On trees with double domination number equal to total domination number plus one

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Abstract

A total dominating set of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D. A vertex of a graph is said to dominate itself and all of its neighbors. A double dominating set of a graph G is a set D of vertices of G such that every vertex of G is dominated by at least two vertices of D. The total (double, respectively) domination number of a graph G is the minimum cardinality of a total (double, respectively) dominating set of G. We characterize all trees with double domination number equal to total domination number plus one.

Keywords: total domination, double domination, tree.
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. Subdivided star is a graph obtained from a star by subdividing each one of its edges.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a total dominating set, abbreviated TDS, of G if every vertex of G has a neighbor in D. The domination (total domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating (total dominating, respectively) set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [1]. For a comprehensive survey of domination in graphs, see [3, 4].

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a double dominating set, abbreviated DDS, of G if every vertex of G is dominated by at least two vertices of D. The double domination number of a graph G, denoted by $\gamma_d(G)$, is the minimum cardinality of a double dominating set of G. The study of double domination in graphs was initiated by Harary and Haynes [2].

A paired dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. The authors of [5] characterized all trees with equal total domination and paired domination numbers.

We characterize all trees with double domination number equal to total domination number plus one.

2 Results

Since the one-vertex graph does not have double dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following four straightforward observations.

Observation 1 *Every support vertex of a graph G is in every $\gamma_t(G)$-set.*

Observation 2 *For every connected graph G of diameter at least three there exists a $\gamma_t(G)$-set that contains no leaf.*

Observation 3 *Every leaf of a graph G is in every $\gamma_d(G)$-set.*

Observation 4 *Every support vertex of a graph G is in every $\gamma_d(G)$-set.*

It is easy to see that $\gamma_d(P_2) = \gamma_t(P_2) = 2$. Now we prove that for every tree different than P_2 the double domination number is greater than the total domination number.

Lemma 5 *For every tree $T \neq P_2$ we have $\gamma_d(T) > \gamma_t(T)$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. Since $T \neq P_2$, we have $\text{diam}(T) \geq 2$. If $\text{diam}(T) = 2$, then T is a star $K_{1,m}$. We have $\gamma_d(T) = m + 1 \geq 2 + 1 > 2 = \gamma_t(T)$. Now let us assume that $\text{diam}(T) = 3$. Thus T is a double star. We have $\gamma_d(T) = n \geq 4 > 2 = \gamma_t(T)$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z mean leaves adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_t(T')$-set. By Observation 1 we have $x \in D'$. Of course, D' is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T')$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $y, z, x \in D$. It is easy to see that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 1 > \gamma_t(T') + 1 \geq \gamma_t(T) + 1 > \gamma_t(T)$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) \geq 3$. Assume that u is adjacent to a leaf, say x. Let $T' = T - T_v$. Let D' be any $\gamma_t(T')$-set. By Observation 1 we have $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $t, x, v, u \in D$. It is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 2 > \gamma_t(T') + 2 \geq \gamma_t(T) + 1 > \gamma_t(T)$.

Now assume that among the descendants of u there is a support vertex, say x, different than v. Let $T' = T - T_v$. Let D' be a $\gamma_t(T')$-set that contains no leaf. The vertex x has to have a neighbor in D', thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $t, v, x \in D$. If $u \in D$, then it is easy to see that $D \setminus \{v, t\}$ is DDS of the tree T'. Now assume that $u \notin D$. Let us observe that $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 1 > \gamma_t(T') + 1 \geq \gamma_t(T)$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_u$. If $T' = P_2$, then $T = P_5$. We have $\gamma_d(P_5) = 4 > 3 = \gamma_t(P_5)$. Now assume that $T' \neq P_2$. Let D' be any $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 3 and 4 we have $t, v \in D$. Observe that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 2 > \gamma_t(T') + 2 \geq \gamma_t(T)$. ■

Now we give a necessary condition for that the double domination number of a tree is equal to its total domination number plus one.

Lemma 6 *If $\gamma_d(T) = \gamma_t(T) + 1$, then for every $\gamma_d(T)$-set D, every vertex of $V(T) \setminus D$ has degree two.*

Proof. Suppose that there exists a $\gamma_d(T)$-set D that does not contain a vertex of T, say x, which has degree different than two. By Observation 3, every leaf belongs to the set D. Therefore $d_T(x) \geq 3$. First assume that some neighbor of x, say y, also does not belong to the set D. By T_1 and T_2 we denote the trees resulting from T by removing the edge xy. Let us observe that each one of those trees has at least three vertices. We define $D_1 = D \cap V(T_1)$ and $D_2 = D \cap V(T_2)$. Let us observe that D_1 is a DDS of the tree T_1 and D_2 is a DDS of the tree T_2. Let D_1' be any $\gamma_t(T_1)$-set and let D_2' be any $\gamma_t(T_2)$-set. By Lemma 5 we have $\gamma_d(T_1) \geq \gamma_t(T_1) + 1$ and $\gamma_d(T_2) \geq \gamma_t(T_2) + 1$. Of course, $D_1' \cup D_2'$ is a TDS of the tree T. Thus $\gamma_t(T) \leq |D_1' \cup D_2'|$. Now we get $\gamma_d(T) = |D| = |D_1 \cup D_2| = |D_1| + |D_2| \geq \gamma_d(T_1) + \gamma_d(T_2) \geq \gamma_t(T_1) + 1 + \gamma_t(T_2) + 1 = |D_1'| + |D_2'| + 2 = |D_1' \cup D_2'| + 2 \geq \gamma_t(T) + 2 > \gamma_t(T) + 1$, a contradiction.

Now assume that all neighbors of x belong to the set D. First assume that there is a neighbor of x, say y, such that each one of the two trees resulting from T by removing the edge xy has at least three vertices. We get a contradiction similarly as when some neighbor of x does not belong to the set D. Now assume that there is no neighbor of x such that each one of the two trees resulting from T by removing the edge between them has at least three vertices. This implies that T is a subdivided star of order at least seven. Let n mean the number of vertices of the tree T. We have $\gamma_d(T) = n-1 = (n+1)/2+1+(n-5)/2 = \gamma_t(T)+1+(n-5)/2 > \gamma_t(T)+1$, a contradiction. ■

We characterize all trees with double domination number equal to total domination number plus one. For this purpose we introduce a family $\mathcal{T} = \{P_3\} \cup \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = \{A_1, A_2, \ldots\}$ and $\mathcal{B} = \{B_1, B_2, \ldots\}$ are families of trees elements of which are given in Figure 1. A tree A_k has $3k + 2$ vertices, and a tree B_k has $3k + 3$ vertices.

Now we prove that for every tree of the family $\mathcal{T}$, the double domination number is equal to the total domination number plus one.

Lemma 7 *If $T \in \mathcal{T}$, then $\gamma_d(T) = \gamma_t(T) + 1$.*

Proof. Of course, $\gamma_d(P_3) = 3 = 2 + 1 = \gamma_t(P_3) + 1$. Let k be a positive integer. For trees A_k and B_k we consider the labeling of the vertices as in Figure 1.

Let D be a $\gamma_t(A_k)$-set that contains no leaf. By Observation 1 we have

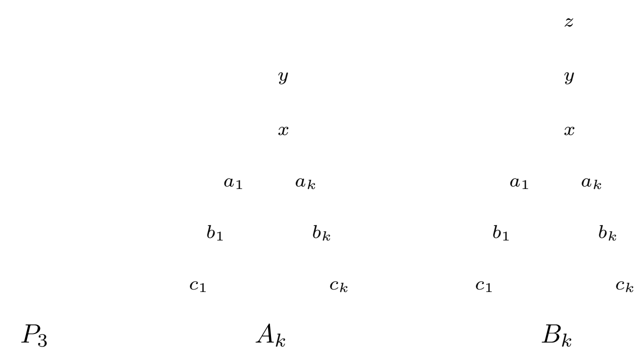

Figure 1: The path P_3, a tree A_k of the family $\mathcal{A}$, and a tree B_k of the family $\mathcal{B}$

$b_1, b_2, \dots, b_k, x \in D$. Since each one of the vertices $b_1, b_2, \dots, b_k$ has to have a neighbor in the set D, we have $a_1, a_2, \dots, a_k \in D$. Therefore $\gamma_t(A_k) \geq 2k+1$. It is easy to observe that $\{b_1, c_1, b_2, c_2, \dots, b_k, c_k, x, y\}$ is a DDS of the tree A_k. Thus $\gamma_d(A_k) \leq 2k+2$. Now we get $\gamma_d(A_k) \leq 2k+2 \leq \gamma_t(A_k)+1$. On the other hand, by Lemma 5 we have $\gamma_d(A_k) \geq \gamma_t(A_k)+1$.

Now let D be a $\gamma_t(B_k)$-set that contains no leaf. By Observation 1 we have $b_1, b_2, \dots, b_k, y \in D$. Since each one of the vertices $b_1, b_2, \dots, b_k, y$ has to have a neighbor in D, we have $a_1, a_2, \dots, a_k, x \in D$. Therefore $\gamma_t(B_k) \geq 2k+2$. It is easy to observe that $\{b_1, c_1, b_2, c_2, \dots, b_k, c_k, x, y, z\}$ is a DDS of the tree B_k. Thus $\gamma_d(B_k) \leq 2k+3$. Now we get $\gamma_d(B_k) \leq 2k+3 \leq \gamma_t(B_k)+1$. This implies that $\gamma_d(B_k) = \gamma_t(B_k)+1$. ■

Now we prove that if the double domination number of a tree is equal to its total domination number plus one, then the tree belongs to the family $\mathcal{T}$.

Lemma 8 *Let T be a tree. If $\gamma_d(T) = \gamma_t(T)+1$, then $T \in \mathcal{T}$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\text{diam}(T) = 1$, then $T = P_2$. We have $\gamma_d(T) = 2 = \gamma_t(T) \neq \gamma_t(T)+1$. If $\text{diam}(T) = 2$, then T is a star $K_{1,m}$. If $T = P_3$, then $T \in \mathcal{T}$. Now assume that T is a star different than P_3. We have $\gamma_d(T) = m+1 \geq 3+1 > 2+1 = \gamma_t(T)+1$. Now let us assume that $\text{diam}(T) = 3$. Thus T is a double star. We have $\gamma_d(T) = n \geq 4 > 3 = 2+1 = \gamma_t(T)+1$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z mean leaves adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_t(T')$-set. By Observation 1 we have $x \in D'$. Of course, D' is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T')$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $y, z, x \in D$. It is easy to see that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 1 = \gamma_t(T) \leq \gamma_t(T')$. This is a contradiction as by Lemma 5 we have $\gamma_d(T') > \gamma_t(T')$. Thus every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) \geq 3$. Assume that u is adjacent to a leaf, say x. Let $T' = T - T_v$. Let D' be any $\gamma_t(T')$-set. By Observation 1 we have $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $t, x, v, u \in D$. It is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2$ $= \gamma_t(T) - 1 \leq \gamma_t(T')$, a contradiction.

Thus every descendant of u is a support vertex. Let x mean a child of u different than v. Let $T' = T - T_v$. Let D' be a $\gamma_t(T')$-set that contains no leaf. The vertex x has to have a neighbor in D', thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $t, v, x \in D$. By Lemma 6 we have $u \in D$. It is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2$ $= \gamma_t(T) - 1 \leq \gamma_t(T')$, a contradiction.

Now assume that $d_T(u) = 2$. Let $T' = T - T_u$. If $T' = P_2$, then $T = P_5$. Obviously, $P_5 = A_1 \in \mathcal{T}$. Now assume that $T' \neq P_2$. Let D' be any $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 3 and 4 we have $t, v \in D$. Observe that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_t(T) - 1 \leq \gamma_t(T') + 1$. This implies that $\gamma_d(T') = \gamma_t(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. If $T' = P_3$, then $T = P_6$. Obviously, $P_6 = B_1 \in \mathcal{T}$. Now assume that $T' \neq P_3$. We distinguish between the following two cases: $T' \in \mathcal{A}$ and $T' \in \mathcal{B}$.

Case 1. $T' \in \mathcal{A}$. Let $T' = A_k$. We consider the labeling of the vertices as in Figure 1. If w corresponds to x, then it is easy to observe that $T = A_{k+1} \in \mathcal{T}$.

Now assume that w corresponds to y. It is easy to see that $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+2$. Now let

D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, t, v \in D$. By Lemma 6 we have $x \in D$. It is easy to see that those vertices do not form a DDS of the tree T. Therefore $\gamma_d(T) \geq 2k+4$. Now we get $\gamma_d(T) \geq 2k+4 > 2k+3 \geq \gamma_t(T)+1$, a contradiction.

Now assume that w corresponds to a_i, for some i. It is easy to see that $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k, x, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+3$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, y, x, t, v \in D$. By Lemma 6 we have $a_i \in D$. Therefore $\gamma_d(T) \geq 2k+5$. Now we get $\gamma_d(T) \geq 2k+5 > 2k+4 \geq \gamma_t(T)+1$, a contradiction.

Now assume that w corresponds to b_i, for some i. Let us observe that $\{a_1, b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, b_i, a_{i+1}, b_{i+1}, \ldots, a_k, b_k, x, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+2$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, y, x, t, v \in D$. Therefore $\gamma_d(T) \geq 2k+4$. Now we get $\gamma_d(T) \geq 2k+4 > 2k+3 \geq \gamma_t(T)+1$, a contradiction.

Now assume that w corresponds to c_i, for some i. Observe that $\{a_1, b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, a_i, a_{i+1}, b_{i+1}, \ldots, a_k, b_k, x, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+2$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_{i-1}, b_{i-1}, c_{i+1}, b_{i+1}, \ldots, c_k, b_k, y, x, t, v \in D$. Observe that adding any one of the remaining vertices to those vertices does not give us a DDS of the tree T. Therefore $\gamma_d(T) \geq 2k+4$. Now we get $\gamma_d(T) \geq 2k+4 > 2k+3 \geq \gamma_t(T)+1$, a contradiction.

Case 2. $T' \in \mathcal{B}$. Let $T' = B_k$. Let us consider the labeling of the vertices as in Figure 1. If w corresponds to x, then it is easy to see that $T = B_{k+1} \in \mathcal{T}$.

Now assume that w corresponds to z. Observe that $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k, z, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+3$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, t, v \in D$. By Lemma 6 we have $x \in D$. Let us observe that adding any one of the remaining vertices to those vertices does not give us a DDS of the tree T. Therefore $\gamma_d(T) \geq 2k+5$. Now we get $\gamma_d(T) \geq 2k+5 > 2k+4 \geq \gamma_t(T)+1$, a contradiction.

Now assume that w corresponds to y. Observe that $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k, y, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+3$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, z, y, t, v \in D$. By Lemma 6 we have $x \in D$. Therefore $\gamma_d(T) \geq 2k+5$. Now we get $\gamma_d(T) \geq 2k+5 > 2k+4 \geq \gamma_t(T)+1$, a contradiction.

Now assume that w corresponds to a_i, for some i. Observe that $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k, x, y, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+4$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, z, y, t, v \in D$. By Lemma 6 we have $x, a_i \in D$. Therefore $\gamma_d(T) \geq 2k+6$. Now we get $\gamma_d(T) \geq 2k+6 > 2k+5 \geq \gamma_t(T)+1$, a contradiction.

Now assume that w corresponds to b_i, for some i. Let us observe that $\{a_1, b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, b_i, a_{i+1}, b_{i+1}, \ldots, a_k, b_k, x, y, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k + 3$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_k, b_k, z, y, t, v \in D$. By Lemma 6 we have $x \in D$. Therefore $\gamma_d(T) \geq 2k + 5$. Now we get $\gamma_d(T) \geq 2k + 5 > 2k + 4 \geq \gamma_t(T) + 1$, a contradiction.

Now assume that w corresponds to c_i, for some i. Let us observe that $\{a_1, b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, a_i, a_{i+1}, b_{i+1}, \ldots, a_k, b_k, x, y, u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq 2k+3$. Now let D be any $\gamma_d(T)$-set. By Observations 3 and 4 we have $c_1, b_1, c_2, b_2, \ldots, c_{i-1}, b_{i-1}, c_{i+1}, b_{i+1}, \ldots, c_k, b_k, z, y, t, v \in D$. By Lemma 6 we have $x \in D$. Observe that adding any one of the remaining vertices to those vertices does not give us a DDS of the tree T. Therefore $\gamma_d(T) \geq 2k+5$. Now we get $\gamma_d(T) \geq 2k+5 > 2k+4 \geq \gamma_t(T)+1$, a contradiction. ■

As an immediate consequence of Lemmas 7 and 8, we have the following characterization of the trees with double domination number equal to total domination number plus one.

Theorem 9 *Let T be a tree. Then $\gamma_d(T) = \gamma_t(T)+1$ if and only if $T \in \mathcal{T}$.*

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