Postprint of: Dettlaff M., Lemańska M., Rodríguez-Velázquez J., Zuazua R., On the super domination number of lexicographic product graphs, DISCRETE APPLIED MATHEMATICS, Vol. 263 (2019), pp. 118-129, DOI: 10.1016/j.dam.2018.03.082

On the super domination number of lexicographic product graphs

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Abstract

The neighbourhood of a vertex v of a graph G is the set $N(v)$ of all vertices adjacent to v in G. For $D \subseteq V(G)$ we define $\overline{D} = V(G) \setminus D$. A set $D \subseteq V(G)$ is called a super dominating set if for every vertex $u \in \overline{D}$, there exists $v \in D$ such that $N(v) \cap \overline{D} = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating sets in G. In this article we obtain closed formulas and tight bounds for the super dominating number of lexicographic product graphs in terms of invariants of the factor graphs involved in the product. As a consequence of the study, we show that the problem of finding the super domination number of a graph is NP-Hard.

Keywords: Domination number; super domination number; domination in graphs; lexicographic product; NP-Hard.
2010 MSC: 05C69, 05C76

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1. Introduction

The *neighbourhood* of a vertex v of a graph G is the set $N(v)$ of all vertices adjacent to v in G. For $D \subseteq V(G)$ we define $\overline{D} = V(G) \setminus D$. A set

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$D \subseteq V(G)$ is *dominating* in G if every vertex in $\overline{D}$ has at least one neighbour in D, *i.e.*, $N(u) \cap D \neq \emptyset$ for every $u \in \overline{D}$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The reader is referred to the books [9, 10] for details on domination in graphs.

The study of super domination in graphs was introduced in [12]. A set $D \subseteq V(G)$ is called a *super dominating set* if for every vertex $u \in \overline{D}$, there exists $v \in D$ such that

$$N(v) \cap \overline{D} = \{u\}. \tag{1}$$

If u and v satisfy (1), then we say that v is an *external private neighbour of u with respect to $\overline{D}$*. The *super domination number* of G, denoted by $\gamma_{\rm sp}(G)$, is the minimum cardinality among all super dominating sets in G. A super dominating set of cardinality $\gamma_{\rm sp}(G)$ is called a $\gamma_{\rm sp}(G)$-set.

In this paper we develop the theory of super domination in lexicographic product graphs. The paper is structured as follows. Section 2 covers basic results on the super domination number of a graph, including a characterization of graphs of order n with $\gamma_{\rm sp}(G) = n-1$. These graphs play an important role **to study** the super domination number of lexicographic product graphs. Section 3 is devoted to the study of the super domination number of lexicographic product graphs. In particular, in Subsection 3.1 we obtain general bounds for the super domination number of lexicographic product graphs in terms of some invariants of the factor graphs involved in the product. In Subsection 3.2 we show that the problem of finding the super domination number of a graph is NP-Hard. We also study several families of graphs for which the bounds obtained previously are achieved. Finally, in Subsection 3.3 we obtain formulas for the super domination number of join graphs.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2. Some remarks on the super domination number

In this section we recall basic properties of the super domination number and give the full characterisation of graphs of order n with $\gamma_{\rm sp}(G) = n - 1$. To begin with, we introduce some notation and terminology. The *closed neighbourhood* of a vertex v is defined as $N[v] = N(v) \cup \{v\}$ and the degree of v is $d(v) = |N(v)|$. If G has n vertices and $d(v) = n - 1$, then v is a *universal* vertex of G.

Theorem 1. [12] *Let G be a graph of order n. Then the following assertions hold.*

- $\gamma_{\rm sp}(G) = 1$ *if and only if* $G \cong K_1$ *or* $G \cong K_2$.
- $\gamma_{\rm sp}(G) = n$ *if and only if G is an empty graph.*
- $\gamma_{\rm sp}(G) \geq \lceil \frac{n}{2} \rceil$.

It is well known that for any graph G without isolated vertices, $1 \leq \gamma(G) \leq \lceil \frac{n}{2} \rceil$, so from the theorem above we have that for any connected graph G,

$$1 \leq \gamma(G) \leq \left\lceil \frac{n}{2} \right\rceil \leq \gamma_{\rm sp}(G) \leq n-1. \tag{2}$$

Graphs with $\gamma_{\rm sp}(G) = n-1$ will play an important role in the study of the super domination number of lexicographic product graphs. In order to characterize these graphs we need to prove the following two lemmas.

Lemma 2. *Let G be a graph of order n. If $\gamma_{\rm sp}(G) = n-1$, then G is P_k-free and C_k-free, for any $k \geq 4$.*

Proof. Suppose that there exists $V' = \{x, y, w, z\} \subseteq V(G)$ such that the subgraph of G induced by V' is isomorphic to a path $P_4 = (x, y, w, z)$ or a cycle $C_4 = (x, y, w, z, x)$. Then $V(G) \setminus \{x, z\}$ is a super dominating set of G, which implies that $\gamma_{\rm sp}(G) \leq n-2$. Therefore, if $\gamma_{\rm sp}(G) = n-1$, then G is P_k-free and C_k-free, for any $k \geq 4$. □

Lemma 3. *Let G be a connected graph of order n. If there is no universal vertex in G, then $\gamma_{\rm sp}(G) \leq n-2$.*

Proof. Suppose that $\gamma_{\rm sp}(G) = n-1$ and G does not have a universal vertex. Let x be a vertex of maximum degree in G. Since $d(x) < n-1$, there exists z such that the distance between x and z is equal to two, and let denote by y a common neighbour of x and z. Suppose now that there exists $w \in N(x)$ such that $yw \notin E(G)$. In such a case, the subgraph induced by the set $\{w, x, y, z\}$ is isomorphic to P_4 or C_4, which is a contradiction with Lemma 2. Hence, $N(x) \subseteq N[y]$. Furthermore, $z \in N(y) \setminus N(x)$, which implies $d(y) \geq d(x) + 1$, which is a contradiction. □

To describe graphs with $\gamma_{\rm sp}(G) = n-1$, we define a family $\mathcal{F}$ of graphs in the following way.

- Let k and k' be two positive integers such that $k' = k$ or $k' = k - 1$.
- Let $\{G_i = (V_i, E_i) : \ i = 1, \ldots, k\}$ be a family of complete graphs.
- Let $\{G'_i = (V'_i, E'_i) : \ i = 1, \ldots, k'\}$ be a family of empty graphs.
- Let $X_1 = \bigcup_{i=1}^{k} E_i$ and $X_2 = \{xy \colon x \in V_i, y \in V'_j \text{ and } 1 \le i \le j \le k'\}$.
- For $k \ge 2$ we define $X_3 = \{xy \colon x \in V_i, y \in V_j \text{ and } 1 \le i < j \le k\}$, while for $k = 1$ we assume that $X_3 = \emptyset$.
- With the notation above, we say that $G \in \mathcal{F}$ if $V(G) = \left(\bigcup_{i=1}^{k} V_i\right) \cup \left(\bigcup_{i=1}^{k'} V'_i\right)$ and $E(G) = X_1 \cup X_2 \cup X_3$ for some integers k and k'.

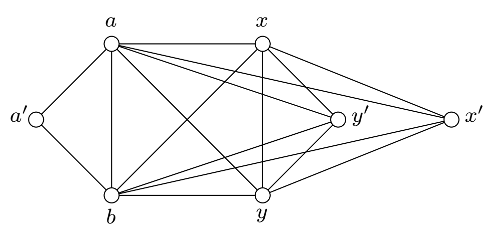

Figure 1: A graph belonging to the family $\mathcal{F}$, where $k = k' = 2$, $V_1 = \{a, b\}$, $V'_1 = \{a'\}$, $V_2 = \{x, y\}$ and $V'_2 = \{x', y'\}$.

Figure 1 shows an example of a graph belonging to the family $\mathcal{F}$. The following remark is a direct consequence of the definition of $\mathcal{F}$.

Remark 4. *Let $G \in \mathcal{F}$. Then the following assertions hold for $x, y \in V(G)$.*

- *If $x \in V_i$, then $N[x] = \left(\bigcup_{j=1}^{k} V_j\right) \cup \left(\bigcup_{j=i}^{k'} V'_j\right)$.*
- *If $y \in V'_j$, then $N(y) = \bigcup_{i=1}^{j} V_i$.*
- *If $x, y \in V_j$, then $N[x] = N[y]$.*
- *If $x, y \in V'_j$, then $N(x) = N(y)$.*
- *If $i < j$, $x \in V_i$ and $y \in V_j$, then $N[y] \subseteq N[x]$.*
- *If $i < j$, $x \in V'_i$ and $y \in V'_j$, then $N(x) \subseteq N(y)$.*

- *If $x \in V_i$ and $y \in V'_j$, then $N(y) \subseteq N(x)$*

Theorem 5. *Let G be a connected graph of order n. Then $\gamma_{\rm sp}(G) = n-1$ if and only if $G \in \mathcal{F}$.*

Proof. From Remark 4 we deduce that if $G \in \mathcal{F}$, then $\gamma_{\rm sp}(G) = n-1$.

From now on we assume that $\gamma_{\rm sp}(G) = n-1$. Thus, by Lemma 3 we can claim that G has at least one universal vertex. Let $\tilde{V}_1$ be the set of universal vertices of G and $\tilde{V}'_1 = \{x \in V(G) \colon N(x) = \tilde{V}_1\}$. If $V(G) = \tilde{V}_1 \cup \tilde{V}'_1$, then $G \in \mathcal{F}$, as the subgraph of G induced by $\tilde{V}_1$ is complete and the subgraph induced by $\tilde{V}'_1$ is empty. Now, if $V(G) \setminus (\tilde{V}_1 \cup \tilde{V}'_1) \neq \emptyset$, then we denoted by H_2 the subgraph of G induced by $V(G) \setminus (\tilde{V}_1 \cup \tilde{V}'_1)$. If the subgraph H_2 has no universal vertex, then $|V(H_2)| \geq 3$ and by Lemma 3 there exists a super dominating set D of H_2 such that $|D| \leq |V(H_2)|-2$, which is a contradiction. So H_2 has at least one universal vertex. Let $\tilde{V}_2$ be the set of universal vertices of H_2 and $\tilde{V}'_2 = \{x \in V(H_2) \colon N(x) = \tilde{V}_2\}$. If $V(H_2) = \tilde{V}_2 \cup \tilde{V}'_2$, then $H_2 \in \mathcal{F}$, which also implies that $G \in \mathcal{F}$[1]. Analogously, if $V(H_2) \setminus (\tilde{V}_2 \cup \tilde{V}'_2) \neq \emptyset$, then we denote by H_3 the subgraph of H_2 induced by $V(H_2) \setminus (\tilde{V}_2 \cup \tilde{V}'_2)$. Next we repeat this process for H_3 to conclude that $H_3 \in \mathcal{F}$ and, since G is a finite graph, we continue the process until $V(H_k) = \tilde{V}_k \cup \tilde{V}'_k$ for some k, where $\tilde{V}'_k$ may be empty, to conclude that $G \in \mathcal{F}$. □

3. Super domination in lexicographic product of graphs

Let G be a graph of order n such that $V(G) = \{u_1, \ldots, u_n\}$ and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be an ordered family formed by n graphs such that H_i corresponds to u_i for every i. The *lexicographic product* of G and $\mathcal{H}$ is the graph $G \circ \mathcal{H}$, such that $V(G \circ \mathcal{H}) = \bigcup_{u_i \in V(G)} (\{u_i\} \times V(H_i))$ and $(u_i, v_r)(u_j, v_s) \in E(G \circ \mathcal{H})$ if and only if $u_i u_j \in E(G)$ or $i = j$ and $v_r v_s \in E(H_i)$. In general, we can construct the graph $G \circ \mathcal{H}$ by taking one copy of each $H_i \in \mathcal{H}$ and joining by an edge every vertex of H_i with every vertex of H_j for every $u_i u_j \in E(G)$. Figure 2 shows the lexicographic product of $P_3 = (u_1, u_2, u_3)$ and the ordered family of graphs $\{P_4, K_2, P_3\}$, and the lexicographic product of $P_4 = (u_1, u_2, u_3, u_4)$ and the family $\{H_1, H_2, H_3, H_4\}$, where $H_1 \cong H_4 \cong K_1$ and $H_2 \cong H_3 \cong K_2$.

[1] Notice that if $V(H_2) = \tilde{V}_2 \cup \tilde{V}'_2$, then $k = 2$ and if $\tilde{V}'_2 = \emptyset$, then $k' = 1$, otherwise $k' = 2$.

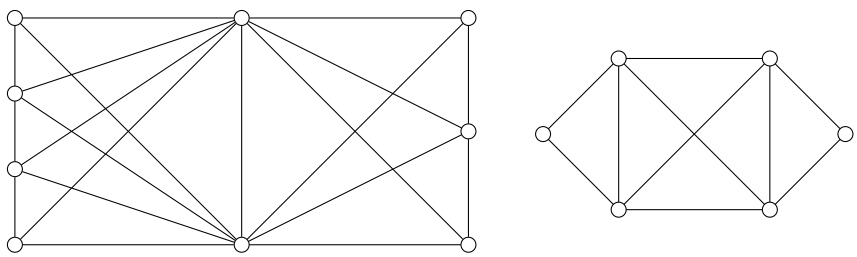

Figure 2: The lexicographic product graphs $P_3 \circ \{P_4, K_2, P_3\}$ and $P_4 \circ \{H_1, H_2, H_3, H_4\}$, where $H_1 \cong H_4 \cong K_1$ and $H_2 \cong H_3 \cong K_2$.

We will restrict our study to two particular cases. First, the traditional lexicographic product graph, where $H_i \cong H$ for every $i \in \{1, \ldots, n\}$, which is denoted as $G \circ H$ for simplicity [7, 11]. The other particular case we will focus on is the join of G and H. The *join graph* $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H [8, 17]. Note that $G + H \cong K_2 \circ \{G, H\}$. The join operation is commutative and associative.

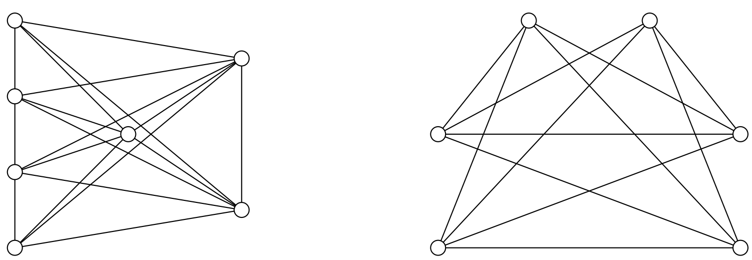

Figure 3: Two join graphs: $P_4 + C_3 \cong K_2 \circ \{P_4, C_3\}$ and $N_2 + N_2 + N_2 \cong K_3 \circ N_2$.

Moreover, complete k-partite graphs,

$$K_{p_1, p_2, \ldots, p_k} \cong K_n \circ \{N_{p_1}, N_{p_2}, \ldots, N_{p_k}\} \cong N_{p_1} + N_{p_2} + \cdots + N_{p_k},$$

are typical examples of join graphs, where N_{p_i} denotes the empty graph of order p_i. The particular case illustrated in Figure 3 (right hand side), is no other than the complete 3-partite graph $K_{2,2,2}$.

Notice that for any $g \in V(G)$ and any graph H, the subgraph of $G \circ H$ induced by $\{g\} \times V(H)$ is isomorphic to H.

Remark 6. *Let G and H be two graphs. Then the following assertions hold.*

- *$G \circ H$ is connected if and only if G is connected.*

- *If $G = G_1 \cup \ldots \cup G_t$, then $G \circ H = (G_1 \circ H) \cup \ldots \cup (G_t \circ H)$.*

According to the remark above, we can restrict ourselves to the case of lexicographic product graphs $G \circ H$ for which G is connected. For basic properties of the lexicographic product of two graphs we suggest the handbook by Hammack, Imrich and Klavžar [7].

A main problem in the study of product of graphs consists in finding exact values or sharp bounds for specific parameters of the product of two graphs and expressing these in terms of invariants of the factor graphs. In particular, we cite the following works on domination theory of lexicographic product graphs. For instance, the domination number was studied in [13, 14], the Roman domination number was studied in [15], the rainbow domination number was studied in [16], while the doubly connected domination number was studied in [1].

To begin our study we need to introduce the following additional notation. Given $g \in V(G)$ and $W \subseteq V(G) \times V(H)$ we define

$$W_g = \{h \in V(H) : (g,h) \in W\}.$$

For simplicity, the neighbourhood of $(g,h) \in V(G \circ H)$ will be denoted by $N(g,h)$ instead of $N((g,h))$.

Lemma 7. *Let G be a graph and let H be a nonempty graph. If W is a $\gamma_{\rm sp}(G \circ H)$-set, then $|W_g| \geq \gamma_{\rm sp}(H)$ for every $g \in V(G)$.*

Proof. Since H is a nonempty graph, if $|W_g| \geq |V(H)| - 1$, then we are done. Assume that $|W_g| \leq |V(H)| - 2$ and let $h, h' \in \overline{W}_g$. Since

$$N(g,h) \cap [(V(G) \setminus \{g\}) \times V(H)] = N(g,h') \cap [(V(G) \setminus \{g\}) \times V(H)],$$

we can conclude that (g,h) (and also (g,h')) has a private neighbour with respect to $\overline{W}$ which belongs to $(\{g\} \times V(H)) \cap W$. Hence, h (and also h') has a private neighbour with respect to $\overline{W}_g$ which belongs to W_g. Therefore, W_g is a super dominating set for H, which implies that $|W_g| \geq \gamma_{\rm sp}(H)$. □

Lemma 8. *Let G and H be two graphs. Let $xx' \in E(G)$ and W a $\gamma_{\rm sp}(G \circ H)$-set. If $\overline{W}_x \neq \emptyset$ and $\overline{W}_{x'} \neq \emptyset$, then $|\overline{W}_x| = |\overline{W}_{x'}| = 1$.*

Proof. Suppose that $(x,y), (x',y'), (x',y'') \in \overline{W}$. Since $xx' \in E(G)$, we have that $N(x',y'') \subseteq N(x',y') \cup N(x,y)$, which is a contradiction. Hence, $|\overline{W}_{x'}| \leq 1$. Therefore, the results follows. □

3.1. General bounds

Recall that an *independent set* of a graph G is a subset $S \subseteq V(G)$ such that no two vertices in S represent an edge of G, *i.e.*, $N(x) \cap S = \emptyset$, for every $x \in S$. The cardinality of a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. We refer to an $\alpha(G)$-set in a graph G as an independent set of cardinality $\alpha(G)$.

Fink, Jacobson [4] defined *k-independent set* of a graph G as a set $S \subseteq V(G)$ such that the subgraph induced by S has maximum degree at most $k-1$, *i.e.*, $|N(x) \cap S| \leq k-1$, for every $x \in S$. The cardinality of a maximum k-independent set of G is called the *k-independence number* of G and is denoted by $\alpha_k(G)$. Obviously any 1-independent set of G is an independent set of G.

Theorem 9. *For any nonempty graph H of order n' and for any graph G of order n,*

$$\gamma_{\mathrm{sp}}(G \circ H) \leq \alpha(G)\gamma_{\mathrm{sp}}(H) + (n - \alpha(G))n'.$$

In particular, if $\gamma_{\mathrm{sp}}(H) = n' - 1$ and $H \not\cong K_{n'}$, then

$$\gamma_{\mathrm{sp}}(G \circ H) \leq nn' - \alpha_2(G).$$

Proof. Let S_1 be an $\alpha(G)$-set and S_2 a $\gamma_{\mathrm{sp}}(H)$-set. We claim that

$$S = \left(\bigcup_{x \in S_1} \{x\} \times S_2 \right) \cup \left(\bigcup_{x \notin S_1} \{x\} \times V(H) \right) \tag{3}$$

is a super dominating set of $G \circ H$. To see this we set $(x, y) \notin S$. Hence, $x \in S_1$ and $y \in \overline{S_2}$, so that there exists $y' \in S_2$ such that $N(y') \cap \overline{S_2} = \{y\}$, which implies that $(x, y') \in S$ and $N(x, y') \cap \overline{S} = \{(x, y)\}$. Thus, S is a super dominating set of $G \circ H$ and, as a consequence,

$$\gamma_{\mathrm{sp}}(G \circ H) \leq |S| = |S_1||S_2| + (n - |S_1|)n' = \alpha(G)\gamma_{\mathrm{sp}}(H) + (n - \alpha(G))n'.$$

Now, let $\gamma_{\mathrm{sp}}(H) = n' - 1$ and $H \not\cong K_{n'}$. In this case we take the sets S_1 and S_2 in a different manner, *i.e.*, we take S_1 as an $\alpha_2(G)$-set and $S_2 = V(H) \setminus \{y\}$, where y is a nonuniversal vertex of H. We claim that the set S defined by (3) is a super dominating set of $G \circ H$. To see this we take $(x, y) \notin S$. Since $x \in S_1$, we have two possibilities, namely, (a) $N(x) \cap S_1 = \emptyset$ or (b) there exists $x' \in V(G) \setminus \{x\}$ such that $N(x) \cap S_1 = \{x'\}$. In case (a), for every $y' \in S_2$ we have that $(x, y') \in S$ and $N(x, y') \cap \overline{S} = \{(x, y)\}$, while in case (b), for every $y' \in V(H) \setminus N[y]$ we have that $(x', y') \in S$ and $N(x', y') \cap \overline{S} = \{(x, y)\}$. Thus, S is a super dominating set of $G \circ H$ and, as a consequence,

$$\gamma_{\rm sp}(G \circ H) \leq |S| = |S_1|(n' - 1) + (n - |S_1|)n' = nn' - \alpha_2(G).$$

Therefore, the result follows. $\square$

As we will show in Theorem 14 and Propositions 16, 17, 19 and 21, the bounds above are achieved by several families of graphs.

Notice that the bound $\gamma_{\rm sp}(G \circ H) \leq \alpha(G)\gamma_{\rm sp}(H) + (n - \alpha(G))n'$ is never better than $\gamma_{\rm sp}(G \circ H) \leq nn' - \alpha_2(G)$, as $\alpha(G) \leq \alpha_2(G)$.

A *vertex cover* of G is a set $X \subseteq V(G)$ such that each edge of G is incident to at least one vertex of X. The *vertex cover number* $\tau(G)$ is the cardinality of a minimum vertex cover of G. A vertex cover of cardinality $\tau(G)$ is called a $\tau(G)$-set. The following well-known result, due to Gallai [5], states the relationship between the independence number and the vertex cover number of a graph.

Theorem 10. [5](Gallai, 1959) *For any graph G of order n, $\alpha(G) + \tau(G) = n$.*

By Theorems 9 and 10 we deduce that

$$\gamma_{\rm sp}(G \circ H) \leq \alpha(G)\gamma_{\rm sp}(H) + \tau(G)n'.$$

Theorem 11. *If G is a graph of order $n \geq 2$ and H is a nonempty graph of order n', then*

$$\gamma_{\rm sp}(G \circ H) \geq n\gamma_{\rm sp}(H).$$

In particular, $\gamma_{\rm sp}(G \circ H) = n\gamma_{\rm sp}(H)$ if and only if $G \cong K_2$, $\gamma_{\rm sp}(H) = n' - 1$ and $H \not\cong K_{n'}$.

Proof. The lower bound is a direct consequence of Lemma 7. Assume that $G \cong K_2$, $\gamma_{\rm sp}(H) = n' - 1$ and $H \not\cong K_{n'}$. By the lower bound we have $\gamma_{\rm sp}(G \circ H) \geq 2(n' - 1)$. Let h be a nonuniversal vertex of H and $V(G) = \{a, b\}$. To show that $\gamma_{\rm sp}(G \circ H) \leq 2(n' - 1)$ we only need to observe that $V(G) \times V(H) \setminus \{(a, h), (b, h)\}$ is a super dominating set for $G \circ H$, *i.e.*, if h' is not adjacent to h in H, then $(a, h') \in N(b, h) \setminus N(a, h)$ and $(b, h') \in N(a, h) \setminus N(b, h)$.

From now on we assume that $\gamma_{\rm sp}(G \circ H) = n\gamma_{\rm sp}(H)$. Let W be a $\gamma_{\rm sp}(G \circ H)$-set. Since $\gamma_{\rm sp}(G \circ H) = n\gamma_{\rm sp}(H)$, from Lemma 7 we deduce that for any $g \in V(G)$, $|W_g| = \gamma_{\rm sp}(H) \leq n' - 1$. Suppose that $G \not\cong K_2$. Let $x \in V(G)$ be a vertex of degree at least two and let $(u,v) \in W$ be a private neighbour of $(x,y) \notin W$ with respect to $\overline{W}$, *i.e.* $N(u,v) \cap \overline{W} = \{(x,y)\}$. Let $x' \in V(G) \setminus \{u\}$ be a neighbour of x and $(x',y') \notin W$. If $u = x$, then $(x',y') \in N(u,v)$, and $\{(x,y),(x',y')\} \subseteq N(u,v) \cap \overline{W}$, which is a contradiction, so that $u \neq x$. Thus, if u has degree one, then for $(u,z) \notin W$ we have $N(u,z) \subset N(x,y) \cup N(x',y')$, which is a contradiction. Otherwise there exists $u' \in N(u) \setminus \{x\}$ and for $(u',z') \notin W$ we have $(u',z') \in N(u,v)$. Thus, $\{(x,y),(u',z')\} \subseteq N(u,v) \cap \overline{W}$, which is a contradiction again. Hence, we can conclude that $G \cong K_2$.

Notice that $H \not\cong K_{n'}$, as $K_2 \circ K_{n'} \cong K_{2n'}$ and $\gamma_{\rm sp}(K_{2n'}) = 2n' - 1 > 2(n'-1) = 2\gamma_{\rm sp}(K_{n'})$.

To conclude the proof suppose that $\gamma_{\rm sp}(H) \leq n' - 2$. In this case, since $\gamma_{\rm sp}(G \circ H) = n\gamma_{\rm sp}(H)$, from Lemma 7 we deduce that for any $g \in V(G)$, $|W_g| = \gamma_{\rm sp}(H) \leq n' - 2$. Let $V(K_2) = \{a, a'\}$. Hence, for any $(a,b_1),(a,b_2),(a',b_3) \notin W$ we have that $N(a,b_1) \subseteq N(a,b_2) \cup N(a',b_3)$, which is a contradiction. Therefore, $\gamma_{\rm sp}(H) = n' - 1$ and so the result follows. $\square$

The following result provides an upper bound on the super domination number of the graph $G \circ H$ in terms of the order and the super domination number of its factors.

Theorem 12. *For any graph G of order $n \geq 2$ and any graph H of order $n' \geq 2$,*

$$\gamma_{\rm sp}(G \circ H) \leq \min\{n(n'-1) + \gamma_{\rm sp}(G), n'(n-1) + \gamma_{\rm sp}(H)\}.$$

Proof. Let S be a $\gamma_{\rm sp}(G)$-set and let $y \in V(H)$. We claim that

$$W = \left(\bigcup_{x \in S} \{x\} \times V(H)\right) \cup \left(\bigcup_{x \notin S} \{x\} \times (V(H) \setminus \{y\})\right)$$

is a super dominating set for $G \circ H$. To see this we only need to observe that for any $(x,y) \in \overline{W}$, there exists $x' \in S$ such that $N(x') \cap \overline{S} = \{x\}$, which implies that $N(x',y') \cap \overline{W} = \{(x,y)\}$ for every $y' \in V(H)$. Hence, $\gamma_{\rm sp}(G \circ H) \leq |W| = n(n'-1) + \gamma_{\rm sp}(G)$.

Now, let S' be a $\gamma_{\rm sp}(H)$-set and let $x' \in V(G)$. We claim that

$$W' = \left(\bigcup_{x \neq x'} \{x\} \times V(H) \right) \cup (\{x'\} \times S')$$

is a super dominating set for $G \circ H$. In this case we only need to observe that $\{x'\} \times S'$ is a super dominating set for the subgraph induced by $\{x'\} \times V(H)$. Hence, $\gamma_{\rm sp}(G \circ H) \leq |W'| = n'(n-1) + \gamma_{\rm sp}(H)$. □

The bound above is tight. For instance, as we will show in Proposition 16, the equality $\gamma_{\rm sp}(G \circ H) = n'(n-1) + \gamma_{\rm sp}(H)$ holds for any graph H with $\gamma_{\rm sp}(H) \leq n' - 2$ and G isomorphic to a complete graph. On the other hand, it is not difficult to check that if $G \cong K_r \odot K_1$ is the corona product K_r times K_1, then $\gamma_{\rm sp}(G) = \frac{n}{2} = r$ and $\gamma_{\rm sp}(G \circ K_{n'}) = 2rn' - r = n(n'-1) + \gamma_{\rm sp}(G)$.

In the next result we obtain an upper bound for $\gamma_{\rm sp}(G \circ N_{n'})$, where $N_{n'}$ is the empty graph of order n'. To state the result, we first need some additional notation and terminology. A set S of vertices is called a 2-*packing* if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The 2-packing number $\rho(G)$ of a graph G is the cardinality of a maximum 2-packing in G. A 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-set. Given a graph G, its *line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are adjacent in G.

Theorem 13. *For any graph G of order n and any integer $n' \geq 2$,*

$$\gamma_{\rm sp}(G \circ N_{n'}) \leq nn' - 2\rho(L(G)).$$

Proof. Let S be a $\rho(L(G))$-set and $V_S = \{x \in V(G) : x \in e \text{ for some } e \in S\}$. Given a fix vertex $y' \in V(H)$ we define

$$W = \left(\bigcup_{x \in V_S} \{x\} \times (V(N_{n'}) \setminus \{y'\}) \right) \cup \left(\bigcup_{x \notin V_S} \{x\} \times V(N_{n'}) \right).$$

Notice that if $(x, y') \notin W$, then $x \in V_S$, which implies that there exists $x' \in N(x)$ such that $\{x, x'\} \in S$. Now, since S is a 2-packing of $L(G)$, for any $y \in V(N_{n'}) \setminus \{y'\}$ we have that (x', y) is a private neighbour of (x, y') with respect to $\overline{W}$, *i.e.*$N(x', y) \cap \overline{W} = \{(x, y')\}$. Hence, W is a super dominating set of $G \circ H$ and so $\gamma_{\rm sp}(G \circ N_{n'}) \leq |W| = nn' - |V_S| = nn' - 2\rho(L(G))$. Therefore, the result follows. □

The bound above is tight. It is achieved, for instance, for $G \cong K_n$. Notice that $\rho(L(K_n)) = 1$, $K_n \circ N_{n'} \cong K_{n',\dots,n'}$ and $\gamma_{\rm sp}(K_n \circ N_{n'}) = nn' - 2 = nn' - 2\rho(L(K_n))$.

3.2. Closed formulas and complexity

In this subsection we obtain closed formulas for the super domination number of lexicographic product graphs and, as a consequence of the study, we show that the problem of computing the super domination number of a graph is NP-Hard.

Theorem 14. *Let G be a graph of order n and maximum degree $\Delta(G)$. If H is a graph of order n' such that $n' - \gamma_{\rm sp}(H) > \Delta(G) + 1$, then*

$$\gamma_{\rm sp}(G \circ H) = \alpha(G)\gamma_{\rm sp}(H) + (n - \alpha(G))n'.$$

Proof. By Theorem 9 we have that $\gamma_{\rm sp}(G \circ H) \le \alpha(G)\gamma_{\rm sp}(H) + (n - \alpha(G))n'$, so that it remains to show that $\gamma_{\rm sp}(G \circ H) \ge \alpha(G)\gamma_{\rm sp}(H) + (n - \alpha(G))n'$.

Let W be a $\gamma_{\rm sp}(G \circ H)$-set and set

$$X = \{x \in V(G) : |W_x| < n'\}.$$

We claim that X is an independent set. To see this, suppose that there are two adjacent vertices x, x' which belong to X. Notice that Lemma 8 leads to $|W_x| = |W_{x'}| = n' - 1$. Hence, for any $\gamma_{\rm sp}(H)$-set S we have that

$$W' = (W \setminus (\{x\} \times V(H))) \cup (\{x\} \times S) \cup \left(\bigcup_{u \in N(x)} \{u\} \times V(H) \right)$$

is a super dominating set for $G \circ H$ and

$$\begin{aligned} |W'| &= |W| + |X \cap N(x)| - (n' - 1 - \gamma_{\rm sp}(H)) \\ &\le |W| + |\Delta(G)| - (n' - 1 - \gamma_{\rm sp}(H)) \\ &< |W|, \end{aligned}$$

which is a contradiction. Thus, X is an independent set and, by Lemma 7 we have that

$$\begin{aligned}
\gamma_{\rm sp}(G \circ H) &= \sum_{u \in X} |W_u| + \sum_{u \notin X} |W_u| \\
&\geq |X|\gamma_{\rm sp}(H) + (n - |X|)n' \\
&= nn' - |X|(n' - \gamma_{\rm sp}(H)) \\
&\geq nn' - \alpha(G)(n' - \gamma_{\rm sp}(H)) \\
&= \alpha(G)\gamma_{\rm sp}(H) + (n - \alpha(G))n',
\end{aligned}$$

as required. □

Fernau and Rodríguez-Velázquez [2, 3] showed that the study of corona product graphs enables us to infer NP-hardness results for computing the (local) metric dimension, based on according NP-hardness results for the (local) adjacency dimension. Our next result shows how the study of lexicographic product graphs enables us to infer an NP-hardness result for computing the super domination number, based on a well known NP-hardness result for the independence number, *i.e.*, since the problem of computing the independence number of a graph is NP-Hard [6], Theorem 14 leads to the following result.

Corollary 15. *The problem of finding the super domination number of a graph is* NP*-Hard.*

Proof. Let G be a graph of order n and maximum degree $\Delta(G)$. By Theorem 14, for any integer $t > \Delta(G) + 1$ we have that

$$\gamma_{\rm sp}\left(G \circ \bigcup_{i=1}^{t} K_2\right) = t(2n - \alpha(G)).$$

Therefore, since the problem of computing the independence number of a graph is NP-Hard, we conclude that the problem of finding the super domination number of a graph is NP-Hard too. □

The remaining results of this subsection concern the case in which we fix the first factor in the lexicographic product.

Proposition 16. *Let H be a noncomplete graph of order n' and let $n \geq 2$ be an integer. Then the following assertions hold.*

- *If $\gamma_{\rm sp}(H) = n' - 1$, then $\gamma_{\rm sp}(K_n \circ H) = nn' - 2$.*

- *If $\gamma_{\rm sp}(H) \le n'-2$, then $\gamma_{\rm sp}(K_n \circ H) = n'(n-1)+\gamma_{\rm sp}(H)$.*

Proof. Since $\alpha(K_n)=1$ and $\alpha_2(K_n)=2$, by Theorem 9 we immediately have that if $\gamma_{\rm sp}(H)=n'-1$, then $\gamma_{\rm sp}(K_n\circ H)\le nn'-2$ and if $\gamma_{\rm sp}(H)\le n'-2$, then $\gamma_{\rm sp}(K_n\circ H)\le n'(n-1)+\gamma_{\rm sp}(H)$.

Now, let W be a $\gamma_{\rm sp}(K_n\circ H)$-set. Notice that, since $K_n\circ H\notin\mathcal{F}$, Theorem 5 leads to $|\overline{W}|\ge 2$. If $(x,y),(x',y')\in\overline{W}$, for $x\ne x'$, then $N(u,v)\subseteq N(x,y)\cup N(x',y')$, for every $(u,v)\in V(K_n)\times V(H)\setminus\{(x,y),(x',y')\}$, which means that $|W|=nn'-2$. Furthermore, if $(x,y),(x,y')\notin W$, then $\overline{W}\subseteq\{x\}\times V(H)$, and by Lemma 7 we have that $|W|\ge (n-1)n'+\gamma_{\rm sp}(H)$. Hence,

$$\gamma_{\rm sp}(K_n\circ H)\ge\min\{nn'-2, n'(n-1)+\gamma_{\rm sp}(H)\}.$$

Thus, if $\gamma_{\rm sp}(H)\le n'-2$, then $\gamma_{\rm sp}(K_n\circ H)\ge n'(n-1)+\gamma_{\rm sp}(H)$. Finally, if $\gamma_{\rm sp}(H)=n'-1$, then $\gamma_{\rm sp}(K_n\circ H)\ge nn'-2$. $\square$

For any complete bipartite graph $K_{r,t}$, where $r\le t$ and $t\ge 2$, we have $\alpha(K_{r,t})=\alpha_2(K_{r,t})=t$. Therefore, we can state the following proposition which shows again that the bounds in Theorem 9 are tight.

Proposition 17. *For any nonempty graph H and any integers r,t, where $r\le t$ and $t\ge 2$,*

$$\gamma_{\rm sp}(K_{r,t}\circ H)=t\gamma_{\rm sp}(H)+rn'.$$

Proof. Since $r\le t$ and $t\ge 2$, we have $\alpha(K_{r,t})=\alpha_2(K_{r,t})=t$. Therefore, Theorem 9 leads to $\gamma_{\rm sp}(K_{r,t}\circ H)\le t\gamma_{\rm sp}(H)+rn'$.

It remains to show that $\gamma_{\rm sp}(K_{r,t}\circ H)\ge t\gamma_{\rm sp}(H)+rn'$. Let $V(K_{r,t})=V_r\cup V_t$ where the vertices in V_r have degree t and the vertices in V_t have degree r. Let W be a $\gamma_{\rm sp}(K_{r,t}\circ H)$-set. By Theorem 5 we have that $|\overline{W}|\ge 2$ and so we can fix $(x,y),(a,b)\in\overline{W}$ and differentiate the following two cases.
Case 1. $x\in V_r$ and $a\in V_t$. By Lemma 8, $|\overline{W}_x|=|\overline{W}_a|=1$ and for any $u\in V(K_{r,t})\setminus\{x,a\}$ we have that $\overline{W}_u=\emptyset$, so that

$$t\gamma_{\rm sp}(H)+rn'\ge|W|\ge(r+t)n'-2. \tag{4}$$

If $t=2$ and $\gamma_{\rm sp}(H)=n'-1$, then (4) leads to $|W|=(r+2)n'-2=2(n'-1)+rn'$, as required. Also, if $\gamma_{\rm sp}(H)\le n'-2$, then (4) leads to $t\le 1$, which is a contradiction. Now, if $t\ge 3$, then (4) leads to $\gamma_{\rm sp}(H)\ge\left\lceil n'-\frac{2}{t}\right\rceil=n'$, which is a contradiction again.
Case 2. $x,a\in V_r$ or $x,a\in V_t$. If $t=2$, then $K_{r,t}\cong C_4$ or $K_{r,t}\cong P_3$, and we are done. If $t\ge 3$, then by Lemma 7 we have that $|W|=\sum_{u\in V_t}|W_u|+\sum_{u\in V_r}|W_u|\ge t\gamma_{\rm sp}(H)+rn'$, as required. $\square$

It is well known that for any integer $n \geq 3$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and it is not difficult to check that $\alpha_2(C_n) = \lfloor \frac{2n}{3} \rfloor$. In order to study the super domination number of $C_n \circ H$ we need to state the following lemma.

Lemma 18. *Let $n \geq 5$ be an integer and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} and the subscripts are taken modulo n. If $S \subseteq V(C_n)$ and $|S| = \lfloor \frac{2n}{3} \rfloor + 1$, then there exists a subscript i such that $\{v_i, v_{i+1}, \ldots, v_{i+4}\} \subseteq S$ or $\{v_i, v_{i+2}, v_{i+3}, v_{i+4}\} \subseteq S$ or $\{v_i, v_{i+1}, v_{i+2}, v_{i+4}\} \subseteq S$.*

Proof. Let $S \subseteq V(C_n)$ such that $|S| = \lfloor \frac{2n}{3} \rfloor + 1$. Suppose that C_n does not contain a path $P = (v_i, v_{i+1}, \ldots, v_{i+4})$ such that all vertices of P belong to S or all but v_{i+1} (or v_{i+3}) belong to S. If no path in this form contains three consecutive vertices of C_n, then S is a 2-independent set, which is a contradiction as $|S| = \lfloor \frac{2n}{3} \rfloor + 1 > \alpha_2(C_n)$. Otherwise, the vertices in $X = S \cap V(P)$ are consecutive and $|X| \leq 4$. Now, if $X = \{x_i, x_{i+1}, x_{i+2}\}$, then $\{x_{i-1}, x_{i-2}, x_{i+3}, x_{i+4}\} \cap S = \emptyset$ and if $X = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$, then $\{x_{i-1}, x_{i-2}, x_{i+4}, x_{i+5}\} \cap S = \emptyset$. Let U be the set of these maximal paths where $|X| = 3$ and, analogously, let U' be the set these paths where $|X| = 4$. We assume that the labelling in all these paths is induced by the labelling in C_n. Next, we can construct a set S^* from S by removing x_{i+2} and adding x_{i+3} for each path in U, and by removing x_{i+2} and adding x_{i+4} for each path in U'. Hence, S^* is a 2-independent set, which is a contradiction, as $|S^*| = |S| = \lfloor \frac{2n}{3} \rfloor + 1 > \alpha_2(C_n)$. □

Proposition 19. *Let H be a nonempty graph of order n' and let $n \geq 4$ be an integer. If $\gamma_{\rm sp}(H) = n' - 1$ and $H \not\cong K_{n'}$, then*

$$\gamma_{\rm sp}(C_n \circ H) = nn' - \left\lfloor \frac{2n}{3} \right\rfloor.$$

Furthermore, if $\gamma_{\rm sp}(H) \leq n' - 2$, then

$$\gamma_{\rm sp}(C_n \circ H) = \left\lfloor \frac{n}{2} \right\rfloor \gamma_{\rm sp}(H) + n' \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Let $V(C_n) = \{x_1, x_2, \ldots, x_n\}$, where x_i is adjacent to x_{i+1} and the subscripts are taken modulo n. Let W be a $\gamma_{\rm sp}(C_n \circ H)$-set and

$$X = \{x \in V(C_n) : |W_x| < n'\}.$$

We first consider the case $\gamma_{\rm sp}(H) = n' - 1$ and $H \not\cong K_{n'}$. By Theorem 9 we have that $\gamma_{\rm sp}(C_n \circ H) \leq nn' - \lfloor \frac{2n}{3} \rfloor$. Suppose $\gamma_{\rm sp}(C_n \circ H) < nn' - \lfloor \frac{2n}{3} \rfloor$.

From Lemma 7 we know that $|\overline{W_x}| = 1$ for every $x \in X$, so that $|\overline{W}| = |X| \geq \left\lfloor \frac{2n}{3} \right\rfloor + 1$. If $n = 4$, then at least three vertices, say $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ belong to $\overline{W}$ and, in such a case, $N(x_1, y_1) \subseteq N(x_2, y_2) \cup N(x_3, y_3)$, so W is not a super dominating set of $C_4 \circ H$, which is a contradiction. If $n \geq 5$, then by Lemma 18 there exists a subscript i such that $\{x_i, x_{i+1}, \ldots, x_{i+4}\} \subseteq X$ or $\{x_i, x_{i+2}, x_{i+3}, x_{i+4}\} \subseteq X$ or $\{x_i, x_{i+1}, x_{i+2}, x_{i+4}\} \subseteq X$. In all these cases $N(x_{i+2}) \subseteq \cup_{j=i}^{i+4} N(x_j)$, so that W is not a super dominating set of $C_n \circ H$, which is a contradiction. Therefore, $\gamma_{\rm sp}(C_n \circ H) = nn' - \left\lfloor \frac{2n}{3} \right\rfloor$.

From now on we assume that $\gamma_{\rm sp}(H) \leq n' - 2$. By Theorem 9 we have that $\gamma_{\rm sp}(C_n \circ H) \leq \left\lfloor \frac{n}{2} \right\rfloor \gamma_{\rm sp}(H) + n' \left\lceil \frac{n}{2} \right\rceil$. We will show that $\gamma_{\rm sp}(C_n \circ H) \geq \left\lfloor \frac{n}{2} \right\rfloor \gamma_{\rm sp}(H) + n' \left\lceil \frac{n}{2} \right\rceil$. If X is an independent set, then by Lemma 7,

$$
\begin{aligned}
|W| =& \sum_{x \in X} |W_x| + \sum_{x \notin X} |W_x| \\
\geq& |X| \gamma_{\rm sp}(H) + (n - |X|) n' \\
=& nn' - |X|(n' - \gamma_{\rm sp}(H)) \\
\geq& nn' - \alpha(C_n)(n' - \gamma_{\rm sp}(H)) \\
=& \left\lfloor \frac{n}{2} \right\rfloor \gamma_{\rm sp}(H) + n' \left\lceil \frac{n}{2} \right\rceil,
\end{aligned}
$$

as required. Suppose that X is not independent. Let S be a maximal subset of X which is composed by consecutive vertices of C_n. If $S = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}$, then

$$
N(x_{i+2}) \subseteq \bigcup_{x_j \in S \setminus \{x_{i+2}\}} N(x_j),
$$

so that we deduce that W is not a super dominating set of $C_n \circ H$, which is a contradiction. Thus, $|S| \leq 4$. We now fix an independent set $S' \subseteq S$ in the following way. If $S = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$, then $S' = \{x_i, x_{i+3}\}$, if $S = \{x_i, x_{i+1}, x_{i+2}\}$, then $S' = \{x_i, x_{i+2}\}$ and, if $S = \{x_i, x_{i+1}\}$, then $S' = \{x_i\}$. Hence, we can construct an independent set $X' \subseteq X$ by replacing every maximal set S defined as above with the corresponding set S'. Since $\gamma_{\rm sp}(H) \leq n' - 2$, by Lemmas 7 and 8 we have that for any S defined as above,

$$
\sum_{x \in S} |W_x| = |S|(n' - 1) \geq (|S| - |S'|) n' + |S'| \gamma_{\rm sp}(H),
$$

which implies that

$$\begin{aligned}|W| &= \sum_{x\in X'} |W_x| + \sum_{x\notin X'} |W_x| \\ &\geq |X'|\gamma_{\rm sp}(H) + (n-|X'|)n' \\ &= nn' - |X'|(n'-\gamma_{\rm sp}(H)) \\ &\geq nn' - \alpha(C_n)(n'-\gamma_{\rm sp}(H)) \\ &= \left\lfloor \frac{n}{2} \right\rfloor \gamma_{\rm sp}(H) + n' \left\lceil \frac{n}{2} \right\rceil,\end{aligned}$$

Therefore, $\gamma_{\rm sp}(C_n \circ H) = \left\lfloor \frac{n}{2} \right\rfloor \gamma_{\rm sp}(H) + n' \left\lceil \frac{n}{2} \right\rceil$. □

For the case of paths we have $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$ and $\alpha_2(P_n) = \left\lceil \frac{2n}{3} \right\rceil$. To complete the study on the super domination of number of $P_n \circ H$ we need to state the following lemma.

Lemma 20. *Let $n \geq 4$ be an integer and $V(P_n) = \{v_1, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} for any $i \in \{1, \ldots, n-1\}$. If $S \subseteq V(P_n)$ and $|S| = \left\lceil \frac{2n}{3} \right\rceil + 1$, then at least one of the following statements hold.*

- *(a) There exists a subscript $i \leq n-4$ such that $\{v_i, v_{i+1}, \ldots, v_{i+4}\} \subseteq S$ or $\{v_i, v_{i+1}, v_{i+2}, v_{i+4}\} \subseteq S$ or $\{v_i, v_{i+2}, v_{i+3}, v_{i+4}\} \subseteq S$.*
- *(b) $\{v_1, v_2, v_3\} \subseteq S$ or $\{v_{n-2}, v_{n-1}, v_n\} \subseteq S$.*

Proof. Suppose $|S| = \left\lceil \frac{2n}{3} \right\rceil + 1$ and conditions (a) and (b) do not hold. If S is a 2-independent set, then $|S| = \left\lceil \frac{2n}{3} \right\rceil + 1 > \alpha_2(P_n)$, which is a contradiction. Otherwise, for any maximal set $X \subseteq S$ composed by consecutive vertices we have $|X| \leq 4$. We consider two cases depending on the cardinality of X:

- $X = \{x_i, x_{i+1}, x_{i+2}\}$. If $i = 2$, then $\{x_1, x_5, x_6\} \cap S = \emptyset$, if $i = n-3$, then $\{x_n, x_{n-4}, x_{n-5}\} \cap S = \emptyset$, while in other cases $\{x_{i-1}, x_{i-2}, x_{i+3}, x_{i+4}\} \cap S = \emptyset$.
- $X = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$. If $i = 2$, then $\{x_1, x_6, x_7\} \cap S = \emptyset$, if $i = n-4$ then $\{x_n, x_{n-5}, x_{n-6}\} \cap S = \emptyset$, while in other cases $\{x_{i-1}, x_{i-2}, x_{i+4}, x_{i+5}\} \cap S = \emptyset$.

Let U be the set of these maximal sets of cardinality $|X| = 3$ and, analogously, let U' be the set these maximal sets of cardinality $|X| = 4$. We assume that the labelling in all these sets is induced by the labelling in P_n. Next, we can construct a set S^* from S by removing x_i and adding x_{i-1} for each set in U, and by removing x_{i+1} and adding x_{i-1} for each set in U'. Hence, S^* is a 2-independent set, which is a contradiction, as $|S^*| = |S| = \left\lceil \frac{2n}{3} \right\rceil + 1 > \alpha_2(P_n)$. $\square$

Proposition 21. *Let H be a nonempty graph of order n' and let $n \geq 2$ be an integer. If $\gamma_{\rm sp}(H) = n' - 1$ and $H \ncong K_{n'}$, then*

$$\gamma_{\rm sp}(P_n \circ H) = nn' - \left\lceil \frac{2n}{3} \right\rceil.$$

Furthermore, if $\gamma_{\rm sp}(H) \leq n' - 2$, then

$$\gamma_{\rm sp}(P_n \circ H) = \left\lceil \frac{n}{2} \right\rceil \gamma_{\rm sp}(H) + n' \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. Let $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, where x_i is adjacent to x_{i+1} for every $i \in \{1, \ldots, n-1\}$. Let W be a $\gamma_{\rm sp}(P_n \circ H)$-set and

$$X = \{x \in V(P_n) : \, |W_x| < n'\}.$$

We first consider the case $\gamma_{\rm sp}(H) = n' - 1$ and $H \ncong K_{n'}$. By Theorem 9 we have that $\gamma_{\rm sp}(P_n \circ H) \leq nn' - \left\lceil \frac{2n}{3} \right\rceil$. It remains to show that $\gamma_{\rm sp}(P_n \circ H) \geq nn' - \left\lceil \frac{2n}{3} \right\rceil$.

The cases $n = 2$ and $n = 3$ were previously discussed in Propositions 16 and 17. From now on we assume that $n \geq 4$. Suppose that $\gamma_{\rm sp}(P_n \circ H) < nn' - \left\lceil \frac{2n}{3} \right\rceil$. Since $\gamma_{\rm sp}(H) = n' - 1$, from Lemma 7 we know that $|\overline{W}_x| = 1$ for every $x \in X$, so that $|\overline{W}| = |X| \geq \left\lceil \frac{2n}{3} \right\rceil + 1$. By Lemma 20 we differentiate the following cases.

(a) There exists a subscript $i \leq n-4$ such that $X_0 = \{v_i, v_{i+1}, \ldots, v_{i+4}\} \subseteq X$ or $X_1 = \{v_i, v_{i+2}, v_{i+3}, v_{i+4}\} \subseteq X$ or $X_2 = \{v_i, v_{i+1}, v_{i+2}, v_{i+4}\} \subseteq X$. For any $l \in \{0, 1, 2\}$ we have that

$$N(v_{i+2}) \subseteq \bigcup_{v_j \in X_l \setminus \{v_{i+2}\}} N(v_j).$$

(b) $\{v_1, v_2, v_3\} \subseteq X$ or $\{v_{n-2}, v_{n-1}, v_n\} \subseteq X$. Thus, $N(v_1) \subseteq N(v_2) \cup N(v_3)$ or $N(v_n) \subseteq N(v_{n-1}) \cup N(v_{n-2})$.

According to the two cases above we conclude that W is not a super dominating set of $P_n \circ H$, which is a contradiction. Therefore, $\gamma_{\rm sp}(P_n \circ H) = nn' - \left\lceil \frac{2n}{3} \right\rceil$.

From now on we assume that $\gamma_{\rm sp}(H) \leq n' - 2$. By Theorem 9 we have that $\gamma_{\rm sp}(P_n \circ H) \leq \left\lceil \frac{n}{2} \right\rceil \gamma_{\rm sp}(H) + n' \left\lfloor \frac{n}{2} \right\rfloor$. We will show that $\gamma_{\rm sp}(P_n \circ H) \geq \left\lceil \frac{n}{2} \right\rceil \gamma_{\rm sp}(H) + n' \left\lfloor \frac{n}{2} \right\rfloor$. Although this part of the proof is completely analogous to the second part of the proof of Proposition 19, we prefer to include it for completeness. If X is an independent set, then by Lemma 7,

$$\begin{aligned}
|W| =& \sum_{x \in X} |W_x| + \sum_{x \notin X} |W_x| \\
\geq& |X| \gamma_{\rm sp}(H) + (n - |X|) n' \\
=& nn' - |X|(n' - \gamma_{\rm sp}(H)) \\
\geq& nn' - \alpha(P_n)(n' - \gamma_{\rm sp}(H)) \\
=& \left\lceil \frac{n}{2} \right\rceil \gamma_{\rm sp}(H) + n' \left\lfloor \frac{n}{2} \right\rfloor,
\end{aligned}$$

as required. Suppose that X is not independent. Let S be a maximal subset of X which is composed by consecutive vertices of P_n. If $S = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}$, then $N(x_{i+2}) \subseteq \cup_{j=1}^{4} N(x_j)$, so that we deduce that W is not a super dominating set of $P_n \circ H$, which is a contradiction. Thus, $|S| \leq 4$. We now fix an independent set $S' \subseteq S$ in the following way. If $S = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$, then $S' = \{x_i, x_{i+3}\}$, if $S = \{x_i, x_{i+1}, x_{i+2}\}$, then $S' = \{x_i, x_{i+2}\}$ and, if $S = \{x_i, x_{i+1}\}$, then $S' = \{x_i\}$. Hence, we can construct an independent set $X' \subseteq X$ by replacing every maximal set S defined as above with the corresponding set S'. Since $\gamma_{\rm sp}(H) \leq n' - 2$, by Lemmas 7 and 8 we have that for any S defined as above,

$$\sum_{x \in S} |W_x| = |S|(n' - 1) \geq (|S| - |S'|) n' + |S'| \gamma_{\rm sp}(H),$$

which implies that

$$\begin{aligned}
|W| =& \sum_{x \in X'} |W_x| + \sum_{x \notin X'} |W_x| \\
\geq& |X'| \gamma_{\rm sp}(H) + (n - |X'|) n' \\
=& nn' - |X'|(n' - \gamma_{\rm sp}(H)) \\
\geq& nn' - \alpha(P_n)(n' - \gamma_{\rm sp}(H)) \\
=& \left\lceil \frac{n}{2} \right\rceil \gamma_{\rm sp}(H) + n' \left\lfloor \frac{n}{2} \right\rfloor,
\end{aligned}$$

Therefore, $\gamma_{\text{sp}}(P_n \circ H) = \left\lceil \frac{n}{2} \right\rceil \gamma_{\text{sp}}(H) + n' \left\lfloor \frac{n}{2} \right\rfloor$. $\square$

3.3. Super domination in join graphs

Since $K_n + K_{n'} = K_{n+n'}$ and $N_n + N_{n'} = K_{n,n'}$, in this section we consider the case of join graphs $G+H$ where G and H are not simultaneously complete nor empty.

Given a graph G, a set $X \subseteq V(G)$ and a vertex $y \in \overline{X}$, we denote the set of external neighbours of y with respect to $\overline{X}$ by

$$F_X(y) = \{x \in X : N(x) \cap \overline{X} = \{y\}\}.$$

Theorem 22. *Let G and H be two nonempty and noncomplete graphs of order n and n', respectively. Then*

$$\gamma_{\text{sp}}(G+H) = \min\{n + n' - 2, n + \gamma_{\text{sp}}(H), n' + \gamma_{\text{sp}}(G)\}.$$

Proof. Let S_1 be a $\gamma_{\text{sp}}(G)$-set and S_2 a $\gamma_{\text{sp}}(H)$-set. Let $g \in V(G)$ and $h \in V(H)$ be nonuniversal vertices of G and H, respectively. It is readily seen that $V(G) \cup S_2$, $V(H) \cup S_1$ and $(V(G) \cup V(H)) \setminus \{g, h\}$ are super dominating sets of $G + H$, so that

$$\gamma_{\text{sp}}(G+H) \leq \min\{n + n' - 2, n + \gamma_{\text{sp}}(H), n' + \gamma_{\text{sp}}(G)\}. \tag{5}$$

Now we take a $\gamma_{\text{sp}}(G+H)$-set W and differentiate the following three cases.

Case 1. $V(G) \cap \overline{W} \neq \emptyset$ and $V(H) \cap \overline{W} \neq \emptyset$. If $g \in V(G) \cap \overline{W}$ and $h \in V(H) \cap \overline{W}$, then $F_W(g) \subseteq V(H)$ and $F_W(h) \subseteq V(G)$, which implies that $\overline{W} = \{g, h\}$. Hence, $\gamma_{\text{sp}}(G+H) = n + n' - 2$.

Case 2. $\overline{W} \subseteq V(G)$. In this case, by analogy to the proof of Lemma 7 we deduce that $|W \cap V(G)| \geq \gamma_{\text{sp}}(G)$, which implies that $|W| \geq n' + \gamma_{\text{sp}}(G)$ and by (5) we deduce that $\gamma_{\text{sp}}(G+H) = n' + \gamma_{\text{sp}}(G)$.

Case 3. $\overline{W} \subseteq V(H)$. This case is analogous to the previous one, so that $\gamma_{\text{sp}}(G+H) = n + \gamma_{\text{sp}}(H)$.

According to the three cases above, the result follows. $\square$

Since $K_n + N_{n'} \in \mathcal{F}$, by Theorem 5 we have that $\gamma_{\text{sp}}(K_n + N_{n'}) = n + n' - 1$. Hence, it remains to study the cases $K_n + H$ and $N_n + H$ where $H \notin \{N_{n'}, K_{n'}\}$.

Theorem 23. *Let H be a graph of order n'. If $H \notin \{N_{n'}, K_{n'}\}$, then for any integer $n \geq 1$,*

$$\gamma_{\text{sp}}(K_n + H) = n + \gamma_{\text{sp}}(H).$$

Proof. Let S be a $\gamma_{\text{sp}}(H)$-set. It is readily seen that $V(K_n) \cup S$ is a super dominating sets of $K_n + H$, so that

$$\gamma_{\text{sp}}(K_n + H) \leq n + \gamma_{\text{sp}}(H) \leq n + n' - 1. \tag{6}$$

Now, let W be a $\gamma_{\text{sp}}(K_n + H)$-set. Since the vertices in $V(K_n)$ are universal vertices of $K_n + H$, if $V(K_n) \cap \overline{W} \neq \emptyset$, then $V(H) \cap \overline{W} = \emptyset$ and, in such a case, $\gamma_{\text{sp}}(K_n + H) = n + n' - 1$, so that (6) leads to $\gamma_{\text{sp}}(K_n + H) = n + \gamma_{\text{sp}}(H)$. On the other hand, if $\overline{W} \subseteq V(H)$, then by analogy to the proof of Lemma 7 we deduce that $|W \cap V(H)| \geq \gamma_{\text{sp}}(H)$, which implies that $|W| \geq n + \gamma_{\text{sp}}(H)$ and by (6) we deduce that $\gamma_{\text{sp}}(K_n + H) = n + \gamma_{\text{sp}}(H)$. Therefore, the result follows. $\square$

Theorem 24. *Let H be a graph of order n'. If $H \notin \{N_{n'}, K_{n'}\}$, then for any integer $n \geq 2$,*

$$\gamma_{\text{sp}}(N_n + H) = \min\{n' + n - 2, n + \gamma_{\text{sp}}(H)\}.$$

Proof. Let W be a $\gamma_{\text{sp}}(N_n + H)$-set. Notice that $|\overline{W} \cap V(N_n)| \leq 1$. Since $H \notin \{N_{n'}, K_{n'}\}$, by Theorem 5 we deduce that $\gamma_{\text{sp}}(N_n + H) \leq n' + n - 2$, which implies that $\overline{W} \cap V(H) \neq \emptyset$. With this fact in mind, and following a procedure analogous to that in the proof of Theorem 22, we conclude the proof. $\square$

Acknowledgements

Research supported in part by the Spanish government under the grant MTM2016-78227-C2-1-P and by the Mexican government under the grant PAPIIT-IN114415-UNAM.

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