



ELSEVIER

Contents lists available at SciVerse ScienceDirect

# Applied Mathematical Modelling

journal homepage: [www.elsevier.com/locate/apm](http://www.elsevier.com/locate/apm)

## Problem of proper decomposition and initialization of acoustic and entropy modes in a gas affected by the mass force

Sergey Leble, Anna Perelomova \*

Gdansk University of Technology, Faculty of Applied Physics and Mathematics, ul. Narutowicza 11/12, 80-233 Gdansk, Poland

### ARTICLE INFO

#### Article history:

Received 16 January 2011

Received in revised form 26 February 2012

Accepted 29 February 2012

Available online 9 March 2012

#### Keywords:

Sound propagation

Non-uniform media

Initialization of hydrodynamic field

### ABSTRACT

The relations connecting perturbations specific for acoustic and entropy modes in an accelerated fluid or in a fluid affected by constant mass force, are derived. They allow to decompose the total vector of perturbations and the overall energy into acoustic and non-acoustic parts uniquely at any instant. In order to do this, three quantities are required, for example total perturbations in entropy, pressure and velocity. The evaluations are made in regard to the content of acoustic and non-acoustic parts of the total energy excluding its kinetic part. In some cases, exact relations may be derived.

© 2012 Elsevier Inc. All rights reserved.

### 1. Introduction

The dynamics of fluids affected by external forces is a very complex problem in general. External forces make the background of waves propagation non-uniform, with background density, temperature and pressure depending on spatial coordinates. That essentially complicates the definition of linear motions (perturbations of infinitely small magnitude) taking place in such non-uniform media. Types of wave motion correspond to the roots of dispersion equation. Even in the simplest case of flow in one dimension, the dispersion relations may be introduced over all wave-length range only if the background pressure and density depend exponentially on the coordinate, see e.g. [1–3]. The number of dispersion equation roots, or branches of possible types of motion (modes), equals the number of governing equations. In one dimension, there are three types of motion: two acoustic branches and, if the thermal conduction of a fluid is ignored, the stationary (entropy, non-wave) mode with zero frequency. In the flows going out of one dimension, the buoyancy waves appear [1,2]. The possibility to distinguish modes analytically and to predict their dynamics, is of importance in the Earth meteorology [1–3] and the Sun atmosphere dynamics applications [4,5], all the more so the numerical treatment is time-consuming and requires large computer power. It may be resolved by means of linear operators uniquely separating different modes in the linear one-dimensional flow of the initially isothermal atmosphere affected by constant gravity force [5,6]. At this stage, the relationships between perturbations in upwards and downwards propagating acoustic waves were established [5]. We call as polarization relations the relationships between fluctuations of physical variables specific for any mode.

We will consider exponentially stratified volumes of an ideal gas with the background constant temperature, affected by constant mass force or moving with constant acceleration, though in one dimension results may be generalized on the case of stable spatial distribution of unperturbed temperature [7]. This study deals with some mathematical aspects of the theory of acoustic-gravity wave generation and propagation. We concentrate on the evaluation of fraction of every mode in the total energy. This problem is of importance in many applications, it clarifies what part of the total energy belongs to acoustics,

\* Corresponding author.

E-mail addresses: [leble@mif.pg.gda.pl](mailto:leble@mif.pg.gda.pl) (S. Leble), [anpe@mif.pg.gda.pl](mailto:anpe@mif.pg.gda.pl) (A. Perelomova).

what kind of perturbations would input mostly in entropy, non-wave energy (Section 4). The evaluations may be done at any instant, moreover, the proportions of different modes in the total energy do not depend on time, so conclusions about the initial fluctuation may be done at the later stages. The special choice of the initial fluctuation in order to produce some kind of mode, becomes possible.

## 2. Conservation equations and dispersion relations

The equations governing fluid in the absence of the first, second viscosity and thermal conduction manifest conservation of momentum, energy and mass. They determine dynamics of all possible types of motion which may take place in a fluid and are generally nonlinear. We start from the linearized conservation equations in terms of variations of pressure and density,  $p'$  and  $\rho'$  from hydrodynamically stable stationary functions  $\bar{p}, \bar{\rho}$ , which are not longer constants in accelerated fluid or in a fluid affected by mass force. The mean flow is absent, so that its velocity equals zero,  $\vec{V}(x, y, z) \equiv \vec{0}$ .

$$\begin{aligned}\frac{\partial \vec{V}}{\partial t} &= -\frac{\vec{\nabla} p'}{\bar{\rho}} + \vec{a} \frac{\rho'}{\bar{\rho}}, \\ \frac{\partial p'}{\partial t} &= -\vec{V} \cdot (\vec{\nabla} \bar{p}) - \gamma \bar{p} (\vec{\nabla} \cdot \vec{V}), \\ \frac{\partial \rho'}{\partial t} &= -\vec{V} \cdot (\vec{\nabla} \bar{\rho}) - \bar{\rho} (\vec{\nabla} \cdot \vec{V}).\end{aligned}\quad (1)$$

The density of a mass force is denoted by  $\vec{a}$ . The acceleration in the direction of axis  $OZ$   $\vec{a} = (0, 0, -g)$  may represent the mass force like gravity ( $a_z = -g$ ). The flow of an ideal gas is considered, which internal energy  $e$  in terms of pressure and density takes the form

$$e = \frac{p}{(\gamma - 1)\rho}, \quad (2)$$

where  $\gamma = C_p/C_v$  denotes the specific heats ratio. Eq. (1) describe gas motion of infinitely small magnitude. The stationary pressure and density in the case of  $a_x = 0$ ,  $a_y = 0$ ,  $a_z \equiv -a$  follow from the zero order equality,

$$\frac{d\bar{p}(z)}{dz} = -a\bar{\rho}(z). \quad (3)$$

The quantities supporting constant temperature  $T_0$  of the background, provide constant internal energy as well. That allows to establish them:

$$\bar{p}(z) = p_0 \exp(-z/H) = \rho_0 a H \exp(-z/H); \quad \bar{\rho}(z) = \rho_0 \exp(-z/H), \quad H = \frac{T_0(C_p - C_v)}{a}. \quad (4)$$

It is convenient to introduce the quantity  $\varphi$  instead of perturbation in density,

$$\varphi' = p' - \gamma \frac{\bar{p}}{\bar{\rho}} \rho'. \quad (5)$$

The quantity

$$\varepsilon = \frac{1}{2} \int dv \left( \bar{\rho} \vec{V}^2 + \frac{p'^2}{\gamma \bar{p}} + \frac{\varphi'^2}{\gamma(\gamma - 1)\bar{p}} \right) \quad (6)$$

is invariant, where

$$v = \{-\infty < x, y < \infty, 0 \leq z \leq h\}, \quad (7)$$

and  $h$  may be infinity. It readily follows from Eqs. (1)–(6), that

$$\frac{\partial \varepsilon}{\partial t} = - \int_{\sigma(v)} dv \vec{\nabla} \cdot (p' \vec{V}) = - \oint_{\sigma(v)} p' \vec{V} d\vec{\sigma} = 0, \quad (8)$$

where  $\sigma$  is a surface circumscriptive the volume  $v$ . The invariance of  $\varepsilon$  manifests the conservation of the total energy of gas. It includes kinetic, barotropic and thermal parts. For  $\varepsilon$  to be invariant, there is a certain freedom of establishing of boundary conditions at  $z = 0$  and  $z = h$ :  $V_z(z = 0) = V_z(z = h) = 0$ ,  $p'$  is any smooth function (impermeability condition across the boundaries), or, for example,  $V_z(z = 0) = 0$ ,  $p'(z = h) = 0$ . Let us use the new set of variables,

$$P = p' \cdot \exp(z/2H), \quad \Phi = \varphi' \cdot \exp(z/2H), \quad \vec{U} = \vec{V} \cdot \exp(-z/2H). \quad (9)$$

Eq. (1) may be rearranged into the following set,

$$\begin{aligned}
\frac{\partial U_x}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial P}{\partial x}, \\
\frac{\partial U_y}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial P}{\partial y}, \\
\frac{\partial U_z}{\partial t} &= \frac{1}{\rho_0} \left( \frac{\gamma - 2}{2\gamma H} - \frac{\partial}{\partial z} \right) P + \frac{\Phi}{\gamma H \rho_0}, \\
\frac{\partial P}{\partial t} &= -\gamma a H \rho_0 \left( \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \right) - a \rho_0 \frac{\gamma - 2}{2} U_z, \\
\frac{\partial \Phi}{\partial t} &= -(\gamma - 1) \rho_0 a U_z,
\end{aligned} \tag{10}$$

which determines the spectral problem

$$\frac{\partial}{\partial t} \Psi(\vec{r}, t) = \mathbf{L} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Psi(\vec{r}, t), \tag{11}$$

where

$$\Psi = (U_x, U_y, U_z, P, \Phi), \tag{12}$$

$\vec{r} = (x, y, z)$  and  $\mathbf{L}$  is the matrix operator including the spatial partial derivatives. The matrix formulation is equivalent to one from [6] but our choice of variables is useful for further consideration, including the energy form, scalar product and hence, the orthogonality of the eigenvectors of operator  $\mathbf{L}$ . The condition of algebraic solvability of Eq. (10) may be established by the Fourier transformation using the basis functions  $\exp(ik_x x + ik_y y + ik_z z)$ ,  $\Psi(\vec{r}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-i\omega t + ik_x x + ik_y y + ik_z z) \psi(k_x, k_y, k_z) dk_x dk_y dk_z + cc$ ,

$$\text{Det} \| i\omega \mathbf{I} - \mathbf{L}(k_x, k_y, k_z) \| = 0, \tag{13}$$

where  $\mathbf{I}$  is the unit matrix,  $\mathbf{L}$  represents the matrix operator  $\mathbf{L}$  in the space of Fourier transforms. There are four wave modes, determined by Eq. (13), which we will denote by indices 1,2,3,4, and the entropy mode, marked by 0. The dispersion relations are well-known, they have been derived earlier (see for example [1]) but we re-derive them from (10) and write down for a reader convenience:

$$\omega_0 = 0. \tag{14}$$

$$\omega_{1,2} = \pm \sqrt{\frac{\gamma a H}{2} \sqrt{k_x^2 + k_y^2 + k_z^2 + \frac{1}{4H^2}} + \sqrt{\left(k_x^2 + k_y^2 + k_z^2 + \frac{1}{4H^2}\right)^2 - \frac{4(\gamma - 1)}{\gamma^2 H^2} (k_x^2 + k_y^2)}},$$

$$\omega_{3,4} = \pm \sqrt{\frac{\gamma a H}{2} \sqrt{k_x^2 + k_y^2 + k_z^2 + \frac{1}{4H^2}} - \sqrt{\left(k_x^2 + k_y^2 + k_z^2 + \frac{1}{4H^2}\right)^2 - \frac{4(\gamma - 1)}{\gamma^2 H^2} (k_x^2 + k_y^2)}}.$$

The entropy mode essentially participates in a gas energy transfer. Propagation of intense sound in a thermoviscous fluid makes this mode enlarge due to losses in acoustic energy, and it is not longer stationary in the nonlinear flow. This nonlinear effect is known as acoustic heating. It is intensively studied in nonlinear acoustics in connection to technical and biomedical applications [8,9]. An intensity of acoustic heating is proportional to the overall absorption of a fluid, including the thermal one. Without account of the entropy mode as one of the primary types of a fluid motion, neither proper subdivision of modes, nor description of overall motion is possible.

Vectors of perturbations  $\Psi$  form the Hilbert space  $L^2(v)$  with the standard scalar product; particularly, for  $\Psi_n$  ( $n = 0, \dots, 4$ ), correspondent to the eigenvalues  $i\omega_n$ , it is

$$\langle \Psi_n, \Psi_m \rangle = \int dv \left( \rho_0 \vec{U}_n \cdot \vec{U}_m^* + \frac{P_n P_m^*}{\gamma a H \rho_0} + \frac{\Phi_n \Phi_m^*}{\gamma(\gamma - 1) a H \rho_0} \right). \tag{15}$$

Therefore, the quantity  $E$  is invariant,

$$E = \int dv \left( \rho_0 |\vec{U}|^2 + \frac{P^2}{\gamma a H \rho_0} + \frac{\Phi^2}{\gamma(\gamma - 1) a H \rho_0} \right), \tag{16}$$

where  $\vec{U}$ ,  $P$  and  $\Phi$  represent a sum of specific perturbations,

$$\vec{U} = \sum_{n=0}^4 \vec{U}_n, P = \sum_{n=0}^4 P_n, \Phi = \sum_{n=0}^4 \Phi_n. \tag{17}$$

The set  $\Psi_n$  ( $n = 0, \dots, 4$ ) form a complete set of eigenvectors. That is true for the self-adjoint boundary conditions. It may be readily established, that  $i\mathbf{L}$  is symmetric in  $L^2(v)$  ( $0 \leq n, m \leq 4$ ):

$$\langle i\mathbf{L}\Psi_n, \Psi_m \rangle - \langle \Psi_m, i\mathbf{L}\Psi_n \rangle = i \oint_{\sigma(v)} \left( P_n^* \vec{U}_m + P_m^* \vec{U}_n \right) d\vec{\sigma}. \quad (18)$$

The most important physically condition of impermeability at the upper and lower boundaries,  $z = 0$  and  $z = h$ , is self-adjoint: for example  $U_z(z = 0) = U_z(z = h) = 0$ . The choice of boundary conditions guarantees the orthogonality of eigenvectors as a direct corollary of Hermiticity of the operator  $L$  in respect to the scalar product (15). The second and third kind (homogeneous) conditions are also admissible.

### 3. Decomposition of a perturbation into acoustic and entropy modes in one-dimensional flow

In the one-dimensional flow along axis  $OZ$  ( $U_x = U_y \equiv 0$ ,  $k_x = k_y \equiv 0$ ), the dispersion relations (14) determine three modes, or, in the other words, possible motions of a gas. Two of them are acoustic, describing sound of opposite direction of propagation (above the cutoff frequency), and the last one is the stationary (or entropy) mode. One should remember about the cutoff frequency in more advanced three-dimensional theory, which experimental evidence and applications is still under discussion (see, for example [10]). In the absence of mass force or acceleration, the entropy mode is isobaric. Eq. (14) are readily simplified:

$$\omega_0 = 0, \omega_1 = \sqrt{\gamma a H} \sqrt{k_z^2 + \frac{1}{4H^2}}, \quad \omega_2 = -\sqrt{\gamma a H} \sqrt{k_z^2 + \frac{1}{4H^2}}. \quad (19)$$

Eq. (10) take the form

$$\begin{aligned} \frac{\partial U_z}{\partial t} &= \frac{1}{\rho_0} \left( \frac{\gamma - 2}{2\gamma H} - \frac{\partial}{\partial z} \right) P + \frac{\Phi}{\gamma H \rho_0}, \\ \frac{\partial P}{\partial t} &= -\gamma a H \rho_0 \left( \frac{\partial U_z}{\partial z} \right) - a \rho_0 \frac{\gamma - 2}{2} U_z, \\ \frac{\partial \Phi}{\partial t} &= -(\gamma - 1) \rho_0 a U_z, \end{aligned} \quad (20)$$

which yield in the invariant relation connecting  $P(z, t)$  and  $\Phi(z, t)$  in both acoustic branches,  $P_a = \frac{1}{\gamma - 1} \left( \frac{\gamma - 2}{2} + \gamma H \frac{\partial}{\partial z} \right) \Phi_a$ , and the link for the stationary entropy mode,  $\Phi_0 = \left( -\frac{\gamma - 2}{2} + \gamma H \frac{\partial}{\partial z} \right) P_0$ . Index  $a$  denote summary acoustic modes. The completeness of the set of eigenvectors allow to represent the total vector of perturbations as a sum of acoustic and entropy vectors at any instant,

$$\Psi(z, t) = \begin{bmatrix} U_z \\ P \\ \Phi \end{bmatrix} = \Psi_1(z, t) + \Psi_2(z, t) + \Psi_3(z, t) \equiv \Psi_a(z, t) + \Psi_0(z, t) = \begin{bmatrix} U_{a,z} \\ \frac{1}{\gamma - 1} \left( \frac{\gamma - 2}{2} + \gamma H \frac{\partial}{\partial z} \right) \Phi_a \\ \Phi_a \end{bmatrix} + \begin{bmatrix} 0 \\ P_0 \\ \left( -\frac{\gamma - 2}{2} + \gamma H \frac{\partial}{\partial z} \right) P_0 \end{bmatrix}. \quad (21)$$

Some relations between eigenvectors Fourier components of have been derived in [4]. Vectors  $\Psi_a(z, t)$  and  $\Psi_0(z, t)$  are orthogonal in accordance to the metric (15),

$$\langle \Psi_a, \Psi_0 \rangle = \int_0^h \left( \rho_0 U_{a,z} U_{0,z} + \frac{P_a P_0}{\gamma a H \rho_0} + \frac{\Phi_a \Phi_0}{\gamma(\gamma - 1) a H \rho_0} \right) dz. \quad (22)$$

Taking a sum  $2\Phi + (\gamma - 2 - 2\gamma H \frac{\partial}{\partial z})P$  and using Eq. (21), one readily reduces all entropy terms in the left-hand side of the ordinary differential equation of the second order,

$$\left( 1 - 4H^2 \frac{\partial^2}{\partial z^2} \right) \Phi_a(z, t) = \frac{2(\gamma - 1)}{\gamma^2} \left( 2\Phi(z, t) + \left( \gamma - 2 - 2\gamma H \frac{\partial}{\partial z} \right) P(z, t) \right) \equiv D(z, t). \quad (23)$$

It is valid at any instant. The solution of (23) takes the form

$$\begin{aligned} \Phi_a(z, t) &= C_1 \exp(-z/2H) + C_2 \exp(z/2H) \\ &+ \frac{1}{4H} \left( \exp(-z/2H) \int_0^z \exp(z'/2H) D(z', t) dx - \exp(z/2H) \int_0^z \exp(-x/2H) D(z', t) dz \right), \end{aligned} \quad (24)$$

where  $C_1, C_2$  denote any real constants. It determines  $P_a(z, t)$  in accordance to Eq. (21), and  $P_0(z) = P(z, t) - P_a(z, t)$ ,  $\Phi_0(z) = \Phi(z, t) - \Phi_a(z, t)$ . That makes possible to conclude about composition of total perturbations in pressure and entropy, and hence of that in the energy. The instant subdivision of perturbations and corresponding energy reproduces the results obtained by means of temporal averaging over the sound period in the case of periodic in time perturbations (see [8]) but it is much more universal because allows to describe precise temporal dynamics of non-periodic perturbations. It allows to

separate energies of different sound branches. Obviously, the most significant difference can appear when pulses or series of pulses are considered.

## 4. Examples

### 4.1. Composition of the total field of exclusively entropy or acoustic parts

The conclusion follow immediately from the relations (21) and completeness of the set of eigenvectors. If

$$\Phi(z, 0) = \left( -\frac{\gamma-2}{2} + \gamma H \frac{d}{dz} \right) P(z, 0), \quad U_z(z, 0) = 0, \quad (25)$$

the total field is represented exclusively by the entropy mode. We write down these equalities in the zero time. It is of importance, that there are valid *at any instant* if they are valid at some instant, for example, at  $t = 0$ . In order to conclude about velocity of the acoustic mode, the knowledge of relation linking it with  $\Phi_a$  is required. It follows from the conservation system (10), but takes complex integro-differential form described by some integro-differential operator  $K$ . Anyway, the relations are asymmetric,  $U_{1,z}(z, 0) = K\Phi_1(z, 0)$ ,  $U_{2,z}(z, 0) = -K\Phi_2(z, 0)$ . It may be concluded from these equalities and Eq. (21), that if  $\Phi_1(z, 0) = -\Phi_2(z, 0)$  and  $P_1(z, 0) = -P_2(z, 0)$ , that is,

$$P(z, 0) = \frac{1}{\gamma-1} \left( \frac{\gamma-2}{2} + \gamma H \frac{d}{dz} \right) \Phi(z, 0), \quad (26)$$

the total field is represented by the entropy mode and acoustic field with non-zero initial velocity  $U_z(z, 0) = U_{1,z}(z, 0) + U_{2,z}(z, 0)$  (hence, the non-zero kinetic energy) and zero initial perturbations  $P(z, 0)$  and  $\Phi(z, 0)$ . To make approximate evaluations of  $K$ , one may restrict by large wavenumbers  $k_z$ ,  $Hk_z \gg 1$  [7]. That makes possible to expand acoustic eigenvalues in the Taylor series and to obtain finally relations for every acoustic branch,

$$U_{z,1}(z, 0) = -\frac{g\gamma^2}{8\rho_0(\gamma-1)(\gamma aH)^{3/2}} \int_0^z \Phi_1(z', 0) dz', \quad U_{z,2}(z, 0) = \frac{g\gamma^2}{8\rho_0(\gamma-1)(\gamma aH)^{3/2}} \int_0^z \Phi_2(z', 0) dz'. \quad (27)$$

The simple conclusion from Eq. (27) is that if  $\int_0^z (\Phi_1(z', 0) + \Phi_2(z', 0)) dz' = 0$ , the part of kinetic energy in the total one is zero.

### 4.2. Energy release

The first example considers pure heating of a gas which may occur at any instant, for definiteness at  $t = 0$ ,

$$P(z, t = 0) = \Phi(z, t = 0) = \Theta(z). \quad (28)$$

Eq. (23) with account for (28) yields

$$\left( 1 + 2H \frac{d}{dz} \right) \Phi_a(z, 0) = \frac{2(\gamma-1)}{\gamma} \Theta(z), \quad (29)$$

with solution

$$\Phi_a(z, 0) = \frac{\gamma-1}{\gamma H} \exp(-z/2H) \int_0^z \exp(z'/2H) \Theta(z') dz'. \quad (30)$$

Introducing the function  $\tilde{\Theta}(z)$ ,  $\Theta(z) = \exp(-z/2H) \frac{d}{dz} (\exp(z/2H) \tilde{\Theta}(z))$ , one readily obtains

$$\begin{aligned} \Phi_a(z, 0) &= \frac{\gamma-1}{\gamma} \tilde{\Theta}(z), \quad P_a(z, 0) = \frac{\gamma-2}{2\gamma} \tilde{\Theta}(z) + H \frac{d}{dz} \tilde{\Theta}(z), \quad \Phi_0(z, 0) = \Phi - \Phi_a \\ &= \frac{2-\gamma}{2\gamma} \tilde{\Theta}(z) + H \frac{d}{dz} \tilde{\Theta}(z), \quad P_0(z, 0) = P - P_a = \frac{1}{\gamma} \tilde{\Theta}(z). \end{aligned} \quad (31)$$

In accordance to Eq. (22), if  $\tilde{\Theta}(0) = \tilde{\Theta}(\infty) = 0$ ,

$$\langle \Psi_a, \Psi_0 \rangle = \frac{1}{\gamma a H \rho_0} \int_0^\infty \left( P_a P_0 + \frac{\Phi_a \Phi_0}{(\gamma-1)} \right) dz = \frac{1}{\gamma^2 a \rho_0} \tilde{\Theta}(z) \Big|_0^\infty = 0. \quad (32)$$

The total energy of sound mode, with exception of its kinetic part, equals product of  $\gamma - 1$  and energy of the entropy mode,

$$\Sigma_a = \frac{1}{\gamma a H \rho_0} \int_0^\infty \left( P_a^2 + \frac{\Phi_a^2}{(\gamma-1)} \right) dz = \frac{H}{\gamma a \rho_0} \int_0^\infty \left( \frac{d}{dz} \tilde{\Theta}(z) \right)^2 dz + \frac{1}{\gamma a H \rho_0} \int_0^\infty \left( \frac{\gamma-1}{\gamma^2} + \left( \frac{\gamma-2}{2\gamma} \right)^2 \right) \tilde{\Theta}^2(z) dz = (\gamma-1) \Sigma_0. \quad (33)$$

### 4.3. Energy release with mass injection

The second example describes the initial mass injection,

$$P(z, t = 0) = \frac{\Phi(z, t = 0)}{1 - \gamma} = \Theta(z). \quad (34)$$

In this case, Eq. (34) with account for (23) yields

$$\left(1 - 2H \frac{d}{dz}\right) \Phi_a(z, 0) = -\frac{2(\gamma - 1)}{\gamma} \Theta(z), \quad (35)$$

with solution

$$\Phi_a(z, 0) = \frac{\gamma - 1}{\gamma H} \exp(z/2H) \int_0^z \exp(-z'/2H) \Theta(z') dz'. \quad (36)$$

As well as in the previous subsection, we introduce the function  $\tilde{\Theta}(z)$ ,  $\Theta(z) = \exp(z/2H) \frac{d}{dz} (\exp(-z/2H) \tilde{\Theta}(z))$ , and use Eq. (21) to obtain at  $t = 0$

$$\begin{aligned} \Phi_a(z, 0) &= \frac{\gamma - 1}{\gamma} \tilde{\Theta}(z), \quad P_a(z, 0) = \frac{\gamma - 2}{2\gamma} \tilde{\Theta}(z) + H \frac{d}{dz} \tilde{\Theta}(z), \quad \Phi_0 = \Phi - \Phi_a \\ &= \frac{(\gamma - 1)(\gamma - 2)}{2\gamma} \tilde{\Theta}(z) + H(1 - \gamma) \frac{d}{dz} \tilde{\Theta}(z), \quad P_0 = P - P_a = \frac{1 - \gamma}{\gamma} \tilde{\Theta}(z). \end{aligned} \quad (37)$$

The acoustic and entropy vectors of perturbations are orthogonal. The acoustic and entropy initial energies take the form

$$\Sigma_a = \frac{H}{\gamma a \rho_0} \int_0^\infty \left(\frac{d}{dz} \tilde{\Theta}(z)\right)^2 dz + \frac{1}{\gamma a H \rho_0} \int_0^\infty \frac{1}{4} \tilde{\Theta}^2(z) dz = \frac{1}{\gamma - 1} \Sigma_0. \quad (38)$$

Their ratio in this case equals  $(\gamma - 1)^{-1}$ . Mathematically, the decomposition is possible due to orthogonality of the correspondent subspaces with respect to scalar product choice (15), and because the energy functional (16) is quadratic. Physically, the decomposition (by measurements) is possible because one specific perturbation determines all other perturbations in the mode at any time. The formalism of relations (17) defines the invertible map in 1, 2, 3 dimensions.

## 5. Conclusions

The relations linking  $U_z$  with  $P$  and  $\Phi$  in sound propagating in accelerated gas, or gas affected by the mass force, are integro-differential. The exact links of excess pressure, density and velocity in unbounded volumes of gas, are derived exactly with regard to one-dimensional flow by one of the authors in [11,12]. The approximate relations in the case of short perturbations (as compared to  $H$ ) have been derived in [13]. In these studies, the projecting operators were derived, which being applied on the total vector of perturbations, decompose the correspondent mode, acoustic or entropy, at any instant. The projectors form the full orthogonal basis with properties:

$$\Pi_1 \cdot \Pi_2 = \Pi_1 \cdot \Pi_3 = \dots = \Pi_3 \cdot \Pi_2 = \mathbf{0}, \quad \Pi_1^2 = \Pi_1, \dots, \Pi_1 + \Pi_2 + \Pi_3 = \mathbf{I}, \quad (39)$$

where  $\mathbf{0}$ ,  $\mathbf{I}$  are zero and unit matrix operators. In general, the completeness of the decomposition (17) explains the importance of the zero-frequency mode in the evaluations of the overall energy and momentum [8,9].

The limit  $a \rightarrow 0$  and therefore  $H \rightarrow \infty$  may be easily traced. Note that the product  $aH$  remains constant (the last equality from Eq. (4)),  $\gamma a H$  is squared sound of velocity in an ideal gas. In the case  $a = 0$ ,  $\varphi$  means the quantity proportional to perturbation in the entropy,  $\varphi' = (\gamma - 1)\rho_0 s'$ . It is identically zero in both acoustic branches and is not longer suitable to be a reference quantity in them. Instead, perturbation in density may be chosen. The reason for which  $\varphi'$  was used in the case non-zero  $a$ , is the simplicity of expression  $P$  in terms of  $\Phi$  including partial derivative with respect to  $z$  but not integral operators.

Two types of initial perturbations considered in Section 4 yield simple ratios of acoustic and entropy energies. Examples do not consider kinetic part of the total energy which associates with acoustic modes. To evaluate it, the expression of velocity in terms of acoustic pressure  $P$  or  $\Phi$  is required. The relation is integro-differential and does not make possible to conclude about part of kinetic energy in the total one in such simple manner as about parts of non-kinetic energy in the examples. The relations for the first and second acoustic modes in terms of excess density,  $R = \rho' \exp(-z/2H)$ , are determined by eigenvectors

$$\begin{aligned} \Psi_1(z, t) &= \begin{bmatrix} U_z \\ \Phi \\ R \end{bmatrix}_1 = \begin{bmatrix} \frac{2\gamma a H}{\rho_0} \int_{-\infty}^{\infty} dz' \left(\frac{\partial}{\partial z'} - \frac{1}{2H}\right) F(z' - z) \\ a(1 - \gamma) \exp(z/2H) \int_z^{\infty} \exp(-z'/2H) dz' \\ 1 \end{bmatrix} R_1(z, t), \\ \Psi_2(z, t) &= \begin{bmatrix} -\frac{2\gamma a H}{\rho_0} \int_{-\infty}^{\infty} dz' \left(\frac{\partial}{\partial z'} - \frac{1}{2H}\right) F(z' - z) \\ a(1 - \gamma) \exp(z/2H) \int_z^{\infty} \exp(-z'/2H) dz' \\ 1 \end{bmatrix} R_2(z, t), \end{aligned} \quad (40)$$

where  $F(z) = \int_0^\infty \frac{\sin(kz)}{\sqrt{k^2 + 1/4H^2}} dk = \frac{2}{\pi} (I_0(z/2H) - L_0(z/2H))$  ( $I_0$  and  $L_0$  denote the modified Bessel function of zeroth order and the Struve function, respectively). These relations linking perturbations of density, pressure and velocity in unbounded space of a gas,  $-\infty < z < \infty$ , were obtained by one of the authors in [11,12].

This study is devoted to an ideal gas. It may be readily generalized in the case of a fluid different from an ideal gas, including liquid, replacing  $\gamma$  by  $c^2 \rho_0/p_0$ , where  $c$  denotes the sound velocity over the fluid without acceleration, and  $p_0$  denotes the unperturbed internal pressure in it. The nonlinearity in conservation equations is not accounted for. The linear projecting is helpful in studies of weakly nonlinear dynamics of a fluid and, in particular, in investigations of modes interaction there. Applying of correspondent projector on the system of conservative equations directly (Eq. (10) supplemented by nonlinear terms) allows to derive coupled dynamic equations for interacting modes. The example of that in unbounded volume of gas was considered in [11,12]. The method elaborated by the authors is successful in the solution of some problems of fluids flows in waveguides [14]. Let us mention that an interaction between waves and the mean flow (that also belongs to the subspace of  $\omega = 0$ ) is studied in [6].

## References

- [1] C. Eckart, Hydrodynamics of Oceans and Atmospheres, Pergamon Press, London, 1960.
- [2] L.M. Brekhovskikh, A.O. Godin, Acoustics of Layered Media, Springer-Verlag, Berlin, 1990.
- [3] J. Pedloski, Geophysical fluid Dynamics, Springer-Verlag, Berlin, 2006.
- [4] P. Souffrin, Astron. Astrophys. 17 (1972) 458.
- [5] G. Marmolino, G. Severino, Astron. Astrophys. 242 (1991) 271.
- [6] R.M. Jones, Phys. Fluids 13 (2001) 1274.
- [7] V. Brezhnev Yu, S.B. Leble, A.A. Perelomova, Phys. Express 41 (1993) 29.
- [8] S. Makarov, M. Ochmann, Acustica 82 (1996) 579.
- [9] A. Perelomova, Phys. Lett. A 357 (2006) 42.
- [10] A. Jimenez, Astrophys. J. 646 (2) (2006) 1398.
- [11] A. Perelomova, Acta Acustica 84 (1998) 1002.
- [12] A. Perelomova, Arch. Acoust. 25 (4) (2000) 451.
- [13] A. Perelomova, Arch. Acoust. 34 (2) (2009) 127.
- [14] S.B. Leble, Nonlinear Waves in Waveguides with Stratification, Springer-Verlag, Berlin, 1990.