# On the partition dimension of trees 

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#### Abstract

Given an ordered partition $\Pi=\left\{P_{1}, P_{2}, \ldots, P_{t}\right\}$ of the vertex set $V$ of a connected graph $G=(V, E)$, the partition representation of a vertex $v \in V$ with respect to the partition $\Pi$ is the vector $r(v \mid \Pi)=\left(d\left(v, P_{1}\right), d\left(v, P_{2}\right), \ldots, d\left(v, P_{t}\right)\right)$, where $d\left(v, P_{i}\right)$ represents the distance between the vertex $v$ and the set $P_{i}$. A partition $\Pi$ of $V$ is a resolving partition of $G$ if different vertices of $G$ have different partition representations, i.e., for every pair of vertices $u, v \in V, r(u \mid \Pi) \neq r(v \mid \Pi)$. The partition dimension of $G$ is the minimum number of sets in any resolving partition of $G$. In this paper we obtain several tight bounds on the partition dimension of trees.


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## 1. Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [17]. After these papers were published several authors developed diverse theoretical works about this topic [3,2, $4-10,14,19$ ]. Slater described the usefulness of these ideas into long range aids to navigation [17]. Also, these concepts have some applications in chemistry for representing chemical compounds $[12,13]$ or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [15]. Other applications of this concept to navigation of robots in networks and other areas appear in $[5,11,14]$. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [16], locating domination [10], resolving domination [1] and resolving partitions $[4,7,8,19]$.

Given a graph $G=(V, E)$ and an ordered set of vertices $S=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ of $G$, the metric representation of a vertex $v \in V$ with respect to $S$ is the vector $r(v \mid S)=\left(d\left(v, v_{1}\right), d\left(v, v_{2}\right), \ldots, d\left(v, v_{k}\right)\right)$, where $d\left(v, v_{i}\right)$ denotes the distance between the vertices $v$ and $v_{i}, 1 \leq i \leq k$. We say that $S$ is a resolving set of $G$ if different vertices of $G$ have different metric representations, i.e., for every pair of distinct vertices $u, v \in V, r(u \mid S) \neq r(v \mid S)$. The metric dimension ${ }^{1}$ of $G$ is the minimum cardinality of any resolving set of $G$, and it is denoted by $\operatorname{dim}(G)$. The metric dimension of graphs is studied in $[3,2,4-6,18]$.

Given an ordered partition $\Pi=\left\{P_{1}, P_{2}, \ldots, P_{t}\right\}$ of the vertices of $G$, the partition representation of a vertex $v \in V$ with respect to the partition $\Pi$ is the vector $r(v \mid \Pi)=\left(d\left(v, P_{1}\right), d\left(v, P_{2}\right), \ldots, d\left(v, P_{t}\right)\right)$, where $d\left(v, P_{i}\right)$, with $1 \leq i \leq t$, represents the distance between the vertex $v$ and the set $P_{i}$, i.e., $d\left(v, P_{i}\right)=\min _{u \in P_{i}}\{d(v, u)\}$. We say that $\Pi$ is a resolving partition of $G$ if different vertices of $G$ have different partition representations, i.e., for every pair of distinct vertices $u, v \in V, r(u \mid \Pi) \neq$

[^0]

Fig. 1. In this tree the vertex 3 is an exterior major vertex of terminal degree two: 1 and 4 are terminal vertices of 3 .


Fig. 2. $\Pi=\{\{1,4,9,12\},\{3,5,8,11\},\{2,6,7,10\}\}$ is a resolving partition.
$r(v \mid \Pi)$. The partition dimension of $G$ is the minimum number of sets in any resolving partition of $G$ and it is denoted by $p d(G)$. The partition dimension of graphs is studied in [4,7,8,18].

## 2. The partition dimension of trees

It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph $G$ we have

$$
\begin{equation*}
p d(G) \leq \operatorname{dim}(G)+1 \tag{1}
\end{equation*}
$$

We know that the partition dimension of any path is two. That is, for any path graph $P$, it follows $p d(P)=\operatorname{dim}(P)+1=2$. A formula for the dimension of trees that are not paths has been established in [5,9,17]. In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a tree $T$ will be called a major vertex of $T$. Any leaf $u$ of $T$ is said to be a terminal vertex of a major vertex $v$ of $T$ if $d(u, v)<d(u, w)$ for every other major vertex $w$ of $T$. The terminal degree of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ of $T$ is an exterior major vertex of $T$ if it has positive terminal degree.

Let $n_{1}(T)$ denote the number of leaves of $T$, and let ex $(T)$ denote the number of exterior major vertices of $T$. We can now state the formula for the dimension of a tree [5,9,17]: if $T$ is a tree that is not a path, then

$$
\begin{equation*}
\operatorname{dim}(T)=n_{1}(T)-\operatorname{ex}(T) \tag{2}
\end{equation*}
$$

As a consequence, if $T$ is a tree that is not a path, then

$$
\begin{equation*}
p d(T) \leq n_{1}(T)-\operatorname{ex}(T)+1 \tag{3}
\end{equation*}
$$

The above bound is tight, it is achieved for the graph in Fig. 1 where $\Pi=\{\{8\},\{4,9\},\{1,2,3,5,6,7\}\}$ is a resolving partition and $\operatorname{pd}(T)=3$. However, there are graphs for which the following bound gives better result than bound (3), for instance, the graph in Fig. 2.

Let $S=\left\{s_{1}, s_{2}, \ldots, s_{k}\right\}$ be the set of exterior major vertices of $T=(V, E)$ with terminal degree greater than one; let $\left\{s_{i 1}, s_{i 2}, \ldots, s_{i l_{i}}\right\}$ be the set of terminal vertices of $s_{i}$ and let $\tau=\max _{1 \leq i \leq k}\left\{l_{i}\right\}$. With the above notation we have the following result.

Theorem 1. For any tree $T$ which is not a path,

$$
p d(T) \leq \kappa+\tau-1
$$

Proof. For a terminal vertex $s_{i j}$ of a major vertex $s_{i} \in S$ we denote by $S_{i j}$ the set of vertices of $T$, different from $s_{i}$, belonging to the $s_{i}-s_{i j}$ path. If $l_{i}<\tau-1$, we assume $S_{i j}=\emptyset$ for every $j \in\left\{l_{i}+1, \ldots, \tau-1\right\}$. Now for every $j \in\{2, \ldots, \tau-1\}$, let
$B_{j}=\cup_{i=1}^{\kappa} S_{i j}$ and, for every $i \in\{1, \ldots, \kappa\}$, let $A_{i}=S_{i 1}$. Let us show that $\Pi=\left\{A, A_{1}, A_{2}, \ldots, A_{\kappa}, B_{2}, \ldots, B_{\tau-1}\right\}$ is a resolving partition of $T$, where $A=V-\left(\left(\cup_{i=1}^{\kappa} A_{i}\right) \cup\left(\cup_{j=2}^{\tau-1} B_{j}\right)\right)$. We consider two different vertices $x, y \in V$. Note that if $x$ and $y$ belong to different sets of $\Pi$, we have $r(x \mid \Pi) \neq r(y \mid \Pi)$.

Case 1: $x, y \in S_{i j}$. If $j=\tau$, then we have that $x, y \in A$ and it follows that $d\left(x, A_{i}\right) \neq d\left(y, A_{i}\right)$. Otherwise, we obtain that $d(x, A)=d\left(x, s_{i}\right) \neq d\left(y, s_{i}\right)=d(y, A)$.

Case 2: $x \in S_{i j}$ and $y \in S_{k l}, i \neq k$. If $j=1$ or $l=1$, then $x$ and $y$ belong to different sets of $\Pi$. So we suppose $j \neq 1$ and $l \neq 1$. Hence, if $d\left(x, A_{i}\right)=d\left(y, A_{i}\right)$, then

$$
\begin{aligned}
d\left(x, A_{k}\right) & =d\left(x, s_{i}\right)+d\left(s_{i}, s_{k}\right)+1 \\
& =d\left(x, A_{i}\right)+d\left(s_{i}, s_{k}\right) \\
& =d\left(y, A_{i}\right)+d\left(s_{i}, s_{k}\right) \\
& =d\left(y, s_{k}\right)+2 d\left(s_{k}, s_{i}\right)+1 \\
& =d\left(y, A_{k}\right)+2 d\left(s_{k}, s_{i}\right) \\
& >d\left(y, A_{k}\right) .
\end{aligned}
$$

Case 3: $x \in S_{i \tau}$ and $y \in A-\cup_{l=1}^{\kappa} S_{l \tau}$. If $d\left(x, A_{i}\right)=d\left(y, A_{i}\right)$, then $d\left(x, s_{i}\right)=d\left(y, s_{i}\right)$. Since $y \notin S_{l \tau}, l \in\{1, \ldots, \kappa\}$, there exists $A_{j} \in \Pi, j \neq i$, such that $s_{i}$ does not belong to the $y-s_{j}$ path. Now let $Y$ be the set of vertices belonging to the $y-s_{j}$ path, and let $v \in Y$ such that $d\left(s_{i}, v\right)=\min _{u \in Y}\left\{d\left(s_{i}, u\right)\right\}$. Hence,

$$
\begin{aligned}
d\left(x, A_{j}\right) & =d\left(x, s_{i}\right)+d\left(s_{i}, v\right)+d\left(v, s_{j}\right)+1 \\
& =d\left(y, s_{i}\right)+d\left(s_{i}, v\right)+d\left(v, s_{j}\right)+1 \\
& =d(y, v)+2 d\left(v, s_{i}\right)+d\left(v, s_{j}\right)+1 \\
& =d\left(y, A_{j}\right)+2 d\left(v, s_{i}\right) \\
& >d\left(y, A_{j}\right) .
\end{aligned}
$$

Case 4: $x, y \in A^{\prime}=A-\cup_{l=1}^{k} S_{l \tau}$. If for some exterior major vertex $s_{i} \in S$, the vertex $x$ belongs to the $y-s_{i}$ path or the vertex $y$ belongs to the $x-s_{i}$ path, then $d\left(x, A_{i}\right) \neq d\left(y, A_{i}\right)$. Otherwise, there exist at least two exterior major vertices $s_{i}, s_{j}$ such that the $x-y$ path and the $s_{i}-s_{j}$ path share more than one vertex (if not, then $x, y \notin A^{\prime}$ ). Let $W$ be the set of vertices belonging to the $s_{i}-s_{j}$ path. Let $u, v \in W$ such that $d(x, u)=\min _{z \in W}\{d(x, z)\}$ and $d(y, v)=\min _{z \in W}\{d(y, z)\}$. We suppose, without loss of generality, that $d\left(s_{i}, u\right)>d\left(v, s_{i}\right)$. Hence, if $d(x, v)=d(y, v)$, then $d(x, u) \neq d(y, u)$, and if $d(x, u)=d(y, u)$, then $d(x, v) \neq d(y, v)$. We have

$$
\begin{aligned}
d\left(x, A_{j}\right) & =d(x, u)+d\left(u, s_{j}\right)+1 \\
& \neq d(y, u)+d\left(u, s_{j}\right)+1 \\
& =d\left(y, A_{j}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
d\left(x, A_{i}\right) & =d(x, v)+d\left(v, s_{i}\right)+1 \\
& \neq d(y, v)+d\left(v, s_{i}\right)+1 \\
& =d\left(y, A_{i}\right)
\end{aligned}
$$

Therefore, for different vertices $x, y \in V$, we have $r(x \mid \Pi) \neq r(y \mid \Pi)$.
One example where $p d(T)=\kappa+\tau-1$ is the tree in Fig. 1.
Any vertex adjacent to a leaf of a tree $T$ is called a support vertex. In the following result $\xi$ denotes the number of support vertices of $T$ and $\theta$ denotes the maximum number of leaves adjacent to a support vertex of $T$.

Corollary 2. For any tree $T$ of order $n \geq 2, p d(T) \leq \xi+\theta-1$.
Proof. If $T$ is a path, then $\xi=2$ and $\theta=1$, so the result follows. Now we suppose $T$ is not a path. Let $v$ be an exterior major vertex of terminal degree $\tau$. Let $x$ be the number of leaves adjacent to $v$ and let $y=\tau-x$. Since $\kappa+y \leq \xi$ and $x \leq \theta$, we deduce $\kappa+\tau \leq \xi+\theta$.

The above bound is achieved, for instance, for the graph of order six composed of two support vertices $a$ and $b$, where $a$ is adjacent to $b$ and four leaves; two of them are adjacent to $a$ and the other two leaves are adjacent to $b$. One example of a graph for which Theorem 1 gives a better result than Corollary 2 is the graph in Fig. 1.

Since the number of leaves, $n_{1}(T)$, of a tree $T$ is bounded below by $\xi+\theta-1$, Corollary 2 leads to the following bound.
Remark 3. For any tree $T$ of order $n \geq 2, p d(T) \leq n_{1}(T)$.
Now we are going to characterize all the trees for which $\operatorname{pd}(T)=n_{1}(T)$. It was shown in [7] that $p d(G)=2$ if and only if the graph $G$ is a path. So by the above remark we obtain the following result.


Fig. 3. A comet graph where $3=\theta=p d(T)<n_{1}(T)$.
Remark 4. Let $T$ be a tree of order $n \geq 4$. If $n_{1}(T)=3$, then $\operatorname{pd}(T)=3$.
Theorem 5. Let $T$ be a tree with $n_{1}(T) \geq 4$. Then $p d(T)=n_{1}(T)$ if and only if $T$ is the star graph.
Proof. If $T=S_{n}$ is a star graph, it is clear that $p d(T)=n_{1}(T)$. Now, let $T=(V, E) \neq S_{n}$, such that $p d(T)=n_{1}(T) \geq 4$. Note that by (3) we have ex $(T)=1$. Let $t=n_{1}(T)$ and let $\Omega=\left\{u_{1}, u_{2}, \ldots, u_{t}\right\}$ be the set of leaves of $T$. Let $u \in V$ be the unique exterior major vertex of $T$. Let us suppose, without loss of generality, $u_{t}$ is a leaf of $T$ such that $d\left(u_{t}, u\right)=\max _{u_{i} \in \Omega}\left\{d\left(u_{i}, u\right)\right\}$.

For the leaves $u_{1}, u_{2}, u_{t} \in \Omega$ let the paths $P=u u_{t 1} u_{t 2}, \ldots, u_{t r_{t}} u_{t}, Q=u u_{11} u_{12}, \ldots, u_{1 r_{1}} u_{1}$ and $R=u u_{21} u_{22}, \ldots, u_{2 r_{2}} u_{2}$. Now, let us form the partition $\Pi=\left\{A_{1}, A_{2}, \ldots, A_{t-2}, A\right\}$, such that $A_{1}=\left\{u_{11}, u_{12}, \ldots, u_{1 r_{1}}, u_{1}, u_{t 2}, u_{t 3}, \ldots, u_{t r_{t}}, u_{t}\right\}, A_{2}=$ $\left\{u_{21}, u_{22}, \ldots, u_{2 r_{2}}, u_{2}, u_{t 1}\right\}, A_{i}=\left\{u_{i}\right\}, i \in\{3, \ldots, t-2\}$ and $A=V-\cup_{i=1}^{t-2} A_{i}$. Let us consider two different vertices $x, y \in V$. Hence, we have the following cases.

Case 1: $x, y \in A_{1}$. Let us suppose $x \in P$ and $y \in Q$. If $d\left(x, A_{2}\right)=d\left(y, A_{2}\right)$, then we have

$$
\begin{aligned}
d(x, A) & =d\left(x, u_{t 1}\right)+1 \\
& =d\left(x, A_{2}\right)+1 \\
& =d\left(y, A_{2}\right)+1 \\
& =d(y, A)+2 \\
& >d(y, A) .
\end{aligned}
$$

Now, if $x, y \in P$ or $x, y \in Q$, then $d(x, A) \neq d(y, A)$.
Case 2: $x, y \in A_{2}$. If $x=u_{t 1}$ or $y=u_{t 1}$, then let us suppose for instance, $x=u_{t 1}$, so we have $d\left(x, A_{1}\right)=1<2 \leq d\left(y, A_{1}\right)$. On the contrary, if $x, y \in R$, then $d(x, A) \neq d(y, A)$.

Case 3: $x, y \in A$. If $d\left(x, A_{1}\right)=d\left(y, A_{1}\right)$, then $t \geq 5$ and there exists a leaf $u_{i}, i \neq 1,2, t-1, t$, such that $d\left(x, A_{i}\right)=$ $d\left(x, u_{i}\right) \neq d\left(y, u_{i}\right)=d\left(y, A_{i}\right)$.

Therefore, for different vertices $x, y \in V$ we have $r(x \mid \Pi) \neq r(y \mid \Pi)$ and $\Pi$ is a resolving partition in $T$, a contradiction.
Let $T$ be the comet graph shown in Fig. 3. A resolving partition for $T$ is $\Pi=\left\{A_{1}, A_{2}, A_{3}\right\}$, where $A_{1}=\{x, t\}, A_{2}=\{y, z\}$ and $A_{3}=\{u, w\}$. In this case, $\theta=p d(T)=3<4=n_{1}(T)$.

Remark 6. For any tree $T$ of order $n \geq 2, p d(T) \geq \theta$.
Proof. Since different leaves adjacent to the same support vertex must belong to different sets of a resolving partition, the result follows.

Other examples where $p d(T)=\theta$ are the star graphs and the graph in Fig. 2.
Theorem 7. Let $T$ be a tree which is not a path. If every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then

$$
p d(T) \leq \max \{\kappa, \tau+1\}
$$

Proof. We suppose $T=(V, E)$ is not a path. Let $S=\left\{s_{1}, s_{2}, \ldots, s_{\kappa}\right\}$ be the set of exterior major vertices of $T$ with terminal degree greater than one and let $B_{i}=\left\{s_{i}\right\}, i=1, \ldots, \kappa$. If $\kappa<\tau+1$, then for $i \in\{\kappa+1, \ldots, \tau+1\}$ we assume $B_{i}=\emptyset$. Let $l_{i}$ be the terminal degree of $s_{i}, i \in\{1, \ldots, \kappa\}$. If $l_{i}<i$, then we denote by $\left\{s_{i 1}, \ldots, s_{i l_{i}}\right\}$ the set of terminal vertices of $s_{i}$. On the contrary, if $l_{i} \geq i$, then the set of terminal vertices of $s_{i}$ is denoted by $\left\{s_{i 1}, \ldots, s_{i i-1}, s_{i i+1}, \ldots, s_{i l_{i}+1}\right\}$. Also, for a terminal vertex $s_{i j}$ of a major vertex $s_{i}$ we denote by $S_{i j}$ the set of vertices of $T$, different from $s_{i}$, belonging to the $s_{i}-s_{i j}$ path. Moreover, we assume $S_{i j}=\emptyset$ for the following three cases: (1) $i=j$, (2) $i \leq l_{i}<\tau$ and $j \in\left\{l_{i}+2, \ldots, \tau+1\right\}$, and (3) $i>l_{i}$ and $j \in\left\{l_{i}+1, \ldots, \tau+1\right\}$. Now, let $t=\max \{\kappa, \tau+1\}$ and let $\Pi=\left\{A_{1}, A_{2}, \ldots, A_{t}\right\}$ be composed of the sets $A_{i}=B_{i} \cup\left(\cup_{j=1}^{\kappa} S_{j i}\right)$, $i=1, \ldots, t$. Since every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then $\Pi$ is a partition of $V$.

Let us show that $\Pi$ is a resolving partition. Let $x, y \in V$ be different vertices of $T$. If $x, y \in A_{i}$, we have the following three cases.


Fig. 4. $\Pi=\{\{1,8,11,14\},\{2,5,12,15\},\{3,6,9,16\},\{4,7,10,13\}\}$ is a resolving partition.
Case 1: $x, y \in S_{j i}$. In this case $d\left(x, A_{j}\right)=d\left(x, s_{j}\right) \neq d\left(y, s_{j}\right)=d\left(y, A_{j}\right)$.
Case 2: $x \in S_{j i}$ and $y \in S_{k i}, j \neq k$. If $d\left(x, A_{k}\right)=d\left(y, A_{k}\right)$ we have $d\left(y, A_{j}\right)>d\left(y, s_{k}\right)=d\left(y, A_{k}\right)=d\left(x, A_{k}\right)>d\left(x, s_{j}\right)=$ $d\left(x, A_{j}\right)$.
Case 3: $x=s_{i}$ and $y \in S_{j i}$. As $s_{i}$ has at least two terminal vertices, there exists a terminal vertex $s_{i l}$ of $s_{i}, l \neq j$, such that $d\left(x, A_{l}\right)=d\left(x, S_{i l}\right)=1$. Hence, $d\left(y, A_{l}\right)>d\left(y, s_{j}\right) \geq 1=d\left(x, A_{l}\right)$. Therefore, for different vertices $x, y \in V$, we have $r(x \mid \Pi) \neq r(y \mid \Pi)$.

The above bound is achieved, for instance, for the graph in Fig. 4.

## 3. On the partition dimension of generalized trees

A cut vertex in a graph is a vertex whose removal increases the number of components of the graph and an extreme vertex is a vertex such that its closed neighborhood forms a complete graph. Also, a block is a maximal biconnected subgraph of the graph. Now, let $\mathfrak{F}$ be the family of sequences of connected graphs $G_{1}, G_{2}, \ldots, G_{k}, k \geq 2$, such that $G_{1}$ is a complete graph $K_{n_{1}}, n_{1} \geq 2$, and $G_{i}, i \geq 2$, is obtained recursively from $G_{i-1}$ by adding a complete graph $K_{n_{i}}, n_{i} \geq 2$, and identifying a vertex of $G_{i-1}$ with a vertex in $K_{n_{i}}$.

From this point we will say that a connected graph $G$ is a generalized tree if and only if there exists a sequence $\left\{G_{1}, G_{2}\right.$, $\left.\ldots, G_{k}\right\} \in \mathfrak{F}$ such that $G_{k}=G$ for some $k \geq 2$. Notice that in these generalized trees every vertex is either a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every $G_{i}$ is isomorphic to $K_{2}$, then $G_{k}$ is a tree, thus justifying the terminology used. In this section we will be centered in the study of partition dimension of generalized trees.

Let $G=(V, E)$ be a generalized tree and let $R_{1}, R_{2}, \ldots, R_{k}$ be the blocks of $G$. A cut vertex $v \in V$ is a support cut vertex if there is at least one block $R_{i}$ of $G$, in which $v$ is the unique cut vertex belonging to the block $R_{i}$. An extreme vertex is an exterior extreme vertex if it is adjacent to only one cut vertex. Let $S=\left\{s_{1}, s_{2}, \ldots, s_{\zeta}\right\}$ be the set of support cut vertices of $G$ and let $\left\{s_{i 1}, s_{i 2}, \ldots, s_{i l_{i}}\right\}$ be the set of exterior extreme vertices adjacent to $s_{i} \in S$. Also, let $Q=\left\{Q_{1}, Q_{2}, \ldots, Q_{\vartheta}\right\}$ be the set of blocks of $G$ which contain more than one cut vertex and more than one extreme vertex and let $\left\{q_{i 1}, q_{i 2}, \ldots, q_{i t_{i}}\right\}$ be the set of extreme vertices belonging to $Q_{i} \in Q$. Now, let $\phi=\max _{1 \leq i \leq \zeta, 1 \leq j \leq \vartheta}\left\{l_{i}, t_{j}\right\}$. With the above notation we have the following result.

## Theorem 8. For any generalized tree $G$,

$$
p d(G) \leq \begin{cases}\zeta+\vartheta+\phi-1, & \text { if } \phi \geq 3 \\ \zeta+\vartheta+1, & \text { if } \phi \leq 2\end{cases}
$$

Proof. For each support cut vertex $s_{i} \in S$, let $A_{i}=\left\{s_{i 1}\right\}$ and for each block $Q_{j} \in Q$, let $B_{j}=\left\{q_{j 1}\right\}$. Let us suppose $\phi \geq 3$. For every $j \in\left\{2, \ldots, l_{i}\right\}$ we take $M_{i j}=\left\{s_{i j}\right\}$ and, if $l_{i}<\phi-1$, then for every $j \in\left\{l_{i+1}, \ldots, \phi-1\right\}$ we consider $M_{i j}=\emptyset$. Analogously, for every $j \in\left\{2, \ldots, t_{i}\right\}$ we take $N_{i j}=\left\{q_{i j}\right\}$ and, if $t_{i}<\phi-1$, then for every $j \in\left\{t_{i+1}, \ldots, \phi-1\right\}$ we consider $N_{i j}=\emptyset$. Now, let $C_{j}=\bigcup_{i=1}^{\max \{\zeta, v\}}\left(M_{i j} \cup N_{i j}\right)$, with $j \in\{2, \ldots, \phi-1\}$.

Let us prove that $\Pi=\left\{A, A_{1}, A_{2}, \ldots, A_{\zeta}, B_{1}, B_{2}, \ldots, B_{\vartheta}, C_{2}, C_{3}, \ldots, C_{\phi-1}\right\}$ is a resolving partition of $G$, where $A=V-$ $\cup_{i=1}^{\zeta} A_{i}-\cup_{i=1}^{\vartheta} B_{i}-\cup_{i=2}^{\phi-1} C_{i}$. To begin with, let $x, y$ be two different vertices of $G$. We have the following cases.

Case 1: $x$ is a cut vertex or $y$ is a cut vertex. Let us suppose, for instance, $x$ is a cut vertex. So there exists an extreme vertex $s_{i 1}$ such that $x$ belongs to a shortest $y-s_{i 1}$ path or $y$ belongs to a shortest $x-s_{i 1}$ path. Hence, we have $d\left(x, A_{i}\right)=d\left(x, s_{i 1}\right) \neq$ $d\left(y, s_{i 1}\right)=d\left(y, A_{i}\right)$.

Case 2: $x, y$ are extreme vertices. If $x, y$ belong to the same block of $G$, then $x, y$ belong to different sets of $\Pi$. On the contrary, if $x, y$ belong to different blocks in $G$, then let us suppose that there exists an extreme vertex $c$ such that $d(x, c) \leq 1$ or $d(y, c) \leq 1$. We can suppose $c \in A_{i}$, for some $i \in\{1, \ldots, \zeta\}$, or $c \in B_{j}$, for some $j \in\{1, \ldots, \vartheta\}$. Without the loss of generality, we suppose that $d(x, c) \leq 1$. Since $x$ and $y$ belong to different blocks of $G$, we have $d(y, c)>1$. So we obtain either $d\left(x, A_{i}\right)=d(x, c) \leq 1<d(y, c)=d\left(y, A_{i}\right)$ or $d\left(x, B_{j}\right)=d(x, c) \leq 1<d(y, c)=d\left(y, B_{j}\right)$.


Fig. 5. $\Pi=\{\{4\},\{7\},\{10\},\{5,8,11\},\{1,2,3,6,9,12\}\}$ is a resolving partition for the generalized tree.
Now, if there exists no such a vertex $c$, then there exist two blocks $H, K \notin Q$ with $x \in H$ and $y \in K$, which contain more than one cut vertex and only one extreme vertex. So $x, y \in A$. Let $u \in H$ be a cut vertex such that $d(y, u)=\max _{v \in H} d(y, v)$. Hence, there exists an extreme vertex $s_{i 1}$ such that $u$ belongs to a shortest $x-s_{i 1}$ path and $d\left(y, s_{i 1}\right)=d(y, u)+d\left(u, s_{i 1}\right)$. As $x, y$ belong to different blocks and $d(y, u)=\max _{v \in H} d(y, v)$ we have $d(y, u) \geq 2$. Thus,

$$
\begin{aligned}
d\left(y, A_{i}\right) & =d\left(y, s_{i 1}\right) \\
& =d(y, u)+d\left(u, s_{i 1}\right) \\
& \geq 2+d\left(u, s_{i 1}\right) \\
& >1+d\left(u, s_{i 1}\right) \\
& =d(x, u)+d\left(u, s_{i 1}\right) \\
& =d\left(x, A_{i}\right) .
\end{aligned}
$$

Hence, we conclude that if $\phi \geq 3$, then for every $x, y \in V, r(x \mid \Pi) \neq r(y \mid \Pi)$. Therefore, $\Pi$ is a resolving partition.
On the other hand, if $\phi \leq 2$, then $\Pi^{\prime}=\left\{A, A_{1}, A_{2}, \ldots, A_{\zeta}, B_{1}, B_{2}, \ldots, B_{\vartheta}\right\}$ is a partition of $V$. Proceeding as above we obtain that $\Pi^{\prime}$ is a resolving partition.

The above bound is achieved, for instance, for the graph in Fig. 5, where $\zeta=3, \vartheta=0$ and $\phi=3$. Also, notice that for the particular case of trees we have $\zeta=\xi, \phi=\theta$ and $\vartheta=0$. So the above result leads to Corollary 2.

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    1 Also called the locating number.

