Advanced Nonlinear Studies **15** (2015), 613–628

On Von Kármán Equations and the Buckling of a Thin Circular Elastic Plate

Joanna Janczewska,* Anita Zgorzelska

Faculty of Applied Physics and Mathematics
Gdańsk University of Technology, 80-233 Gdańsk, Poland
e-mail: janczewska@mif.pg.gda.pl, azgorzelska@mif.pg.gda.pl

Hanna Guze

Mathematics Teaching and Distance Learning Centre
Gdańsk University of Technology, 80-233 Gdańsk, Poland
e-mail: hanna.guze@pg.gda.pl

Received 11 January 2015
Communicated by Jean Mawhin

Abstract

We shall be concerned with the buckling of a thin circular elastic plate simply supported along a boundary, subjected to a radial compressive load uniformly distributed along its boundary. One of the main engineering concerns is to reduce deformations of plate structures. It is well known that von Kármán equations provide an established model that describes nonlinear deformations of elastic plates. Our approach to study plate deformations is based on bifurcation theory. We will find critical values of the compressive load parameter by reducing von Kármán equations to an operator equation in Hölder spaces with a nonlinear Fredholm map of index zero. We will prove a sufficient condition for bifurcation by the use of a gradient version of the Crandall-Rabinowitz theorem due to A.Yu. Borisovich and basic notions of representation theory. Moreover, applying the key function method by Yu.I. Sapronov we will investigate the shape of bifurcation branches.

1991 Mathematics Subject Classification. 35B32, 74G20, 74K20, 74B20.
Key words. bifurcation, von Kármán equations, elastic plate, nonlinear Fredholm operator, variational gradient, variational approach, group action

*The author acknowledges the support of National Science Centre of Poland (grant no. 2011/03/B/ST1/04533).

1 Introduction

Following J.N. Reddy in [27]: ”A plate is a flat structural element with planform dimensions much larger then its thickness and is subjected to loads that cause bending deformation in addition to stretching. Street manhole covers, table tops, side panels and roofs of buildings and transportation vehicles, glass window panels, turbine disks, bulkheads and tank bottoms provide familiar examples of plate structures.” This is enough to explain a wide-spread interest in the engineering and applied sciences in the study of elastic plate models. Particularly, von Kármán equations provide an estabished model that describes nonlinear deformations of elastic plates. These equations have attracted a great deal of interest in the literature (see [10, 11, 12, 14, 20, 21, 23, 26, 27, 32]). The derivation of equations goes back to the work of Theodore von Kármán (see [18]). In this paper we will be concerned with nonlinear deformations of a thin circular elastic plate simply supported along a boundary, subjected to a radial compressive load λ uniformly distributed along its boundary (see Fig. 1).

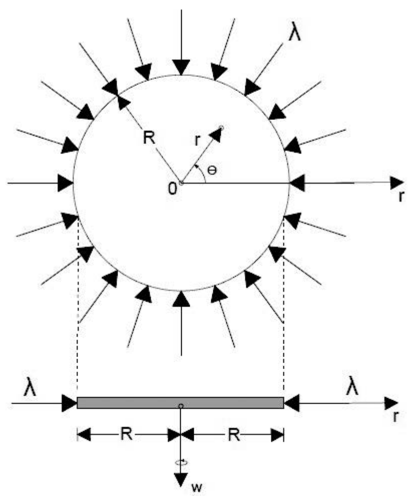

Figure 1: A circular plate under uniform radial compressive load

If the load is zero then the plate is undeformed and flat. If the load increases then the plate buckles. One of the main engineering concerns is to reduce deformations of plate structures.

Let us consider a corresponding mathematical model. To this end, introduce the Cartesian coordinate system (u, v, w) as in the picture (see Fig. 2). The unloaded plate is a disc in $\mathbb{R}^2$ of variables (u, v), centered at the origin, of radius R. Without loss of generality we will assume that $R = 1$. Set

$$\Omega = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}.$$

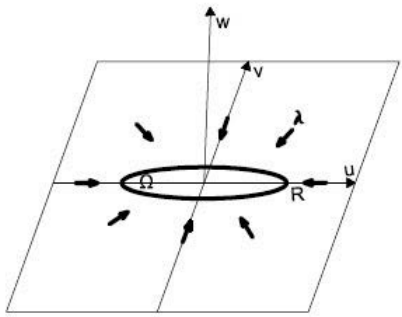

Figure 2: A plate in Cartesian coordinates

A vertical displacement of the plate in a point (u, v) will be denoted by w. The buckling of the plate will be described by a deflection function $w = w(u, v)$ defined in $\overline{\Omega}$. The deformation of the plate causes the appearance of internal stresses, including membrane stresses, which can be described by Airy's stress function $\sigma = \sigma(u, v)$. A pair $(w(u, v), \sigma(u, v))$ is called an equilibrium form of the plate.

Let $C^{m,\mu}(\overline{\Omega})$, where $m \in \mathbb{N} \cup \{0\}$ and $\mu \in (0, 1)$, denote the Hölder space consisting of functions $f \in C^m(\overline{\Omega})$ such that f and all partial derivatives of f up to the order m satisfy in Ω the Hölder condition of exponent μ. It is well known that $C^{m,\mu}(\overline{\Omega})$ with the norm given by

$$\begin{aligned} \|f\|_{C^{m,\mu}(\overline{\Omega})} &= \max_{0\le|\alpha|\le m} \sup_{\substack{(u_1,v_1)\neq(u_2,v_2) \\ (u_i,v_i)\in\Omega}} \frac{|D^\alpha f(u_1, v_1) - D^\alpha f(u_2, v_2)|}{((u_1 - u_2)^2 + (v_1 - v_2)^2)^{\frac{\mu}{2}}} + \\ &+ \|f\|_{C^m(\overline{\Omega})}, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha_i \in \mathbb{N} \cup \{0\}$ and $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} u \partial^{\alpha_2} v}$, is a Banach space.

Let us assume that functions $w(u, v)$ and $\sigma(u, v)$ are $C^{4,\mu}$-smooth. They may be found as solutions of von Kármán equations

$$\begin{cases} \Delta^2 w - [w, \sigma] + 2\lambda\Delta w = 0, \\ \Delta^2\sigma + \frac{1}{2}[w, w] = 0 \qquad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a compressive load parameter, Δ is the Laplace operator, and $[\cdot, \cdot]$ denotes von Kármán bracket (sometimes called the Monge-Ampere form) given by

$$[w, \sigma] = \frac{\partial^2 w}{\partial u^2}\frac{\partial^2\sigma}{\partial v^2} - 2\frac{\partial^2 w}{\partial u\partial v}\frac{\partial^2\sigma}{\partial u\partial v} + \frac{\partial^2 w}{\partial v^2}\frac{\partial^2\sigma}{\partial u^2}.$$

The system of 4th order partial differential equations (1.1) is derived from equilibrium conditions of all internal forces acting on infinitely small element of the plate. Forces perpendicular to the midplane of undeformed plate gives the first equation. Equilibrium conditions of membrane forces (forces acting in the midplane) yields the second one. We will assume that the plate is simply supported along the boundary $\partial\Omega$. It means that functions $w(u, v)$ and $\sigma(u, v)$ satisfy the following boundary conditions

$$\begin{cases} \Delta w = w = 0, \\ \Delta\sigma = \sigma = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For each $\lambda > 0$, the boundary value problem (1.1)-(1.2) possesses the trivial solution $w_0(u, v) = \sigma_0(u, v) \equiv 0$ defined on $\overline{\Omega}$. This solution corresponds to the unbuckled plate (see Fig. 2). If λ reaches a first critical value λ_1 then the plate may buckle.

1.1 Bifurcation theory: a tool in studying deformations of elastic plates

Bifurcation theory is one of the most powerful tools used in studying deformations of elastic beams, plates and shells. Numerous works have been devoted to the study of bifurcation in von Kármán problems (see for instance [1, 2, 3, 5, 6, 7, 8, 15, 16, 17, 19].

Let X, Y be Banach spaces and U be an open subset of $\mathbb{R}$. Let us consider an operator equation

$$F(x, \lambda) = 0, \tag{1.3}$$

where $F \colon X \times U \to Y$ is a continuous map. Assume that the equation (1.3) possesses the set of trivial solutions

$$\Gamma = \{(x_0, \lambda)\colon \text{ for all } \lambda \in U\}.$$

At some critical values of λ this family may split into several new branches. It is of interest to know whether there exists a branching of new solutions of the equation (1.3) at a certain value of λ. Bifurcation theory provides answers for this question.

Definition 1.1 $\lambda_0 \in U$ is called a bifurcation point of (1.3) if there exists another branch of solutions $(x(t), \lambda(t))$, depending on $|t| < \varepsilon$, with $\lambda(0) = \lambda_0$ and $x(0) = x_0$.

If the Fréchet derivative of F with respect to x at the point (x_0, λ_0) exists and is a Fredholm map of index zero then, by implicit function theorem, the necessary condition for bifurcation at λ_0 is that $\dim\ker F'_x(x_0, \lambda_0) > 0$.

General bifurcation theorems provide sufficient conditions for λ_0 to be a bifurcation point (see for example [9, 13, 22, 24, 25, 28, 29, 30, 31]).

Methods of bifurcation theory are applied to mathematical physics, in particular, mechanics of elastic constructions, hydromechanics and Lagrangian systems.
A broad class of problems arising in applications is modeled by a nonlinear functional equation like (1.3), where a parameter λ has a physical interpretation. For example, λ can be a value of loading or an elasticity coefficient in some elasticity problems, the Rayleigh number in hydrodynamics.

1.2 Work plan

One of the main engineering concerns is to reduce deformations of plate structures. From the mathematical point of view, this means the study of bifurcation from the set of trivial solutions of (1.1)-(1.2).

Our purpose is to prove a sufficient condition for $\lambda \in \mathbb{R}_+$ to be a bifurcation point of the von Kármán equations (1.1)-(1.2).

We are going to establish critical values of the compressive load parameter λ in (1.1) by reducing the boundary problem (1.1)-(1.2) to an operator equation like (1.3) with a nonlinear Fredholm map of index 0 for which classical and modern bifurcation theory may be applied. For example, we are going to use a gradient version of the Crandall-Rabinowitz theorem due to A.Yu. Borisovich, basic notions of representation theory and the method of key function due to Yu.I. Sapronov.

To our knowledge, there are no previous papers on the boundary problem (1.1)-(1.2) that use a ”gradient structure” of von Kármán equations. The study of von Kármán equations with applications to their gradient structure started with [15] and has been continued in [5, 6, 7, 16, 17]. However, all the works cited above deal with two or three parameter von Kármán equations (models other than our).

The paper is organized as follows. In Section 2 we write the boundary value problem (1.1)-(1.2) in the form of an operator equation like (1.3) in suitable Banach spaces, where a parameter λ is a value of the compressive load. We prove that F is a nonlinear Fredholm map of index 0, and zeros of F are critical points of an energy functional E. Moreover, we state a sufficient condition for λ to be a bifurcation point of the von Kármán equations (1.1)-(1.2). In Section 3 we find critical values of the compressive load parameter. Section 4 provides a detailed proof of a sufficient condition for bifurcation. Finally, in Section 5 we determine the shape of branches of nontrivial solutions of (1.1)-(1.2).

2 Fredholm’s and variational nature of von Kármán equations

We start with introducing the notation. Set

$$C^{4,\mu}_{0,0}(\overline{\Omega}) = \{f \in C^{4,\mu}(\overline{\Omega}) \colon \Delta f = f = 0 \text{ on } \partial\Omega\}$$

and

$$C^{2,\mu}_{0}(\overline{\Omega}) = \{f \in C^{2,\mu}(\overline{\Omega}) \colon f = 0 \text{ on } \partial\Omega\}.$$

To simplify notation, we use the same letters: X, Y, H, Γ, F and E in an abstract and our case. Let

$$X = C^{4,\mu}_{0,0}(\overline{\Omega}) \times C^{4,\mu}_{0,0}(\overline{\Omega}) \text{ and } Y = C^{0,\mu}(\overline{\Omega}) \times C^{0,\mu}(\overline{\Omega}).$$

The norm of an element $x = (w, \sigma)$ in X and Y is defined by coordinates as the maximum of norms of w and σ in $C^{4,\mu}(\overline{\Omega})$ and $C^{0,\mu}(\overline{\Omega})$ respectively. The operator $F \colon X \times \mathbb{R}_+ \to Y$ is defined by

$$F(x,\lambda) = \left(\Delta^2 w - [w,\sigma] + 2\lambda\Delta w, -\Delta^2\sigma - \frac{1}{2}[w,w]\right), \tag{2.1}$$

where $x = (w,\sigma)$. We see at once that the operator equation

$$F(x,\lambda) = 0 \tag{2.2}$$

is equivalent to the boundary value problem (1.1)-(1.2). Define

$$\Gamma = \{(0,\lambda) \in X \times \mathbb{R}_+ : \lambda \in \mathbb{R}_+\}. \tag{2.3}$$

Γ is said to be the set of trivial solutions of (2.2) corresponding to the undeflected plate.

The operator F is easily seen to be C^∞-smooth. What is more, an easy computation shows that

$$F'_x(x,\lambda)h = (\Delta^2 z - [z,\sigma] - [w,\eta] + 2\lambda\Delta z, -\Delta^2\eta - [w,z]), \tag{2.4}$$

where $x = (w,\sigma)$ and $h = (z,\eta)$.

Lemma 2.1 *For each $\lambda \in \mathbb{R}_+$, $F'_x(0,\lambda)\colon X \to Y$ is a Fredholm map of index zero.*

Proof. Fix $\lambda \in \mathbb{R}_+$. Substituting $x_0 = (0,0)$ into (2.4) we obtain

$$F'_x(0,\lambda)h = (\Delta^2 z + 2\lambda\Delta z, -\Delta^2\eta)$$

for each $h = (z,\eta) \in X$. Thus $F'_x(0,\lambda)\colon X \to Y$ can be written as the sum

$$F'_x(0,\lambda)h = A(h) + B(h),$$

where $A, B\colon X \to Y$ are given by

$$A(h) = (\Delta^2 z, -\Delta^2\eta) \ \text{ and } \ B(h) = (2\lambda\Delta z, 0). \tag{2.5}$$

Now, to finish the proof, it is sufficient to show that $A\colon X \to Y$ is an isomorphism, and $B\colon X \to Y$ is a completely continuous map.

It is well known that the Laplace operator $\Delta\colon C_0^{m,\mu}(\overline{\Omega}) \to C^{m-2,\mu}(\overline{\Omega})$ is an isomorphism, where $m \geq 2$ and $\mu \in (0,1)$. Hence $\Delta^2\colon C_{0,0}^{4,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$ is an isomorphism, and, in consequence, A is an isomorphism of X onto Y. Let us remark that $B\colon X \to Y$ is completely continuous. Let $B_1 : C_{0,0}^{4,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$ be defined by

$$B_1(z) = 2\lambda\Delta z.$$

Then

$$B(z,\eta) = (B_1(z), 0).$$

By the diagram

$$C_{0,0}^{4,\mu}(\overline{\Omega}) \xrightarrow{J} C_0^{2,\mu}(\overline{\Omega}) \xrightarrow{W=2\lambda\Delta} C^{0,\mu}(\overline{\Omega}),$$

where J is the natural embedding of $C^{4,\mu}_{0,0}(\overline{\Omega})$ into $C^{2,\mu}_{0}(\overline{\Omega})$, we have $B_1 = W \circ J$. Since J is completely continuous and W is continuous, we conclude that B_1 and B are completely continuous. This is the end of the proof.

Write

$$H = L^2(\Omega) \times L^2(\Omega).$$

It is known that, under the scalar product

$$((w,\sigma),(z,\eta))_H = \frac{1}{\pi}\iint_\Omega (wz + \sigma\eta)dudv,$$

H is a Hilbert space. The functional $E\colon X \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\begin{aligned} E(x,\lambda) &= \frac{1}{2\pi}\iint_\Omega \Big((\Delta w)^2 - (\Delta\sigma)^2 - [w,w]\sigma\Big)\,dudv + \\ &\quad -\frac{1}{2\pi}\iint_\Omega 2\lambda\left(\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2\right)dudv. \end{aligned} \tag{2.6}$$

In fact, E is the total energy of the plate and is C^∞-smooth. Moreover, it is easy to check that

$$E'_x(x,\lambda)h = (F(x,\lambda),h)_H \tag{2.7}$$

for all $x, h \in X$ and $\lambda \in \mathbb{R}_+$. Hence we say that the map F is a *variational gradient* of the functional E with respect to the inner product in H.

We can now formulate our main result.

Theorem 2.1 *The necessary and sufficient condition for* $\lambda_0 \in \mathbb{R}_+$ *to be a bifurcation point of the equation* (2.2) *is* $\dim\ker F'_x(0,\lambda_0) > 0$.

With respect to the dimension of the space $\ker F'_x(0,\lambda)$ at $\lambda \in \mathbb{R}_+$, we can divide bifurcation points into simple if $\dim\ker F'_x(0,\lambda) = 1$ and multiple if $\dim\ker F'_x(0,\lambda) > 1$.

Our proof of Theorem 2.1 is based on a gradient version of the Crandall-Rabinowitz theorem on simple bifurcation points. The original theorem was proved by Michael G. Crandall and Paul H. Rabinowitz in [13]. A gradient one is due to Andrey Yu. Borisovich (see [4]) and deals with the case when F is a variational gradient as above. For the convenience of the reader we state this theorem.

Theorem 2.2 *Assume that H is a Hilbert space with a scalar product* $(\cdot,\cdot)_H$. *Let X and Y be Banach spaces continuously embedded in H. Suppose that a C^r-operator* $F\colon X_\delta(0) \times \mathbb{R}_\delta(\lambda_0) \to Y$ *and a C^{r+1}-functional* $E\colon X_\delta(0) \times \mathbb{R}_\delta(\lambda_0) \to \mathbb{R}$, *where* $r \geq 2$, *satisfy the following conditions:*

1. $F(0,\lambda) = 0$ *for all* $\lambda \in \mathbb{R}_\delta(\lambda_0)$,
2. $\dim\ker F'_x(0,\lambda_0) = 1$, $F'_x(0,\lambda_0)e = 0$, $(e,e)_H = 1$,

3. $\operatorname{codim} \operatorname{im} F'_x(0, \lambda_0) = 1$,

4. *$E'_x(x, \lambda)h = (F(x, \lambda), h)_H$ for all $(x, \lambda) \in X_\delta(0) \times \mathbb{R}_\delta(\lambda_0)$ and $h \in X$,*

5. $E'''_{xx\lambda}(0, \lambda_0)(e, e, 1) \neq 0$.

Then λ_0 is a bifurcation point of the equation $F(x, \lambda) = 0$. In fact, the solution set of this equation in some neighbourhood of the point $(0, \lambda_0)$ consists of the curve $\Gamma = \{(0, \lambda) \colon \lambda \in \mathbb{R}_\delta(\lambda_0)\}$ and a C^{r-2}-curve Λ, intersecting only at $(0, \lambda_0)$. Moreover, if $r \geq 3$, the curve Λ can be parametrized by a variable t, $|t| < \varepsilon$, as

$$\Lambda = \{(x(t), \lambda(t)) \colon t \in \mathbb{R}_\varepsilon(0)\}, \quad x(0) = 0, \quad \lambda(0) = \lambda_0 \quad x'(0) = e.$$

Here and subsequently, $X_\delta(0)$, $\mathbb{R}_\delta(\lambda_0)$ and $\mathbb{R}_\varepsilon(0)$ denote respective balls in X and $\mathbb{R}$.

3 The study of linearized problem

From now on, for each $\lambda \in \mathbb{R}_+$, $N(\lambda)$ stands for $\ker F'_x(0, \lambda)$.

Definition 3.1 A number $\lambda > 0$ is called a critical value of parameter if

$$\dim N(\lambda) > 0.$$

The aim of this section is to find all critical values of parameter λ. To do this, fix $\lambda > 0$ and consider the equation

$$F'_x(0, \lambda)h = 0. \tag{3.1}$$

It follows that the equation (3.1) is equivalent to the boundary value problem

$$\begin{cases} \Delta^2 z + 2\lambda \Delta z = 0, & \\ \Delta^2 \eta = 0 & \text{in } \Omega, \\ \Delta z = z = 0, & \\ \Delta \eta = \eta = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $h = (z, \eta)$. Since $\Delta \colon C_0^{2,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$ is an isomorphism, the boundary value problem (3.2) can be replaced by the following one

$$\begin{cases} \Delta z + 2\lambda z = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$N(\lambda) = \ker(\Delta + 2\lambda I) \times \{0\}, \tag{3.3}$$

where $\Delta + 2\lambda I \colon C_0^{2,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$ and I is the natural embedding of $C_0^{2,\mu}(\overline{\Omega})$ into $C^{0,\mu}(\overline{\Omega})$.

For each $k \in \mathbb{N} \cup \{0\}$, let $J_k \colon \mathbb{R} \to \mathbb{R}$ denote the kth Bessel function of 1st kind. The function J_k is given by

$$J_k(s) = \frac{1}{\pi}\int_0^{\pi} \cos(s \sin t - kt)dt.$$

Let $(r, \varphi) \in [0, 1] \times [0, 2\pi)$ denote the polar coordinates of a point $(u, v) \in \overline{\Omega}$.

It is well known that a number $-2\lambda < 0$ is an eigenvalue of the Laplace operator $\Delta\colon C_0^{2,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$ if and only if $\sqrt{2\lambda} > 0$ is a zero of one of Bessel's functions of 1st kind. If $J_0(\sqrt{2\lambda}) = 0$ then $\dim \ker(\Delta + 2\lambda I) = 1$ and $\ker(\Delta + 2\lambda I)$ is spanned by the function $J_0(\sqrt{2\lambda} r)$. If there exists $k \in \mathbb{N}$ such that $J_k(\sqrt{2\lambda}) = 0$ then $\dim \ker(\Delta + 2\lambda I) = 2$ and $\ker(\Delta + 2\lambda I)$ is spanned by: $J_k(\sqrt{2\lambda} r)\cos(k\varphi)$ and $J_k(\sqrt{2\lambda} r)\sin(k\varphi)$.

Summarizing. $\lambda > 0$ is a critical value of parameter if and only if $\sqrt{2\lambda} > 0$ is a zero of one of the Bessel functions J_k.

By the above, we get a complete description of $N(\lambda)$. Namely, we have three possible cases.

- If $J_k(\sqrt{2\lambda}) \neq 0$ for each $k \in \mathbb{N} \cup \{0\}$ then $N(\lambda)$ is a trivial space.
- If $J_0(\sqrt{2\lambda}) = 0$ then $\dim N(\lambda) = 1$ and $N(\lambda)$ is spanned by $(J_0(\sqrt{2\lambda} r), 0)$.
- If there is $k \in \mathbb{N}$ such that $J_k(\sqrt{2\lambda}) = 0$ then $\dim N(\lambda) = 2$ and $N(\lambda)$ is spanned by $(J_k(\sqrt{2\lambda} r)\cos(k\varphi), 0)$ and $(J_k(\sqrt{2\lambda} r)\sin(k\varphi), 0)$.

4 The proof of Theorem 2.1

The proof of Theorem 2.1 is based on Theorem 2.2 and will be divided into two cases. The first case when $\dim N(\lambda) = 1$ and the second when $\dim N(\lambda) = 2$.

4.1 Simple bifurcation points

Theorem 4.1 *Let $\lambda_0 \in \mathbb{R}_+$ be a critical value of compressive load parameter such that $\dim N(\lambda_0) = 1$. Set $e \in N(\lambda_0)$ such that $(e, e)_H = 1$. Then λ_0 is a bifurcation point of the equation* (2.2). *In fact, the solution set of this equation in some neighbourhood of the point* $(0, \lambda_0)$ *consists of the trivial family* (2.3) *and a C^∞-curve*

$$\Lambda = \{(x(t), \lambda(t))\colon t \in \mathbb{R}_\varepsilon(0)\}, \quad x(0) = 0, \quad \lambda(0) = \lambda_0 \quad x'(0) = e,$$

intersecting only at $(0, \lambda_0)$.

Proof. We only need to show that $E'''_{xx\lambda}(0, \lambda_0)(e, e, 1) \neq 0$. Differentiating (2.7) with respect to x we have

$$E''_{xx}(x, \lambda)(h, g) = (F'_x(x, \lambda)h, g)_H \tag{4.1}$$

for each $\lambda \in \mathbb{R}_+$ and for all $x, h, g \in X$. Substituting (2.4) into (4.1) we get

$$E''_{xx}(x,\lambda)(h,g) = \frac{1}{\pi}\iint_\Omega (\Delta^2 z - [z,\sigma] - [w,\eta] + 2\lambda\Delta z)z_1 dudv + \\ +\frac{1}{\pi}\iint_\Omega (-\Delta^2\eta - [w,z])\eta_1 dudv,$$

where $x = (w,\sigma)$, $h = (z,\eta)$ and $g = (z_1,\eta_1)$. From (3.3) it follows that $e = (e_1, 0)$ and $\Delta e_1 = -2\lambda_0 e_1$. By the above, we see at once that

$$E'''_{xx\lambda}(0,\lambda_0)(e,e,1) = \frac{1}{\pi}\iint_\Omega 2(\Delta e_1)e_1 dudv = -\frac{4\lambda_0}{\pi}\iint_\Omega e_1^2 dudv = \\ -4\lambda_0(e,e)_H = -4\lambda_0 < 0.$$

This is the end of the proof.

4.2 Multiple bifurcation points

4.2.1 The action of $\mathbb{Z}_2$ on $\mathbb{R}^2$

Let $\mathbb{Z}_2$ denote the finite group of $\{0,1\}$ with mod 2 addition. Let $GL(\mathbb{R}^2)$ denote the group of linear automorphisms of $\mathbb{R}^2$ with composition of maps. The homomorphism $\varrho\colon \mathbb{Z}_2 \to GL(\mathbb{R}^2)$ defined by $\varrho(0) = I$, $\varrho(1) = S_u$, where $I(u,v) = (u,v)$ and $S_u(u,v) = (u,-v)$ is a linear representation of $\mathbb{Z}_2$ in $GL(\mathbb{R}^2)$.

Definition 4.1 A set $U \subset \mathbb{R}^2$ is said to be $\mathbb{Z}_2$-invariant if $(u,v) \in U$ implies $(u,-v) \in U$.

Definition 4.2 Let $U \subset \mathbb{R}^2$ be a $\mathbb{Z}_2$-invariant set. A function $f\colon U \to \mathbb{R}^k$ $(k \in \mathbb{N})$ is called $\mathbb{Z}_2$-equivariant if $f(u,v) = f(u,-v)$ for each $(u,v) \in U$.

Set $Z \subset \{f\colon f\colon U \to \mathbb{R}^k\}$, where U is a $\mathbb{Z}_2$-invariant subset in $\mathbb{R}^2$. The subset of all $\mathbb{Z}_2$-equivariant functions in Z will be denoted by $Z^{\mathbb{Z}_2}$. Remark that if Z is a linear space then $Z^{\mathbb{Z}_2}$ is a linear subspace in Z.

Definition 4.3 Let $E_1, E_2 \subset \{f\colon f\colon U \to \mathbb{R}^k\}$ be real Banach spaces such that if $f \in E_i$ then $f \circ S_u \in E_i$ for $i = 1,2$.

- A map $P\colon E_1 \to E_2$ is said to be $\mathbb{Z}_2$-equivariant if for each $f \in E_1$,

$$P(f \circ S_u) = P(f) \circ S_u.$$

- A map $P\colon E_1 \times S \to E_2$ $(S \subset \mathbb{R})$ is called $\mathbb{Z}_2$-equivariant if for each parameter $s \in S$, $P(\cdot, s)\colon E_1 \to E_2$ is $\mathbb{Z}_2$-equivariant.

The restriction of $P\colon E_1 \to E_2$ $(P\colon E_1 \times S \to E_2)$ to the space $E_1^{\mathbb{Z}_2}$ $(E_1^{\mathbb{Z}_2} \times S)$ will be denoted by $P^{\mathbb{Z}_2}$.

4.2.2 Multiple bifurcation with $\mathbb{Z}_2$-symmetries

Lemma 4.1 *Fix $\lambda \in \mathbb{R}_+$. If there exists $k \in \mathbb{N}$ such that $J_k(\sqrt{2\lambda}) = 0$ then $\dim N(\lambda)^{\mathbb{Z}_2} = 1$ and $N(\lambda)^{\mathbb{Z}_2}$ is spanned by $(J_k(\sqrt{2\lambda}r)\cos(k\varphi), 0)$.*

Proof. Let $x = (w, 0) \in N(\lambda)$. By assumption it follows that there are real constants C_1 and C_2 such that

$$w(u, v) = C_1 J_k(\sqrt{2\lambda}r)\cos(k\varphi) + C_2 J_k(\sqrt{2\lambda}r)\sin(k\varphi)$$

for all $(u, v) \in \overline{\Omega}$. In addition, if x is $\mathbb{Z}_2$-equivariant then $w(u, v) = w(u, -v)$, and so

$$C_1 J_k(\sqrt{2\lambda}r)\cos(k\varphi) + C_2 J_k(\sqrt{2\lambda}r)\sin(k\varphi) = \\ C_1 J_k(\sqrt{2\lambda}r)\cos(k\varphi) - C_2 J_k(\sqrt{2\lambda}r)\sin(k\varphi)$$

for all $r \in [0, 1]$ and $\varphi \in [0, 2\pi)$. Hence we get $2C_2 J_k(\sqrt{2\lambda}r)\sin(k\varphi) = 0$, which implies $C_2 = 0$, and consequently

$$w(u, v) = C_1 J_k(\sqrt{2\lambda}r)\cos(k\varphi).$$

This is the end of the proof.

Lemma 4.2 *The map $F : X \times \mathbb{R}_+ \to Y$ given by* (2.1) *is $\mathbb{Z}_2$-equivariant.*

Proof. We check at once that

(*i*) $\Delta(w \circ S_u) = (\Delta w) \circ S_u$ for each $w \in C^2(\overline{\Omega})$,

(*ii*) $\Delta^2(w \circ S_u) = (\Delta^2 w) \circ S_u$ for each $w \in C^4(\overline{\Omega})$,

(*iii*) $[w \circ S_u, \sigma \circ S_u] = [w, \sigma] \circ S_u$ for all $w, \sigma \in C^2(\overline{\Omega})$.

Combining (*i*) – (*iii*) with (2.1) we have

$$F(x \circ S_u, \lambda) = F(x, \lambda) \circ S_u$$

for all $x \in X$ and $\lambda \in \mathbb{R}_+$. This is the end of the proof.

By Lemma 4.2 we conclude that if $(x, \lambda) \in X^{\mathbb{Z}_2} \times \mathbb{R}_+$ then

$$F(x, \lambda) = F(x \circ S_u, \lambda) = F(x, \lambda) \circ S_u.$$

Therefore F maps $X^{\mathbb{Z}_2} \times \mathbb{R}_+$ into $Y^{\mathbb{Z}_2}$.

Convergence in Hölder spaces $C^{m,\mu}(\overline{\Omega})$ implies pointwise convergence. It follows that $X^{\mathbb{Z}_2}$ and $Y^{\mathbb{Z}_2}$ are Banach spaces.

Lemma 4.3 *For each $\lambda \in \mathbb{R}_+$, $(F^{\mathbb{Z}_2})'_x(0, \lambda) : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ is a Fredholm map of index zero.*

Proof. Fix $\lambda \in \mathbb{R}_+$. For each $(z, \eta) \in X$ we have

$$F'_x(0, \lambda)(z, \eta) = A(z, \eta) + B(z, \eta),$$

where $A\colon X \to Y$ and $B\colon X \to Y$ are defined by (2.5). We have proved that A is an isomorphism and B is completely continuous. As $\Delta\colon C^2(\overline{\Omega}) \to C(\overline{\Omega})$ and $\Delta^2\colon C^4(\overline{\Omega}) \to C(\overline{\Omega})$ are $\mathbb{Z}_2$-equivariant, A and B maps $X^{\mathbb{Z}_2}$ into $Y^{\mathbb{Z}_2}$. Hence

$$(F^{\mathbb{Z}_2})'_x(0, \lambda) = A^{\mathbb{Z}_2} + B^{\mathbb{Z}_2}.$$

Since $Y^{\mathbb{Z}_2}$ is a closed subspace in Y, $B^{\mathbb{Z}_2}\colon X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ is completely continuous. To remains to prove that $A^{\mathbb{Z}_2}$ is onto $Y^{\mathbb{Z}_2}$.

Let $(f, g) \in Y^{\mathbb{Z}_2}$. There is $(z, \eta) \in X$ such that $A(z, \eta) = (f, g)$. We get $A(z, \eta) = (\Delta^2 z, -\Delta^2\eta) = (\Delta^2 z, -\Delta^2\eta) \circ S_u = (\Delta^2(z \circ S_u), -\Delta^2(\eta \circ S_u)) = A(z \circ S_u, \eta \circ S_u)$. It follows that $(z, \eta) = (z \circ S_u, \eta \circ S_u) = (z, \eta) \circ S_u$, and, in consequence, $(z, \eta) \in X^{\mathbb{Z}_2}$. This is the end of the proof.

Applying Lemma 4.2 and the equality (2.7) we conclude that

$$(E^{\mathbb{Z}_2})'_x(x, \lambda)h = (F^{\mathbb{Z}_2}(x, \lambda), h)_H$$

for all $x, h \in X^{\mathbb{Z}_2}$ and $\lambda \in \mathbb{R}_+$.

Theorem 4.2 *Let $\lambda_0 \in \mathbb{R}_+$ be a critical value of compressive load parameter such that* $\dim N(\lambda_0) = 2$. *Set $e \in N(\lambda_0)^{\mathbb{Z}_2}$ such that $(e, e)_H = 1$. Then λ_0 is a bifurcation point of the equation*

$$F^{\mathbb{Z}_2}(x, \lambda) = 0. \tag{4.2}$$

In fact, the solution set of this equation in some neighbourhood of the point $(0, \lambda_0)$ in $X^{\mathbb{Z}_2} \times \mathbb{R}_+$ consists of the trivial family $\Gamma = \{0\} \times \mathbb{R}_+$ and a C^∞-curve

$$\hat{\Lambda} = \{(x(t), \lambda(t))\colon t \in \mathbb{R}_\varepsilon(0)\}, \quad x(0) = 0, \quad \lambda(0) = \lambda_0 \quad x'(0) = e,$$

intersecting only at $(0, \lambda_0)$.

What is left is to show that $(E^{\mathbb{Z}_2})'''_{xx\lambda}(0, \lambda_0)(e, e, 1) \neq 0$. The proof is similar to that of Theorem 4.1 and for this reason it will be omitted.

5 Postcritical bifurcation

The last section is devoted to the study of the shape of nontrivial branching at a bifurcation point λ_0 of (2.2). We indicate how the key function method due to Yu.I. Sapronov may be used to investigate bifurcation in the von Kármán equations (1.1)-(1.2).

Fix a critical value of compressive load parameter $\lambda_0 \in \mathbb{R}_+$. Let $i = \dim N(\lambda_0)$. Applying the finite dimensional reduction of Lyapunov-Schmidt type we can reduce the problem of bifurcation from trivial solutions of (2.2) to the problem of bifurcation from trivial solutions of a suitable equation

$$\psi(\xi, \lambda) = 0,$$

where $(\xi, \lambda) \in V \subset \mathbb{R}^i \times \mathbb{R}_+$ and V is a neighbourhood of $(0, \lambda_0)$ in $\mathbb{R}^i \times \mathbb{R}_+$. The function $\psi\colon V \to \mathbb{R}^i$ has the same smoothness as F. Moreover, there is a function $\phi\colon V \to \mathbb{R}$ such that

$$\nabla_\xi \phi(\xi, \lambda) = \psi(\xi, \lambda)$$

and ϕ is called a key function. By the use of the method introduced by Yu.I. Sapronov (see [28, 29]) we may study the shape of nontrivial branching of solutions at λ_0 by analyzing coefficients of the Taylor series of ϕ at $(0, \lambda_0)$.

Taking into account a gradient structure of (2.2) we can express the key function method in terms of derivatives of E at $(0, \lambda_0)$ as follows.

Theorem 5.1 *Let the assumptions of Theorem 2.2 hold with $r \geq 3$. Define*

$$\begin{aligned} C_1 &= E'''_{xx\lambda}(0, \lambda_0)(e, e, 1), \\ C_2 &= E'''_{xxx}(0, \lambda_0)(e, e, e), \\ C_3 &= E^{(4)}_{xxxx}(0, \lambda_0)(e, e, e, e) + 3E'''_{xxx}(0, \lambda_0)(e, e, y), \end{aligned}$$

where y is a unique solution of the equation

$$F'_x(0, \lambda_0)y - (y, e)_H e = -F''_{xx}(0, \lambda_0)(e, e). \tag{5.1}$$

(i) If $C_1 \neq 0$ then the solution set of the equation

$$F(x, \lambda) = 0$$

in a small neighbourhood of $(0, \lambda_0)$ consists of the trivial family

$$\Gamma = \{(0, \lambda)\colon \lambda \in \mathbb{R}_\delta(\lambda_0)\}$$

and a C^{r-2}-smooth curve Λ, intersecting only at $(0, \lambda_0)$.

(ii) If $C_1 \neq 0$ and $C_2 \neq 0$ then Λ can be parametrized as follows

$$x(\lambda) = D_2(\lambda - \lambda_0)e + o(|\lambda - \lambda_0|), \quad \lambda \in \mathbb{R}_\varepsilon(\lambda_0),$$

where $D_2 = -2C_1/C_2$. Then λ_0 is said to be a transcritical bifurcation point (see Fig. 3).

(iii) If $C_1 \neq 0$, $C_2 = 0$ and $C_3 \neq 0$ then the parametrization of Λ depends on the signs of C_1 and C_3. Set $D_3 = -6C_1/C_3$.

If $C_1 \cdot C_3 > 0$ then

$$x^\pm(\lambda) = \pm\sqrt{|D_3|}(\lambda_0 - \lambda)^{\frac{1}{2}} e + o\left(|\lambda_0 - \lambda|^{\frac{1}{2}}\right), \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0].$$

Then λ_0 is said to be a subcritical bifurcation point (see Fig. 3).

If $C_1 \cdot C_3 < 0$ then

$$x^{\pm}(\lambda) = \pm\sqrt{D_3}(\lambda - \lambda_0)^{\frac{1}{2}} e + o\left(|\lambda - \lambda_0|^{\frac{1}{2}}\right), \quad \lambda \in [\lambda_0, \lambda_0 + \varepsilon).$$

Then λ_0 is said to be a postcritical bifurcation point (see Fig. 3).

For the proof of Theorem 5.1 we refer the reader to [17].

Theorem 5.2 *Let $\lambda_0 \in \mathbb{R}_+$ be a critical value of compressive load parameter such that* $\dim N(\lambda_0) = 1$. *Then λ_0 is a postcritical bifurcation point of the equation* (2.2).

Proof. Set $e = (e_1, 0) \in N(\lambda_0)$. According to Theorem 5.1 we have to compute C_1, C_2 and C_3. From what has already been proved (compare the proof of Theorem 4.1), we have

$$C_1 = -4\lambda_0 < 0.$$

An easy computation shows that

$$F''_{xx}(0, \lambda_0)(e, e) = (0, -[e_1, e_1])$$

and

$$F'''_{xxx}(0, \lambda_0)(e, e, e) = 0.$$

Applying (2.7) we obtain

$$C_2 = (F''_{xx}(0, \lambda_0)(e, e), e)_H = 0$$

and

$$C_3 = (F'''_{xxx}(0, \lambda_0)(e, e, e), e)_H + 3(F''_{xx}(0, \lambda_0)(e, e), y) = -\frac{3}{\pi} \iint_{\Omega} [e_1, e_1] y_2 du dv,$$

where $y = (y_1, y_2)$ is a unique solution of (5.1). By the definition of y we deduce that

$$-\Delta^2 y_2 = [e_1, e_1].$$

Hence

$$C_3 = \frac{3}{\pi} \iint_{\Omega} (\Delta^2 y_2) y_2 du dv = \frac{3}{\pi} \iint_{\Omega} (\Delta y_2)^2 du dv > 0.$$

In consequence, $C_1 \cdot C_3 < 0$, and so λ_0 is a postcritical bifurcation point. This is the end of the proof.

Theorem 5.3 *Let $\lambda_0 \in \mathbb{R}_+$ be a critical value of compressive load parameter such that* $\dim N(\lambda_0) = 2$. *Then λ_0 is a postcritical bifurcation point of the equation* (4.2).

The proof is similar to that of Theorem 5.2 with F and E replaced by $F^{\mathbb{Z}_2}$ and $E^{\mathbb{Z}_2}$. The details are left to the reader.

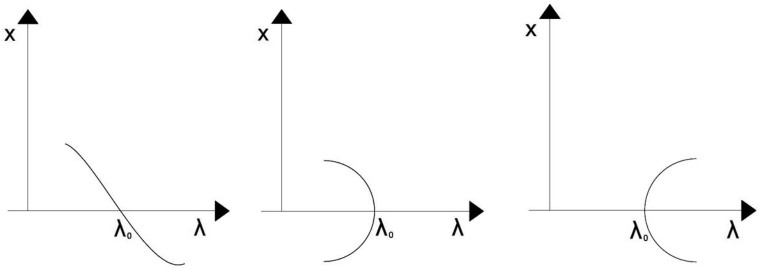

Figure 3: Transcritical, subcritical and postcritical bifurcation

References

[1] S.S. Antman, Nonlinear Problems of Elasticity, Appl. Math. Sci. 107, Springer-Verlag, New York, 1995.

[2] M.S. Berger, *On von Kármán's equations and the buckling of a thin elastic plate, I. The clamped plate*, Commun. Pure Appl. Math. **20** (1967), 687–719.

[3] M.S. Berger, P.C. Fife, *On von Kármán's equations and the buckling of a thin elastic plate, II. Plate with general edge conditions*, Commun. Pure Appl. Math. **21** (1968), 227–241.

[4] A.Yu. Borisovich, Functional-topological Properties of the Plateau Operator and Applications to the Study of Bifurcations in Problems of Geometry and Hydrodynamics, in: Minimal Surfaces, Adv. Soviet Math. 15, Amer. Math. Soc., Providence, RI, 1993, pp. 287–330.

[5] A.Yu. Borisovich, J. Janczewska, *Stable and unstable bifurcation in the von Kármán problem for a circular plate*, Abstr. Appl. Anal. **2005** (2005), no. 8, 889–899.

[6] A.Yu. Borisovich, J. Dymkowska, Cz. Szymczak, *Buckling and postcritical behaviour of the elastic infinite plate strip resting on linear elastic foundation*, J. Math. Anal. Appl. **307** (2005), no. 2, 480–495.

[7] A.Yu. Borisovich, J. Dymkowska, Cz. Szymczak, *Bifurcations in von Kármán problem for rectangular, thin, elastic plate resting on elastic foundation of Winkler type*, AMRX Appl. Math. Res. Express **2006** (2006), Art. ID 82959, 24 pp.

[8] C.S. Chien, M.S. Chen, *Multiple bifurcation in the von Kármán equations*, SIAM J. Sci. Comput. **6** (1997), 1737–1766.

[9] S.N. Chow, J.K. Hale, Methods of Bifurcation Theory, Springer-Verlag, New York-Berlin, 1982.

[10] I. Chueshow, I. Lasiecka, Von Karman Evolution Equations, Well-posedness and Long-Time Dynamics, SMM, Springer, New York, 2010.

[11] P. Ciarlet, P. Rabier, Les Equations de von Kármán, Lect. Notes Math. 826, Springer-Verlag, Berlin, 1980.

[12] P. Ciarlet, Mathematical Elasticity, Theory of Shells, North-Holland, Amsterdam, 2000.

[13] M.G. Crandall, P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal. **8** (1971), 321–340.

[14] E.J. Hölder, D. Schaffer, *Boundary conditions and mode jumping in the von Kármán equations*, SIAM J. Math. Anal. **3** (1984), 446–457.

[15] J. Janczewska, *Bifurcation in the solution set of the von Kármán equations of an elastic disc lying on an elastic foundation*, Ann. Polon. Math. **77** (2001), no. 1, 53–68.

[16] J. Janczewska, *The necessary and sufficient condition for bifurcation in the von Kármán equations*, NoDEA Nonlinear Differential Equations Appl. **10** (2003), no. 1, 73–94.

[17] J. Janczewska, *Multiple bifurcation in the solution set on the von Kármán equations with* S^1*-symmetries*, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), no. 1, 109–126.

[18] T. Von Kármán, Festigkeitsprobleme in Maschinenbau, Encyklopedie der Mathematischen Wissenschaften, 4, Leipzig, (1910), 348–352.

[19] H.B. Keller, J.B. Keller, E.L. Reiss, *Buckled states of circular plates*, Quart. Appl. Math. **20** (1962/1963), 55–65.

[20] E.M. Kramer, *The von Kármán equations, the stress function and elastic ridges in high dimensions*, J. Math. Phys. **2** (1997), 831–846.

[21] J. Lagnese, Boundary Stabilization of Thin Plates, SIAM, Philadelphia, 1989.

[22] J.E. Marsden, On the Geometry of the Lyapounov-Schmidt Procedure, Lect. Notes Math. 755, Springer-Verlag, 1979.

[23] N.F. Morozov, Selected Two-Dimensional Problems of Elasticity Theory, Univ. Press, Leningrad, 1978.

[24] L. Nirenberg, Topics in Nonlinear Functional Analysis, Courant Inst. Math. Sci., 1974.

[25] T. Poston, I. Stewart, Catastrophe Theory and Its Applications, Pitman Lecture Notes, 1978.

[26] B. Rao, *Marguerre-von Kármán equations and membrane model*, Nonl. Anal. **8** (1995), 1131–1140.

[27] J.N. Reddy, Energy Principles and Variational Methods in Applied Mechanics, John Wiley & Sons, Inc., Hoboken, New Jersey, 2002.

[28] Yu.I. Sapronov, Branching of Solutions of Smooth Fredholm Equations, Lect. Notes Math. 1108, Springer-Verlag, 1982.

[29] Yu.I. Sapronov, *Finite-dimensional reduction in smooth extremal problems*, Usp. Mat. Nauk **1** (1996), 101–132.

[30] C.A. Stuart, An Introduction to Bifurcation Theory Based on Differential Calculus, R.J. Knops (ed.), Pitman Lecture Notes, 1979.

[31] V.A. Trenogin, M.M. Vainberg, Branching Theory of Solutions of Nonlinear Equations, Nauka, 1996.

[32] I.I. Voronovich, Mathematical Problems in the Nonlinear Theory of Shells, Nauka, 1989.