S} \to \mathbb{R}_+ \cup \{0\}$ defined by

$$\rho_{so.\sum}(\mathbf{Q}, \mathbf{T}) = \sum_{m,l=1}^{\infty} \frac{1}{2^{m+l}} \|Q^{m,m+1}e^{(l)} - T^{m,m+1}e^{(l)}\|_1,$$

where $\{e^{(m)}\}$ is a standard basis in ℓ^1.

Clearly $\rho_{n.\sup}$ generates the strongest topology and $\rho_{so.\sum}$ generates the weakest. Note that metrics $\rho_{n.\sum}$ and $\rho_{so.\sup}$ cannot be compared. Indeed, consider $\mathbf{Q}_j = (Q_j^{m,m+1})_{j\geqslant 1} \in \mathscr{S}$ defined as follows:

$$Q_j^{m,m+1} = \begin{cases} Q, & \text{if } 1 \leqslant m < j, \\ I, & \text{if } m \geqslant j, \end{cases}$$

where I stands for the identity operator and $\mathbf{Q} = (Q)_{m\geqslant 1}$ is such that $Q \neq I$. Then

$$\begin{aligned} \rho_{n.\sum}(\mathbf{Q}_j, \mathbf{Q}) &= \sum_{m=1}^{\infty} \frac{1}{2^m} |||Q_j^{m,m+1} - Q^{m,m+1}||| \\ &= \sum_{m=1}^{j-1} \frac{1}{2^m} |||Q - Q||| + \sum_{m=j}^{\infty} \frac{1}{2^m} |||I - Q||| \\ &= \frac{1}{2^{j-1}} |||I - Q||| \to 0 \text{ as } j \to \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \rho_{so.\sup}(\mathbf{Q}_j, \mathbf{Q}) &= \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_m \|Q_j^{m,m+1} e^{(l)} - Q^{m,m+1} e^{(l)}\|_1 \\ &= \sum_{l=1}^{\infty} \frac{1}{2^l} \|e^{(l)} - Qe^{(l)}\|_1 > 0. \end{aligned}$$

Thus, $\rho_{so.\sup}(\mathbf{Q}_j, \mathbf{Q}) \nrightarrow 0$ as $j \to \infty$. It follows that $\rho_{n.\sum}$ is not stronger than $\rho_{so.\sup}$.

Now let us define $\mathbf{Q}_j = (Q_j^{m,m+1})_{j\geqslant 1} \in \mathscr{S}$ as follows:

$$Q_j^{m,m+1} e^{(l)} = \begin{cases} e^{(l)}, & \text{if } 1 \leqslant l \leqslant j, \\ e^{(1)}, & \text{if } l > j, \end{cases}$$

that is, $Q_j^{m,m+1}\underline{x} = (x_1 + \sum_{k=j+1}^{\infty} x_k, x_2, \ldots, x_j, 0, \ldots)$ for any $\underline{x} = (x_1, x_2, \ldots)$. Note that $\mathbf{Q}_j = (Q_j)_{m\geqslant 1} = (Q_j, Q_j, \ldots)$. Consider $\mathbf{I} = (I, I, \ldots) \in \mathscr{S}$, where I stands for the identity operator. Observe that

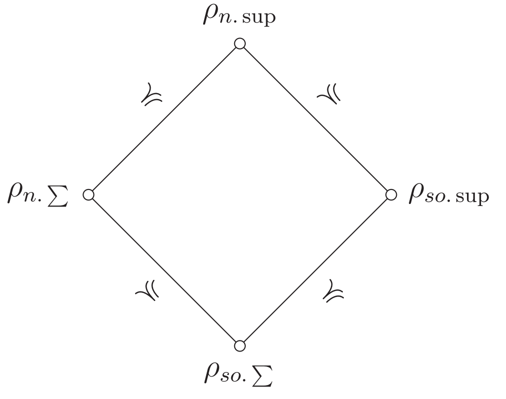
$$\begin{aligned} \rho_{so.\sup}(\mathbf{Q}_j, \mathbf{I}) &= \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_m \|Q_j^{m,m+1} e^{(l)} - Ie^{(l)}\|_1 \\ &= \sum_{l=1}^{\infty} \frac{1}{2^l} \|Q_j^{m,m+1} e^{(l)} - e^{(l)}\|_1 \\ &= \sum_{l=j+1}^{\infty} \frac{1}{2^l} \|e^{(1)} - e^{(l)}\|_1 = \sum_{l=j+1}^{\infty} \frac{1}{2^l} \cdot 2 = \frac{1}{2^{j-1}} \to 0 \text{ as } j \to \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \rho_{n.\sum}(\mathbf{Q}_j, \mathbf{I}) &= \sum_{m=1}^{\infty} \frac{1}{2^m} |||Q_j^{m,m+1} - I||| \\ &= \left(\sum_{m=1}^{\infty} \frac{1}{2^m} \right) |||Q_j - I||| = 1 \cdot 2 = 2 \nrightarrow 0 \text{ as } j \to \infty. \end{aligned}$$

Thus $\rho_{so.\sup}$ is not stronger than $\rho_{n.\sum}$. It follows that the metrics $\rho_{n.\sum}$ and $\rho_{so.\sup}$ are not comparable.

The relationships between the considered metrics are illustrated in the diagram below:



2. Norm mixing

This paper is dedicated to the geometric structure of the sets of those operators $\mathbf{Q} \in \mathscr{S}$ which have asymptotically stationary density (we call them mixing). Of course we have different types of mixing depending on considered topologies. In this section we examine the strongest case, the norm mixing. We start with

Definition 2.1. A nonhomogeneous Markov chain $\mathbf{Q}$ is said to be norm mixing, if there exists a one-dimensional (stochastic) projection $P \in \mathbf{S}$ such that for every m we have

$$\lim_{n\to\infty} \||Q^{m,n} - P|\| = 0.$$

The set of all norm mixing Markov chains is denoted by $\mathscr{S}_{nm}$.

Remark 2.2. A mixing Markov chain is sometimes called norm asymptotically stable. Equivalently it may be defined by

$$\lim_{n\to\infty} \sup_{\underline{x}\in\mathscr{D}} \|Q^{m,n}\underline{x} - \mathbf{p}\|_1 = 0,$$

where $\mathbf{p} \in \mathscr{D}$ is a fixed probabilistic vector (then each row of the limit matrix P coincides with $\mathbf{p}$).

The following theorem shows that norm mixing nonhomogeneous Markov chains are rare, which is the opposite of the homogeneous case (cf. [2, Theorem 2.4]). This supports what was remarked on by Iosifescu [5] that norm mixing is not a "natural" concept for nonhomogeneous Markov chains and that $\mathscr{S}_{nm}$ is a very restricted class (see [5, Remark 4]).

Theorem 2.3. *The set $\mathscr{S}_{nm}^c$ of all Markov chains which are not norm mixing is $\rho_{n.\sup}$ topology dense subset of $\mathscr{S}$. Moreover, in this case its interior $\mathrm{Int}\mathscr{S}_{nm}^c \neq \emptyset$.*

Proof. We will show that

$$\forall_{\mathbf{Q}\in\mathscr{S}} \quad \forall_{\varepsilon>0} \quad \exists_{\mathbf{Q}_*\in\mathscr{S}_{nm}^c} \qquad \rho_{n.\sup}(\mathbf{Q}, \mathbf{Q}_*) < 2\varepsilon.$$

Given an arbitrary $\mathbf{Q} \in \mathscr{S}$ and $0 < \varepsilon < 1$ consider a convex combination

$$Q_*^{m,m+1} = (1-\varepsilon)Q^{m,m+1} + \varepsilon R^{m,m+1},$$

where $\mathbf{R} \in \mathscr{S}$ is defined as follows: for any vector $\underline{x} = (x_1, x_2, \ldots) \in \mathscr{D}$,

$$R^{m,m+1}\underline{x} = (\underbrace{0, \ldots, 0}_{m}, x_1, x_2, \ldots).$$

Then

$$\begin{aligned}\rho_{n.\sup}(\mathbf{Q}_*, \mathbf{Q}) &= \sup_m |||(1-\varepsilon)Q^{m,m+1} + \varepsilon R^{m,m+1} - Q^{m,m+1}||| \\ &= \varepsilon \sup_m |||Q^{m,m+1} - R^{m,m+1}||| \leqslant 2\varepsilon.\end{aligned}$$

It remains to show that $\mathbf{Q}_* \notin \mathscr{S}_{nm}$. Suppose that, on the contrary, there exists $\mathbf{p} \in \mathscr{D}$ such that $\lim_{n\to\infty} Q_*^{m,n}\mathbf{p} = \mathbf{p}$. Since $\mathbf{p} \in \mathscr{D}$ then there exists $M \in \mathbb{N}$ such that

$$\sum_{j=1}^{M} \mathbf{p}_j > 1 - \varepsilon.$$

Hence

$$\sum_{j=1}^{M} (Q_*^{m,n}\mathbf{p})_j \longrightarrow \sum_{j=1}^{M} \mathbf{p}_j > 1 - \varepsilon, \quad n \to \infty.$$

On the other hand it follows from the definition of $\mathbf{Q}_*$ that

$$\sum_{j=1}^{M} (Q_*^{m,n+1}\mathbf{p})_j \leqslant 1 - \varepsilon,$$

when m is large enough, which is a contradiction. Indeed, if $m > M$, then

$$\begin{aligned}\sum_{j=1}^{M} (Q_*^{m,n+1}\mathbf{p})_j &= 1 - \sum_{j=M+1}^{\infty} (Q_*^{m,n+1}\mathbf{p})_j \\ &= 1 - \sum_{j=M+1}^{\infty} \left(Q_*^{n,n+1}(Q_*^{m,n}\mathbf{p})\right)_j \\ &= 1 - \sum_{j=M+1}^{\infty} (((1-\varepsilon)Q^{n,n+1} + \varepsilon R^{n,n+1})(Q_*^{m,n}\mathbf{p}))_j \\ &\leqslant 1 - \varepsilon \sum_{j=M+1}^{\infty} (R^{n,n+1}(Q_*^{m,n}\mathbf{p}))_j = 1 - \varepsilon.\end{aligned}$$

It follows that $\mathscr{S}_{nm}^c$ is $\rho_{n.\sup}$ dense in $\mathscr{S}$ (in particular $\mathscr{S}_{nm}^c$ is $\rho_{n.\sum}$ dense).

It remains to show that $Int\mathscr{S}_{nm}^c \neq \emptyset$ for the $\rho_{n.\sup}$ topology. For this consider the open ball

$$K(\mathbf{R}, 1) = \{\mathbf{T} \in \mathscr{S} : \rho_{n.\sup}(\mathbf{T}, \mathbf{R}) < 1\},$$

where as before

$$R^{m,m+1}\underline{x} = (\underbrace{0, \ldots, 0}_{m}, x_1, x_2, \ldots).$$

We will show that $K(\mathbf{R}, 1) \subseteq \mathscr{S}_{nm}^c$. In fact, if $\mathbf{T} \in K(\mathbf{R}, 1)$, then for some $\varepsilon > 0$

$$\sup_{\underline{x}\in\mathscr{D}} \|T^{m-1,m}\underline{x} - R^{m-1,m}\underline{x}\|_1 \leqslant \rho_{n.\sup}(\mathbf{T}, \mathbf{R}) = 1 - \varepsilon.$$

In particular, for every $m > M \geqslant 1$ and $\underline{x} \in \mathscr{D}$,

$$\sum_{j=1}^{M}(T^{0,m}\underline{x})_j \leqslant \rho_{n.\sup}(\mathbf{T},\mathbf{R}) = 1-\varepsilon.$$

It follows that

$$\sup_{M\in\mathbb{N}} \limsup_{m\to\infty} \sum_{j=1}^{M}(T^{0,m}\underline{x})_j \leqslant \rho_{n.\sup}(\mathbf{T},\mathbf{R}) = 1-\varepsilon < 1,$$

and therefore $\mathbf{T}$ has no invariant densities. Hence $\mathbf{T} \in \mathscr{S}_{nm}^{c}$. □

Topologies on $\mathscr{S}$ generated by $\rho_{n.\sup}$ and $\rho_{n.\sum}$ differ. In fact, we have

Proposition 2.4. *The set $\mathscr{S}_{nm}$ is $\rho_{n.\sum}$ dense in $\mathscr{S}$.*

Proof. Let $\mathbf{T} \in \mathscr{S}$ and $\varepsilon > 0$ be taken arbitrarily. We find $M \in \mathbb{N}$ such that $\frac{1}{2^{M-1}} < \varepsilon$. Define

$$T_{\varepsilon}^{m,m+1} = \begin{cases} T^{m,m+1}, & \text{if } m \leqslant M,\\ E, & \text{if } m > M,\end{cases}$$

where $E\underline{x} = \left(\left(\sum_{j=1}^{\infty} x_j\right), 0, 0, \ldots\right)$. Clearly $\mathbf{E} = (E^{m,m+1})_{m\geqslant 1} \in \mathscr{S}$ (where for every $m \in \mathbb{N}$, $E^{m,m+1} = E$) is a stochastic projection (and it is norm mixing). We find

$$\lim_{n\to\infty} \sup_{\underline{x}\in\mathscr{D}} \|T_{\varepsilon}^{m,n}\underline{x} - (1,0,\ldots)\|_1 = 0.$$

It follows that

$$\forall_{m\in\mathbb{N}} \qquad \lim_{n\to\infty} |||T_{\varepsilon}^{m,n} - E||| = 0.$$

Hence the Markov chain $\mathbf{T}_{\varepsilon}$ is norm mixing. Obviously,

$$\begin{aligned}\rho_{n.\sum}(\mathbf{T},\mathbf{T}_{\varepsilon}) &= \sum_{m=1}^{M}\frac{1}{2^m}|||T^{m,m+1} - T_{\varepsilon}^{m,m+1}||| + \sum_{m=M+1}^{\infty}\frac{1}{2^m}|||T^{m,m+1} - E|||\\ &\leqslant 2\cdot\frac{1}{2^M} = \frac{1}{2^{M-1}} < \varepsilon.\end{aligned}$$

We conclude that $\mathbf{T}_{\varepsilon} \in \mathscr{S}_{nm}$. □

The metric $\rho_{n.\sup}$ is much more relevant concerning the geometric structure of $\mathscr{S}$. It will be used in the sequel.

Definition 2.5. A (nonhomogeneous) Markov chain $\mathbf{Q}$ is said to be norm almost mixing, if

$$\forall_{m\in\mathbb{N}} \qquad \lim_{n\to\infty} \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1 = 0.$$

The set of all norm almost mixing Markov chains is denoted by $\mathscr{S}_{nam}$.

In [6] norm almost mixing is called norm completely mixing. The reader will find the following theorem as a generalization of genericity of norm completely mixing for homogeneous Markov chains (cf. [6, Theorem 3]).

Theorem 2.6. *$\mathscr{S}_{nam}$ is a dense G_δ subset of $\mathscr{S}$ in both $\rho_{n.\sup}$ and $\rho_{n.\sum}$ topologies.*

Proof. First we will show that the set $\mathscr{S}_{nam}$ is a $\rho_{n.\sup}$ dense subset of $\mathscr{S}$, i.e. we will show that

$$\forall_{\mathbf{Q}\in\mathscr{S}} \quad \forall_{\varepsilon>0} \quad \exists_{\mathbf{Q}_\varepsilon\in\mathscr{S}_{nam}} \qquad \rho_{n.\sup}(\mathbf{Q},\mathbf{Q}_\varepsilon) < 2\varepsilon$$

(the denseness in $\rho_{n.\sum}$ metric follows from Proposition 2.4 or from the fact that $\rho_{n.\sum} \leqslant \rho_{n.\sup}$).

Given an arbitrary $\mathbf{Q} \in \mathscr{S}$ and $0 < \varepsilon < 1$ consider a convex combination

$$Q_\varepsilon^{n,n+1} = (1-\varepsilon)Q^{n,n+1} + \varepsilon E,$$

where E is such as in the proof of the Proposition 2.4 (clearly $\rho_{n.\sup}(\mathbf{Q},\mathbf{Q}_\varepsilon) < 2\varepsilon$). By convexity $\mathbf{Q}_\varepsilon \in \mathscr{S}$. For any pair of vectors $\underline{x}, \underline{y} \in \mathscr{D}$ we have

$$\begin{aligned}\|Q_\varepsilon^{n-1,n}\underline{x} - Q_\varepsilon^{n-1,n}\underline{y}\|_1 &= (1-\varepsilon)\|Q^{n-1,n}\underline{x} - Q^{n-1,n}\underline{y}\|_1\\ &= (1-\varepsilon)\|Q^{n-1,n}(\underline{x}-\underline{y})\|_1\\ &\leqslant (1-\varepsilon)\|\underline{x}-\underline{y}\|_1.\end{aligned}$$

Moreover,

$$\begin{aligned}\|Q_\varepsilon^{m,n}\underline{x} - Q_\varepsilon^{m,n}\underline{y}\|_1 &= \|Q^{n-1,n}(Q^{m,n-1}\underline{x} - Q^{m,n-1}\underline{y})\|_1\\ &\leqslant (1-\varepsilon)\|Q^{m,n-1}\underline{x} - Q^{m,n-1}\underline{y}\|_1, \quad \underline{x},\underline{y}\in\mathscr{D}.\end{aligned}$$

Iterating the last inequality for any $\underline{x}, \underline{y} \in \mathscr{D}$ we have

$$\|Q_\varepsilon^{m,n}\underline{x} - Q_\varepsilon^{m,n}\underline{y}\|_1 \leqslant (1-\varepsilon)^{n-m}\|\underline{x}-\underline{y}\|_1,$$

and so

$$\|Q_\varepsilon^{m,n}\underline{x} - Q_\varepsilon^{m,n}\underline{y}\|_1 \leqslant 2(1-\varepsilon)^{n-m}.$$

As the above inequality holds true for any pair of vectors $\underline{x}, \underline{y} \in \mathscr{D}$, then

$$\sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q_\varepsilon^{m,n}\underline{x} - Q_\varepsilon^{m,n}\underline{y}\|_1 \leqslant 2(1-\varepsilon)^{n-m}.$$

Therefore,

$$\lim_{n\to\infty} \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q_\varepsilon^{m,n}\underline{x} - Q_\varepsilon^{m,n}\underline{y}\|_1 = 0.$$

Thus $\mathscr{S}_{nam}$ is a dense subset of $\mathscr{S}$ in both $\rho_{n.\sum}$ and $\rho_{n.\sup}$ metrics.

It remains to show that $\mathscr{S}_{nam}$ is a G_δ subset of $\mathscr{S}$. Observe that the sequence $\|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1$ is nonincreasing. Indeed,

$$\begin{aligned}\|Q^{m,n+1}\underline{x} - Q^{m,n+1}\underline{y}\|_1 &= \|Q^{n,n+1}(Q^{m,n}\underline{x}) - Q^{n,n+1}(Q^{m,n}\underline{y})\|_1\\ &\leqslant \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1.\end{aligned}$$

It follows that the sequence $\sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1$ is nonincreasing. Therefore, we obtain that

$$\begin{aligned}\mathscr{S}_{nam} &= \left\{\mathbf{Q}\in\mathscr{S} : \forall_{m\in\mathbb{N}} \lim_{n\to\infty} \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1 = 0\right\}\\ &= \bigcap_{m=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=m+1}^{\infty}\left\{\mathbf{Q}\in\mathscr{S} : \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1 < \frac{1}{k}\right\}.\end{aligned}$$

To finish the proof we only need to notice that for fixed $m < n$ the function

$$\mathscr{S} \ni \mathbf{Q} \mapsto \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1$$

is $\rho_{n.\sum}$ continuous. Hence, $\mathscr{S}_{nam}$ is a G_δ set for the metric $\rho_{n.\sum}$ (so it is a G_δ set for any stronger metric like $\rho_{n.\sup}$). □

3. Strong operator topology mixing

In this section we study the strong operator topology mixing. We begin with

Definition 3.1. A nonhomogeneous Markov chain $\mathbf{Q}$ is said to be strong almost mixing if

$$\forall_m \;\; \forall_i \;\; \forall_j \qquad \lim_{n\to\infty} \|Q^{m,n}_{i.} - Q^{m,n}_{j.}\|_1 = 0.$$

The set of all strong almost mixing Markov chains is denoted by $\mathscr{S}_{sam}$.

Theorem 3.2 (Schur)**.** *A sequence $(x_n) \subset \ell^1$ is weakly convergent if and only if it converges in norm, i.e. weak and norm convergence of sequences are equivalent.*

Clearly, we have

Corollary 3.3. *A nonhomogeneous Markov chain $\mathbf{Q}$ is strong almost mixing if*

$$\forall_m \;\; \forall_i \;\; \forall_j \qquad w - \lim_{n\to\infty} (Q^{m,n}_{i.} - Q^{m,n}_{j.}) = 0.$$

The strong almost mixing property means that the rows of the matrix $(Q^{m,n}_{ij})_{i,j\in\mathbb{N}}$ tend to be the same. Obviously $\mathscr{S}_{nam} \subset \mathscr{S}_{sam}$. We easily obtain the following:

Theorem 3.4. *The set $\mathscr{S}_{sam}$ is a $\sum\sum$ strong operator topology (i.e. in $\rho_{so.\sum}$) dense G_δ subset of $\mathscr{S}$.*

Proof. It remains to show that $\mathscr{S}_{sam}$ is a strong operator topology G_δ. For this notice that

$$\begin{aligned}\mathscr{S}_{sam} &= \left\{\mathbf{Q} \in \mathscr{S} : \forall_{m\in\mathbb{N}} \;\; \forall_{i\in\mathbb{N}} \;\; \forall_{j\in\mathbb{N}} \qquad \lim_{n\to\infty} \|Q^{m,n}_{i.} - Q^{m,n}_{j.}\|_1 = 0\right\} \\ &= \bigcap_{m=1}^{\infty}\bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcap_{l=1}^{\infty}\bigcap_{N=1}^{\infty} \bigcup_{n>\max\{N,m\}} \left\{\mathbf{Q} \in \mathscr{S} : \|Q^{m,n}_{i.} - Q^{m,n}_{j.}\|_1 < \frac{1}{l}\right\}\end{aligned}$$

(we notice that $n \mapsto \|Q^{m,n}_{i.} - Q^{m,n}_{j.}\|_1 = \|Q^{m,n}e^{(i)} - Q^{m,n}e^{(j)}\|_1$ is nonincreasing). To end the proof observe that for fixed $m < n$ the function

$$\mathscr{S} \ni \mathbf{Q} \mapsto \|Q^{m,n}_{i.} - Q^{m,n}_{j.}\|_1$$

is continuous for the metric $\rho_{so.\sum}$. Therefore, $\mathscr{S}_{sam}$ is a G_δ set for the metric $\rho_{so.\sum}$. Since the metric $\rho_{so.\sup}$ is stronger than $\rho_{so.\sum}$, it follows that $\mathscr{S}_{sam}$ is a G_δ set for $\rho_{so.\sup}$ as well. □

Definition 3.5. A nonhomogeneous Markov chain $\mathbf{Q}$ is said to be strong mixing, if

$$\exists_{\mathbf{p}_0\in\mathscr{D}} \;\; \forall_m \;\; \forall_i \qquad \lim_{n\to\infty} \|Q^{m,n}_{i.} - \mathbf{p}_0\|_1 = 0.$$

The set of all strong mixing Markov chains is denoted by $\mathscr{S}_{sm}$.

Clearly $\mathscr{S}_{nm} \subseteq \mathscr{S}_{sm}$.
We easily observe the following

Corollary 3.6. *If a nonhomogeneous Markov chain is strong mixing, then it is strong almost mixing.*

Theorem 3.7. *The set $\mathscr{S}_{sm}^c$ of all Markov chains which are not strong mixing is $\rho_{so.\,\sup}$ topology dense subset of $\mathscr{S}$.*

Proof. Given an arbitrary $\mathbf{Q} \in \mathscr{S}$ and $0 < \varepsilon < 1$ consider a convex combination

$$Q_*^{m,m+1} = (1-\varepsilon)Q^{m,m+1} + \varepsilon R^{m,m+1},$$

where $\mathbf{R} \in \mathscr{S}$ as before is defined as follows: for any vector $\underline{x} = (x_1, x_2, \ldots) \in \mathscr{D}$,

$$R^{m,m+1}\underline{x} = (\underbrace{0, \ldots, 0}_{m}, x_1, x_2, \ldots).$$

Then

$$\rho_{so.\,\sup}(\mathbf{Q}_*, \mathbf{Q}) = \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_m \|(1-\varepsilon)Q^{m,m+1}e^{(l)} + \varepsilon R^{m,m+1}e^{(l)} - Q^{m,m+1}e^{(l)}\|_1$$
$$= \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_m \|Q^{m,m+1}e^{(l)} - R^{m,m+1}e^{(l)}\|_1 \leqslant 2\varepsilon.$$

Similar arguments to those used towards the proof of the Theorem 2.3 imply that $\mathbf{Q}_* \notin \mathscr{S}_{sm}$. Indeed, suppose that, on the contrary, there exists $\mathbf{p}_0 \in \mathscr{D}$ such that for every $m \in \mathbb{N}$ and every $\mathbf{p} \in \mathscr{D}$, $\lim_{n\to\infty} \|Q_*^{m,n}\mathbf{p} - \mathbf{p}_0\|_1 = 0$. Since $\mathbf{p}_0 \in \mathscr{D}$ then there exists $M \in \mathbb{N}$ such that

$$\sum_{j=1}^{M} \mathbf{p}_{0j} > 1 - \varepsilon.$$

Hence

$$\sum_{j=1}^{M} (Q_*^{m,n}\mathbf{p})_j \longrightarrow \sum_{j=1}^{M} \mathbf{p}_{0j} > 1 - \varepsilon, \quad n \to \infty.$$

On the other hand it follows from the definition of $\mathbf{Q}_*$ that

$$\sum_{j=1}^{M} (Q_*^{m,n+1}\mathbf{p})_j \leqslant 1 - \varepsilon,$$

when m is large enough, which is a contradiction. □

4. Ergodic principle

This section is devoted to the ergodic principle for nonhomogeneous Markov chains and quadratic stochastic processes and the relation between them. We recall results presented in [3]. We begin with

Definition 4.1. A Markov chain $\mathbf{Q}$ is said to satisfy the ergodic principle if

$$\lim_{n\to\infty} |Q_{ik}^{m,n} - Q_{jk}^{m,n}| = 0$$

is valid for every $i, j, k, m \in \mathbb{N}$.

Notice that the above definition states that the sequence $(Q_{i\cdot}^{m,n} - Q_{j\cdot}^{m,n})_{n\in\mathbb{N}}$ converges to 0 in weak* topology in ℓ^1.

Remark 4.2. There are a few more relevant works in the literature dealing with the topic of limit behavior of nonhomogeneous Markov chains (see e.g. [4,5]). The reader should be warned that authors do not always use the same names for the same notions, e.g. in [5], weak ergodicity is what we refer to as norm almost mixing and strong ergodicity is what we call norm mixing.

Ganikhodjaev et al. (see [3, Theorem 2.2]) discussed relations between the following conditions: For a nonhomogeneous Markov chain **Q**:

(i) **Q** satisfies the ergodic principle.
(ii) For every $i, j, m \in \mathbb{N}$ the following relation holds:

$$\lim_{n\to\infty} \|Q^{m,n}e^{(i)} - Q^{m,n}e^{(j)}\|_1 = 0.$$

(iii) For every $\varphi, \psi \in \mathscr{D}$ and $m \in \mathbb{N}$ the following relation holds:

$$\lim_{n\to\infty} \|Q^{m,n}\varphi - Q^{m,n}\psi\|_1 = 0.$$

Note that all three conditions are not equivalent in general. Clearly (ii) and (iii) are equivalent and they imply (i). However, (i) is essentially weaker and does not imply (ii) and (iii), as the ergodic principle is concerned with weak* convergence (and therefore the Schur theorem is not applicable). Obviously in the finite dimension case all three conditions are equivalent. Note that (ii) is the strong almost mixing condition.

In fact, repeating arguments used in the proof of Theorem 2.2 [3] the following generalization of equivalence of the conditions (ii) and (iii) may be shown. We have

Theorem 4.3. *Let **Q** be a nonhomogeneous Markov chain. The following conditions are equivalent:*

(i) ***Q** is strong mixing.*
(ii) *There exists $\boldsymbol{p}_0 \in \mathscr{D}$ such that for every $m \in \mathbb{N}$ and every $\boldsymbol{p} \in \mathscr{D}$*

$$\lim_{n\to\infty} \|Q^{m,n}\boldsymbol{p} - \boldsymbol{p}_0\|_1 = 0.$$

Ganikhodjaev et al. [3] have proved the following theorem:

Theorem 4.4 [3]**.** *Let **Q** be a Markov process. If there exists a number $k_0 \in \mathbb{N}$ and a sequence $\{\lambda_n\}$, $0 < \lambda_n < 1$ for every $n \in \mathbb{N}$, satisfying the conditions*

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \tag{1}$$

$$\sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(1-\lambda_k)}{(1-\lambda_j)} \to 0 \text{ as } n \to \infty \tag{2}$$

and such that

$$Q_{ik_0}^{n-1,n} \geqslant \lambda_n \quad \text{for all } i, n \in \mathbb{N}, \tag{3}$$

then the Markov process satisfies the ergodic principle.

We will generalize the result above by showing that the condition (2) is not essential. Moreover, in (3) the state k_0 is not necessarily fixed (i.e. may depend on each step n).

Recall that a Banach lattice E is called an $AL-space$ if its norm is additive, i.e. if $\|x+y\| = \|x\|+\|y\|$ whenever $x \in E, x \geqslant 0$ and $y \in E, y \geqslant 0$.

Remark 4.5. The norm $\|\cdot\|_1$ on the cone $\ell^1_+ = \{\underline{x} = (x_n) : \|\underline{x}\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty, x_n \geqslant 0\}$ is additive. Therefore ℓ^1 is an $AL - space$.

Note that if $\underline{x}, \underline{y} \in \mathscr{D}$ then

$$\|\underline{x} - \underline{x} \wedge \underline{y}\|_1 = 1 - \|\underline{x} \wedge \underline{y}\|_1$$

and

$$\|\underline{x} - \underline{y}\|_1 = 2(1 - \|\underline{x} \wedge \underline{y}\|_1),$$

where $\underline{x} \wedge \underline{y} = \min\{\underline{x}, \underline{y}\}$.

Theorem 4.6. *Let $\mathbf{Q}$ be a Markov chain. If there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$, $0 \leqslant \lambda_n < 1$, satisfying (1) and such that for some sequence of states k_n*

$$Q^{n-1,n}_{ik_n} \geqslant \lambda_n \quad \text{for all } i, n \in \mathbb{N}, \tag{4}$$

then $\mathbf{Q}$ is norm almost mixing (and therefore $\mathbf{Q}$ satisfies the ergodic principle).

Proof. First we observe that for every $\underline{x}, \underline{y} \in \mathscr{D}$ and every natural number n we have

$$\|Q^{n-1,n}\underline{x} - Q^{n-1,n}\underline{y}\|_1 \leqslant (1 - \lambda_n)\|\underline{x} - \underline{y}\|_1 \leqslant 2(1 - \lambda_n).$$

Applying (4) we obtain

$$\|Q^{n-1,n}\underline{x} \wedge Q^{n-1,n}\underline{y}\|_1 \geqslant \lambda_n$$

for all $n \in \mathbb{N}$ and all $\underline{x}, \underline{y} \in \mathscr{D}$. Therefore, repeating arguments from [1],

$$\begin{aligned}
&\|Q^{n-1,n}\underline{x} - Q^{n-1,n}\underline{y}\|_1 \\
&= \|Q^{n-1,n}(\underline{x} - \underline{x} \wedge \underline{y}) - Q^{n-1,n}(\underline{y} - \underline{x} \wedge \underline{y})\|_1 \\
&= \left\| Q^{n-1,n}\left(\frac{\underline{x} - \underline{x} \wedge \underline{y}}{1 - \|\underline{x} \wedge \underline{y}\|_1}\right) - Q^{n-1,n}\left(\frac{\underline{y} - \underline{x} \wedge \underline{y}}{1 - \|\underline{x} \wedge \underline{y}\|_1}\right)\right\|_1 (1 - \|\underline{x} \wedge \underline{y}\|_1) \\
&= \|Q^{n-1,n}\underline{u} - Q^{n-1,n}\underline{v}\|_1(1 - \|\underline{x} \wedge \underline{y}\|_1) \\
&= 2(1 - \|\underline{x} \wedge \underline{y}\|_1)(1 - \|Q^{n-1,n}\underline{u} \wedge Q^{n-1,n}\underline{v}\|_1) \\
&\leqslant 2(1 - \lambda_n)(1 - \|\underline{x} \wedge \underline{y}\|_1) \\
&= (1 - \lambda_n)\|\underline{x} - \underline{y}\|_1,
\end{aligned}$$

where

$$\underline{u} = \frac{\underline{x} - \underline{x} \wedge \underline{y}}{1 - \|\underline{x} \wedge \underline{y}\|_1}, \quad \underline{v} = \frac{\underline{y} - \underline{x} \wedge \underline{y}}{1 - \|\underline{x} \wedge \underline{y}\|_1} \quad \text{and} \quad \|\underline{u}\|_1 = \|\underline{v}\|_1 = 1.$$

Therefore,

$$\begin{aligned}
\sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1 &= \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{n-1,n}(Q^{m,n-1}\underline{x}) - Q^{n-1,n}(Q^{m,n-1}\underline{y})\|_1 \\
&\leqslant (1 - \lambda_n) \sup_{\underline{x},\underline{y}\in\mathscr{D}} \|Q^{m,n-1}\underline{x} - Q^{m,n-1}\underline{y}\|_1.
\end{aligned}$$

Iterating the last inequality we have

$$\sup_{\underline{x},\underline{y}\in\mathcal{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1 \leqslant \sup_{\underline{x},\underline{y}\in\mathcal{D}} (1-\lambda_n)(1-\lambda_{n-1})\dots(1-\lambda_{m+1})\|\underline{x}-\underline{y}\|_1 = 2\prod_{j=m+1}^{n}(1-\lambda_j).$$

Because $(\lambda_n)_{n\in\mathbb{N}}$, $0 \leqslant \lambda_n < 1$, $n \in \mathbb{N}$, satisfies (1), then

$$\prod_{j=m+1}^{n}(1-\lambda_j) \to 0 \text{ as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty}\sup_{\underline{x},\underline{y}\in\mathcal{D}} \|Q^{m,n}\underline{x} - Q^{m,n}\underline{y}\|_1 = 0$$

which completes the proof. □

The above theorem gives us a constructive method for norm approximation of nonhomogeneous Markov chain $\mathbf{Q} \in \mathcal{S}$ by norm almost mixing Markov chains. In fact, given $\mathbf{Q} \in \mathcal{S}$ and any control sequence $0 \leqslant \varepsilon_n \to 0$ such that $\sum_{n=1}^{\infty}\varepsilon_n = \infty$, consider a convex combination

$$\widetilde{Q}_{\underline{\varepsilon}}^{n,n+1} = (1-\varepsilon_n)Q^{n,n+1} + \varepsilon_n E,$$

where $\mathbf{E} = (E^{m,m+1})_{m\geqslant 1} \in \mathcal{S}$ (here for every $m \in \mathbb{N}$, $E^{m,m+1} = E$) is defined as follows: for any vector $\underline{x} = (x_1, x_2, \dots) \in \mathcal{D}$,

$$E\underline{x} = \left(\left(\sum_{j=1}^{\infty}x_j\right), 0, 0, \dots\right).$$

We get

$$\||\widetilde{Q}_{\underline{\varepsilon}}^{n,n+1} - Q^{n,n+1}|\| \leqslant 2\varepsilon_n \to 0,$$

hence asymptotically $\widetilde{Q}_{\underline{\varepsilon}}^{n,n+1}$ is shadowing $Q^{n,n+1}$. Clearly, $(\widetilde{Q}_{\underline{\varepsilon}}^{n,n+1})_{n\geqslant 1} \in \mathcal{S}_{nam}$.

We will now discuss the limit behavior of quadratic stochastic processes. We will use the concept considered in [3]. We start with

Definition 4.7. The family of functions $\mathbf{P} = \{P_{ij,k}^{[s,t]} : i,j,k \in \mathbb{N}, s,t \in \mathbb{R}_+, t-s \geqslant 1\}$ is said to be a quadratic stochastic process (QSP) if for fixed $s, t \in \mathbb{R}_+$ it satisfies the following conditions:

(i) $P_{ij,k}^{[s,t]} \geqslant 0$, $\sum_{k=1}^{\infty} P_{ij,k}^{[s,t]} = 1$ for any $i,j,k \in \mathbb{N}$.

(ii) $P_{ij,k}^{[s,t]} = P_{ji,k}^{[s,t]}$ for any $i,j,k \in \mathbb{N}$.

(iii) for any initial distribution $\underline{x}^{(0)} \in \mathcal{D}$, $\underline{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots)$ and $s < r < t$ such that $t - r \geqslant 1$, $r - s \geqslant 1$ one of the following equations is satisfied:

(iiiA) $P_{ij,k}^{[s,t]} = \sum_{m,l=1}^{\infty} P_{ij,m}^{[s,r]} P_{ml,k}^{[r,t]} x_l^{(r)}$,

(iiiB) $P_{ij,k}^{[s,t]} = \sum_{m,l,g,h=1}^{\infty} P_{im,l}^{[s,r]} P_{jg,h}^{[s,r]} P_{lh,k}^{[r,t]} x_m^{(s)} x_g^{(s)}$,

where $x_k^{(r)} = \sum_{i,j=1}^{\infty} P_{ij,k}^{[0,r]} x_i^{(0)} x_j^{(0)}$.

We will consider discrete time QSP, i.e. $\mathbf{P} = \{P_{ij,k}^{[s,t]}\}$, where $s, t \in \mathbb{N}$.

Definition 4.8. A QSP **P** is said to satisfy the ergodic principle if

$$\lim_{n\to\infty} \left| P_{ij,k}^{[m,n]} - P_{uv,k}^{[m,n]} \right| = 0$$

is valid for every $i, j, u, v, k \in \mathbb{N}$ and arbitrary $m \in \mathbb{N}$.

It is known that certain Markov chains can be defined by means of QSP (see [3]). Let

$$H_{ij}^{m,n} := \sum_{l=1}^{\infty} P_{il,j}^{[m,n]} x_l^{(m)}, \quad i, j \in \mathbb{N}.$$

Theorem 4.9 [3]. *If* **P** *is a QSP, then* $\mathbf{H} = \{H_{ij}^{m,n}\}$ *is a Markov chain.*

Ganikhodjaev et al. (see [3, Theorem 2.6]) discussed the relation between the QSP **P** and the Markov chain **H**. In fact, taking our previous remark into consideration, they proved the following:

Theorem 4.10. *Let* **P** *be a QSP. The following conditions are equivalent:*

(i) **P** *is strong almost mixing, i.e.*

$$\forall_{m\in\mathbb{N}} \quad \forall_{i,j,u,v\in\mathbb{N}} \qquad \lim_{n\to\infty} \| P_{ij,\cdot}^{[m,n]} - P_{uv,\cdot}^{[m,n]} \|_1 = 0.$$

(ii) *The Markov chain* **H** *is strong almost mixing.*

The following generalization of Theorem 3.4 [3] is a direct application of our Theorem 4.6. We have

Theorem 4.11. *Let* **P** *be a QSP. If there exists a sequence* $(\lambda_n)_{n\in\mathbb{N}}$, $0 \leqslant \lambda_n < 1$, *satisfying (1), and such that for some sequence of states* k_n

$$P_{il,k_n}^{[n-1,n]} \geqslant \lambda_n \quad \textit{for all} \;\; i, l, n \in \mathbb{N},$$

then **P** *is strong almost mixing (and therefore* **P** *satisfies the ergodic principle).*

Proof. It is sufficient to note that

$$H_{ik_n}^{n-1,n} = \sum_{l=1}^{\infty} P_{il,k_n}^{[n-1,n]} x_l^{(n-1)} \geqslant \sum_{l=1}^{\infty} \lambda_n x_l^{(n-1)} = \lambda_n,$$

and then use Theorem 4.10. $\square$

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