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Periodic Solutions of Generalized Lagrangian Systems with Small Perturbations

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Abstract
In this paper we study the generalized Lagrangian system with a small perturbation. We assume the main term in the system to have a maximum, but do not suppose any condition for perturbation term. Then we prove the existence of a periodic solution via Ekeland's principle. Moreover, we prove a convergence theorem for periodic solutions of perturbed systems.

Keywords Periodic solution · Trudinger's function · Ekeland's variational principle · Palais–Smale condition · Lagrangian system · Orlicz–Sobolev space

AMS Subject Classification Primary 34C25; Secondary 37J46 · 49J35

1 Introduction and Main Results

In this paper we prove the existence of periodic solutions for the second order Hamiltonian systems

$$\begin{cases} \frac{d}{dt}\left(\nabla\Phi(\dot{q}(t))\right) + V_q(t, q(t)) = \lambda W_q(t, q(t)),\ t \in [0, T], \\ q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0, \end{cases} \tag{1}$$

where $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and $W\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are C^1-smooth, T-periodic with respect to $t \in \mathbb{R}$, $n \geq 1$, $T > 0$, λ is a real small parameter and $\Phi\colon \mathbb{R}^n \to [0, \infty)$ is a G-function in the sense of Trudinger, i.e. $\Phi(0) = 0$, Φ is C^1-smooth, coercive, convex and symmetric, and $\nabla\Phi \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. Here and subsequently $V_q : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $W_q : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ denote the gradient maps of V and W, respectively, with respect to $q \in \mathbb{R}^n$. From now on $(\cdot,\cdot)\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ stands for the standard inner product in $\mathbb{R}^n$ and $|\cdot|\colon \mathbb{R}^n \to [0, \infty)$ is the Euclidean norm. We assume the conditions below:

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(a) there exists a constant $\alpha > 0$ such that

$$V(t,q) + \alpha|q|^2 \leq V(t,0)$$

for all $t \in [0,T]$ and $q \in \mathbb{R}^n$;

(Δ_2) there is a constant $L > 0$ such that

$$\Phi(2q) \leq L\Phi(q)$$

for each $q \in \mathbb{R}^n$;

(∇_2) there exists a constant $l > 0$ such that

$$\Phi(lq) \geq 2l\Phi(q)$$

for each $q \in \mathbb{R}^n$.

Our assumptions imply that the action functional corresponding to the system (1) with $\lambda = 0$ satisfies the Palais–Smale condition (Lemma 2.1 in Sect. 2). Let us also remark that $q \equiv 0$ is a solution of (1) for $\lambda = 0$. Our aim is to prove the existence of periodic solutions of (1) for $|\lambda|$ small enough without any extra conditions on W.

Let us consider the Orlicz space

$$L^\Phi(0,T;\mathbb{R}^n) = \left\{q : \mathbb{R} \to \mathbb{R}^n : q \text{ is } T\text{-periodic, measurable, } \int_0^T \Phi(q(t))dt < \infty\right\}$$

with the Luxemburg norm

$$\|q\|_\Phi = \inf\left\{v > 0 \colon \int_0^T \Phi\left(\frac{q(t)}{v}\right)dt \leq 1\right\}.$$

It is well-known that $L^\Phi(0,T;\mathbb{R}^n)$ is a Banach space (cf. [11]). As Φ is Δ_2-regular and ∇_2-regular, $L^\Phi(0,T;\mathbb{R}^n)$ is separable and reflexive (cf. [1]). Moreover, it is not difficult to show that

$$\|q\|_\Phi \leq 1 + \int_0^T \Phi(q(t))dt, \quad q \in L^\Phi(0,T;\mathbb{R}^n). \tag{2}$$

Proposition 1.1 (cf. [3], Lem. 3.16) *Let q_k be a sequence in $L^\Phi(0,T;\mathbb{R}^n)$ and $q \in L^\Phi(0,T;\mathbb{R}^n)$. If $q_k \to q$ almost everywhere in $(0,T)$ and $\int_0^T \Phi(q_k(t))dt \to \int_0^T \Phi(q(t))dt$ then $q_k \to q$ in $L^\Phi(0,T;\mathbb{R}^n)$.*

The mixed Orlicz–Sobolev space $W_T^{1,\Phi}$ is the space of functions $q \in L^2(0,T;\mathbb{R}^n)$ having a weak derivative $\dot{q} \in L^\Phi(0,T;\mathbb{R}^n)$. Let us recall that, if $q \in W_T^{1,\Phi}$,

$$q(t) = \int_0^t \dot{q}(s)ds + c$$

and $q(0) = q(T)$. The norm over $W_T^{1,\Phi}$ is defined by

$$\|q\|^2 = \|q\|_2^2 + \|\dot{q}\|_\Phi^2,$$

where

$$\|q\|_2 = \left(\int_0^T |q(t)|^2 dt\right)^{\frac{1}{2}}.$$

It is easy to verify that $W_T^{1,\Phi}$ is a reflexive Banach space.

Proposition 1.2 (cf. [8], Prop. 2.1) *There exists a positive constant C_Φ such that for $q \in W_T^{1,\Phi}$,*

$$\|q\|_\infty \leq C_\Phi \|q\|, \tag{3}$$

where $\|q\|_\infty = \max_{t\in[0,T]} |q(t)|$.

By Proposition 2.3 of [8], the imbedding of $W_T^{1,\Phi}$ in $C(0, T; \mathbb{R}^n)$, with its natural norm $\|\cdot\|_\infty$, is compact. We are now ready to state the announced result.

Theorem 1.3 *Let $V(t, q)$ and $W(t, q)$ be C^1-smooth on $\mathbb{R} \times \mathbb{R}^n$, T-periodic in t, and $\Phi(q)$ be a G-function. Under the assumptions* (*a*), (Δ_2), (∇_2)*, the following assertions hold.*

(i) *There is a positive number λ_0 such that the system* (1) *has a solution q_λ when $|\lambda| \leq \lambda_0$.*
(ii) *For any sequence λ_j converging to zero, along a subsequence q_{λ_j} converges to zero in $W_T^{1,\Phi}$.*

Let us emphasize that we mean by solution of (1) an absolutely continuous function in $L^2(0, T; \mathbb{R}^n)$ that satisfies (1) weakly. If we require that Φ is not only convex but stricly convex, then q_λ has a classical first derivative. There are many important examples of Φ satisfying our assumptions. If we set $\Phi(q) = \frac{1}{2}|q|^2$, $q \in \mathbb{R}^n$, we obtain the classical second order Hamiltonian systems. Applications of fundamental techniques of critical point theory to the existence of periodic solutions of second order Hamiltonian systems were presented e.g. in [9]. If we set $\Phi(q) = \frac{1}{p}|q|^p$, $q \in \mathbb{R}^n$, $1 < p < \infty$, we get the one-dimensional p-Laplacian. Nonlinear perturbations of this operator have been studied recently e.g. in [2, 5, 6]. Variational systems involving p-Laplacian occur naturally in a variety of settings in physics and engineering [2]. Moreover, let us remind an anisotropic example $\Phi(q) = \sum_{i=1}^n a_i |q_i|^{p_i}$, $1 < p_i < \infty$, $a_i > 0$, $q = (q_1, q_2, \ldots, q_n)$, which has been investigated e.g. in [4, 10].

2 Proof of Theorem 1.3

We shall prove Theorem 1.3. Our approach is based on Ekeland's variational principle. For (1) with $\lambda = 0$, we define the Lagrangian functional by

$$I_0(q) = \int_0^T (\Phi(\dot{q}(t)) - V(t, q(t)))\, dt, \tag{4}$$

where Φ and V satisfy our assumptions. Then I_0 is well-defined in $W_T^{1,\Phi}$ and becomes a C^1-functional (cf. [8], Prop. 2.10). Moreover, I_0 is bounded from below. Using (a), we get

$$I_0(q) \geq \int_0^T -V(t, q(t))dt \geq \int_0^T -V(t, 0)dt =: V_0. \tag{5}$$

From an easy calculation, we also see that

$$I_0'(q)v = \int_0^T \big((\nabla\Phi(\dot{q}(t)), \dot{v}(t)) - (V_q(t, q(t)), v(t))\big)\, dt, \tag{6}$$

where $q, v \in W_T^{1,\Phi}$.

Lemma 2.1 *I_0 satisfies the Palais–Smale condition.*

Proof Let q_k be any sequence in $W_T^{1,\Phi}$ such that $I_0(q_k)$ is bounded and $I_0'(q_k)$ converges to zero in $\left(W_T^{1,\Phi}\right)^*$. By (a) and (2), we obtain

$$\begin{aligned} I_0(q) &\geq \|\dot{q}\|_\Phi - 1 + \alpha \int_0^T |q(t)|^2 dt + \int_0^T -V(t, 0)dt \\ &= \|\dot{q}\|_\Phi - 1 + \alpha\|q\|_2^2 + V_0. \end{aligned} \tag{7}$$

As $I_0(q_k)$ is bounded, there is $C > 0$ such that $|I_0(q_k)| \leq C$ for each $k \in \mathbb{N}$. We thus get

$$\|\dot{q}_k\|_\Phi - 1 + \alpha\|q_k\|_2^2 + V_0 \leq C \tag{8}$$

for each $k \in \mathbb{N}$. Hence q_k is bounded in $W_T^{1,\Phi}$. Since $W_T^{1,\Phi}$ is reflexive, there is a subsequence of q_k that converges weakly to some $q \in W_T^{1,\Phi}$. We keep denoting this subsequence by q_k. By the compact imbedding, q_k converges to q in $C(0, T; \mathbb{R}^n)$ and, in consequence, q_k converges to q in $L^2(0, T; \mathbb{R}^n)$. Moreover, since the modulus function increases essentially more slowly than Φ near infinity $\dot{q}_k$ goes to $\dot{q}$ in $L^1(0, T; \mathbb{R})$, and hence, along a subsequence $\dot{q}_k$ goes to $\dot{q}$ almost everywhere in $(0, T)$. Without loss of generality we denote this subsequence by q_k. According to the above remarks, we have

$$|I_0'(q_k)(q_k - q)| \leq \|I_0'(q_k)\|_{\left(W_T^{1,\Phi}\right)^*} \|q_k - q\| \to 0,$$

$$\int_0^T \big(V_q(t, q_k(t)), q_k(t) - q(t)\big)\,dt \to 0,$$

and consequently,

$$\int_0^T (\nabla\Phi(\dot{q}_k(t)), \dot{q}_k(t) - \dot{q}(t))\,dt = I_0'(q_k)(q_k - q) + \int_0^T \big(V_q(t, q_k(t)), q_k(t) - q(t)\big)\,dt \to 0 \tag{9}$$

as $k \to \infty$. As Φ is convex,

$$\Phi(x) - \Phi(x - y) \le (\nabla\Phi(x), y)$$

for each $x, y \in \mathbb{R}^n$. From this it follows that

$$\int_0^T \Phi(\dot{q}_k(t))dt - \int_0^T \Phi(\dot{q}(t))dt \le \int_0^T (\nabla\Phi(\dot{q}_k(t)), \dot{q}_k(t) - \dot{q}(t))dt,$$
$$\int_0^T \Phi(\dot{q}_k(t))dt \le \int_0^T \Phi(\dot{q}(t))dt + \int_0^T (\nabla\Phi(\dot{q}_k(t)), \dot{q}_k(t) - \dot{q}(t))dt.$$

Letting $k \to \infty$ we obtain

$$\limsup_{k\to\infty} \int_0^T \Phi(\dot{q}_k(t))dt \le \int_0^T \Phi(\dot{q}(t))dt.$$

On the other hand, by Fatou's lemma

$$\liminf_{k\to\infty} \int_0^T \Phi(\dot{q}_k(t))dt \ge \int_0^T \Phi(\dot{q}(t))dt.$$

Therefore

$$\lim_{k\to\infty} \int_0^T \Phi(\dot{q}_k(t))dt = \int_0^T \Phi(\dot{q}(t))dt,$$

and finally, by Proposition 1.1, $\dot{q}_k \to \dot{q}$ in $L^\Phi(0, T; \mathbb{R}^n)$. Since $q_k \to q$ in $L^2(0, T; \mathbb{R}^n)$ and $\dot{q}_k \to \dot{q}$ in $L^\Phi(0, T; \mathbb{R}^n)$, we have $q_k \to q$ in $W_T^{1,\Phi}$, which completes the proof. □

We now choose a function such that $0 \le h(x) \le 1$ in $\mathbb{R}^n$, $h(x) = 1$ for $|x| \le C_\Phi$ and $h(x) = 0$ for $|x| \ge 2C_\Phi$, where C_Φ is given by (3). We define

$$I_\lambda(q) = \int_0^T (\Phi(\dot{q}(t)) - V(t, q(t)) + \lambda h(q(t))W(t, q(t)))\,dt, \tag{10}$$

where $q \in W_T^{1,\Phi}$. Then a critical point of I_λ is a solution of

$$\begin{cases} \frac{d}{dt}\left(\nabla\Phi(\dot{q}(t))\right) + V_q(t,q(t)) = \lambda h(q(t))W_q(t,q(t)) + \lambda\nabla h(q(t))W(t,q(t)) \\ q(0)-q(T)=\dot{q}(0)-\dot{q}(T)=0. \end{cases} \tag{11}$$

Our plan to prove Theorem 1.3 is as follows. First, we find a critical point q_λ of I_λ. Next, we show that $\|q_\lambda\|_\infty \le C_\Phi$ for $|\lambda|$ small enough. Then $h(q_\lambda)=1$, $\nabla h(q_\lambda)=0$ and therefore q_λ becomes a solution of (1). Set

$$C_0 = \max\{W(t,q)\colon t \in [0,T] \wedge |q| \le 2C_\Phi\}.$$

We have

$$I_\lambda(q) = I_0(q) + \lambda\int_0^T h(q(t))W(t,q(t))dt \ge V_0 - |\lambda| T C_0,$$

and so I_λ is bounded from below. Using the same arguments as in Lemma 2.1 with the fact that $h(q)W(t,q)$ and its gradient with respect to q are bounded, we get the next lemma.

Lemma 2.2 *For each $\lambda \in \mathbb{R}$, I_λ satisfies the Palais–Smale condition.*

Applying Ekeland's variational principle we conclude that I_λ has a minimum on $W_T^{1,\Phi}$. It follows that there is $q_\lambda \in W_T^{1,\Phi}$ such that

$$I_\lambda(q_\lambda) = \inf_{q\in W_T^{1,\Phi}} I_\lambda(q) \ \wedge \ I'_\lambda(q_\lambda) = 0.$$

Since

$$I_0(q) - |\lambda| T C_0 \le I_\lambda(q) \le I_0(q) + |\lambda| T C_0$$

for each $q \in W_T^{1,\Phi}$, we obtain $I_\lambda(q_\lambda) \to V_0$ as $\lambda \to 0$.

Lemma 2.3 *Let λ_m be a sequence converging to zero and let the functional I_{λ_m} reach a minimum at the point q_{λ_m}. Then a subsequence of q_{λ_m} converges to zero in $W_T^{1,\Phi}$.*

Proof By definition,

$$I_{\lambda_m}(q_{\lambda_m}) = \inf_{q\in W_T^{1,\Phi}} I_{\lambda_m}(q) \ \wedge \ I'_{\lambda_m}(q_{\lambda_m}) = 0,$$

and hence q_{λ_m} is a solution of (11) with λ replaced by λ_m. Using the same argument as in the proof of Lemma 2.1, by the boundedness of $I_{\lambda_m}(q_{\lambda_m})$, we can conclude that q_{λ_m} is bounded in $W_T^{1,\Phi}$ and a subsequence of q_{λ_m} converges to a limit q_0 in $W_T^{1,\Phi}$. Then q_0 satisfies that $I_0(q_0) = V_0$ and $I'_0(q_0) = 0$, i.e. $q_0 \equiv 0$. □

Lemma 2.4 *There is* $\lambda_0 > 0$ *such that for* $|\lambda| \leq \lambda_0$ *we have* $\|q_\lambda\|_\infty \leq C_\Phi$.

Proof Suppose on the contrary to our claim that there is a sequence λ_m converging to zero such that $\|q_{\lambda_m}\|_\infty > C_\Phi$. By Lemma 2.3 it follows that there is a subsequence of q_{λ_m} going to zero in $W_T^{1,\Phi}$. Without loss of generality we will denote this subsequence by q_{λ_m}. Thus for m large enough, $\|q_{\lambda_m}\| \leq 1$, and consequently $\|q_{\lambda_m}\|_\infty \leq C_\Phi$, by (3). A contradiction occurs. □

The lemma above will be used to find a solution of (1). We are now in a position to prove Theorem 1.3.

Proof *(Proof of Theorem 1.3)* Choose $\lambda_0 > 0$ that satisfies Lemma 2.4. Let I_λ reach a minimum at q_λ with $|\lambda| \leq \lambda_0$. Then $\|q_\lambda\|_\infty \leq C_\Phi$. For this reason $h(q_\lambda) = 1$, $\nabla h(q_\lambda) = 0$, and consequently q_λ becomes a solution of (1). Let λ_j be a sequence converging to zero. From Lemma 2.3 it follows that a subsequence of q_{λ_j} converges to zero in $W_T^{1,\Phi}$, which completes the proof. □

We conclude our work by explaining the regularity of solutions of (1) in case that Φ is strictly convex. We set for $|\lambda| \leq \lambda_0$ and $t \in [0, T]$,

$$x_\lambda(t) = \nabla\Phi\left(\dot{q}_\lambda(t)\right).$$

Let us note that

$$\dot{x}_\lambda(t) = \frac{d}{dt}\left(\nabla\Phi\left(\dot{q}_\lambda(t)\right)\right) = -V_q(t, q_\lambda(t)) + \lambda W_q(t, q_\lambda(t)),$$

and so it is continuously differentiable. It is known that if Φ is strictly convex then $\nabla\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible and its inverse map $(\nabla\Phi)^{-1} = \nabla\Phi^*$ is continuous (Corollary 4.1.3 in [7]), where Φ^* denotes the Fenchel transform of Φ defined by

$$\Phi^*(y) = \sup_{x\in\mathbb{R}^n}\left((x, y) - \Phi(x)\right).$$

Hence $\dot{q}_\lambda(t) = (\nabla\Phi)^{-1}(x_\lambda(t))$ is continuously differentiable too. Finally, if $\nabla\Phi^*$ is C^1 then q_λ is C^2, i.e. a classical solution. These additional assumptions are satisfied for $\Phi(x) = \frac{1}{p}|x|^p$, $1 < p \leq 2$.

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