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Proper gradient otopies

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ABSTRACT

We prove that the inclusion of the space of proper gradient local maps into the space of proper local maps induces a bijection between the sets of the respective otopy classes of these maps.

0. Introduction

J.C. Becker and D.H. Gottlieb have introduced an extremely useful generalization of the concept of homotopy called *otopy* (see for instance [3,4,7]). The main advantage of using otopies is that otopy relates maps with not necessarily the same domain (the *local maps* of Definition 1.1). Furthermore, otopy theory turns out to be fruitful in equivariant degree theory (see [2,6]).

Our main result is the following theorem: the inclusion of the space of proper gradient local maps into the space of all proper local maps induces a bijection between the sets of connected components of these spaces i.e. between the respective otopy classes of local maps. We expected this result to be true (see [1, Remark 2.2]), but were not able to prove at that moment due to many technical difficulties. The result may be regarded as a version of Parusiński's Theorem (see [9] for details) and as a special (simplest) case of the following conjecture: the above inclusion is a (weak) homotopy equivalence.

It is worth pointing out that the advantage of using proper local maps instead of all local maps is that the space of proper local maps is a "very nice" metrizable space. In fact, it is homeomorphic to the space of based continuous maps of the n-sphere into itself.

However, the proof of our main result is more difficult compared to that concerning all local maps presented in [1].

The paper is arranged as follows. Section 1 presents some preliminaries from otopy theory for local maps. Section 2 contains a discussion of our result with some comments. This result is proved in Section 4. In Section 3 we introduce canonical, elementary and standard maps. Sections 5, 6 and 7 contain proofs of key lemmas needed in Section 4. Finally, Appendix A presents three technical facts used in Section 5.

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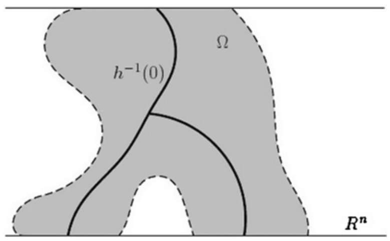


Fig. 1. Domain of the otopy and its zeros.

1. Preliminaries

The notation $A \Subset B$ means that A is a compact subset of B.

Definition 1.1. A continuous map $f : U \to \mathbb{R}^n$ is called a *local map* if

- U is an open subset of $\mathbb{R}^n$,
- $f^{-1}(0) \Subset U$.

We will often write such maps as pairs (f, U), pointing out their domains.

The set-theoretic union of two local maps f and g with disjoint domains will be denoted by $f \sqcup g$. Recall that a local map (f, U) is called *gradient* if there is a C^1-function $\varphi : U \to \mathbb{R}$ such that $f = \nabla\varphi$. The function φ is called the *potential* of f. A local map is *proper* if preimages of compact subsets are compact. In the sequel, we will also use the following equivalent characterization of proper local maps: for every sequence $\{x_k\} \subset U$ with no accumulation point in U we have $\lim_{k\to\infty} \|f(x_k)\| = \infty$.

We will consider the set of all local maps, denoted by $\mathcal{F}(n)$, and the following subsets:

$$\mathcal{F}_\nabla(n) := \{ f \in \mathcal{F}(n) \mid f \text{ is gradient} \},$$
$$\mathcal{P}(n) := \{ f \in \mathcal{F}(n) \mid f \text{ is proper} \},$$
$$\mathcal{P}_\nabla(n) := \mathcal{F}_\nabla(n) \cap \mathcal{P}(n).$$

Immediately from the above definitions we obtain the following commutative diagram of inclusions:

$$\begin{array}{ccc} \mathcal{P}_\nabla(n) & \hookrightarrow & \mathcal{P}(n) \\ \downarrow & & \downarrow \\ \mathcal{F}_\nabla(n) & \hookrightarrow & \mathcal{F}(n) \end{array} \tag{1.1}$$

Let $I = [0, 1]$.

Definition 1.2. A continuous map $h : \Omega \to \mathbb{R}^n$ is called an *otopy* if

- Ω is an open subset of $\mathbb{R}^n \times I$,
- $h^{-1}(0) \Subset \Omega$,

see Fig. 1.

Given an otopy (h, Ω) we can define for each $t \in I$ sets $\Omega_t = \{x \in \mathbb{R}^n \mid (x, t) \in \Omega\}$ and maps $h_t : \Omega_t \to \mathbb{R}^n$ with $h_t(x) = h(x, t)$. Note that from the above h_t may be the empty map.

Definition 1.3. If (h, Ω) is an otopy, we say that (h_0, Ω_0) and (h_1, Ω_1) are *otopic* (written $h_0 \sim h_1$ or $(h_0, \Omega_0) \sim (h_1, \Omega_1)$).

Remark 1. Of course, otopy gives an equivalence relation on $\mathcal{F}(n)$. The set of otopy classes of local maps will be denoted by $\mathcal{F}[n]$. Observe that if (f, U) is a local map and V is an open subset of U such that $f^{-1}(0) \subset V$, then (f, U) and $(f|_V, V)$ are otopic. In particular, if $f^{-1}(0) = \emptyset$ then (f, U) is otopic to the empty map.

Apart from the usual otopies, we will consider otopies that satisfy some additional conditions, namely:

- *gradient* i.e. $h(x,t) = \nabla_x \chi(x,t)$ for some not necessarily continuous function χ such that χ_t is C^1 for each $t \in I$,
- *proper* i.e. h is proper,
- *proper gradient*.

The sets of respective otopy classes in $\mathcal{F}_\nabla(n)$, $\mathcal{P}(n)$, $\mathcal{P}_\nabla(n)$ will be denoted by $\mathcal{F}_\nabla[n]$, $\mathcal{P}[n]$, $\mathcal{P}_\nabla[n]$. We will abbreviate *proper gradient otopic* to *pg-otopic*.

Remark 2. In [1] we use the stronger definition of the gradient otopy. However, it is easy to see that the results of [1] are not affected if in [1] we replace the stronger definition with the above weaker one. Moreover, the replacement of the definition is mainly motivated by the following expectation: paths in $\mathcal{P}_\nabla(n)$ should bijectively correspond to proper gradient otopies (see below for the definition of topology in $\mathcal{P}_\nabla(n)$).

Let $\Sigma^n = R^n \cup \{*\}$ denote the one-point compactification of $\mathbb{R}^n$. It is a pointed space with base point $*$. We will write $\mathcal{M}_*\Sigma^n$ for the set of pointed continuous maps from Σ^n to Σ^n. With every map $f \in \mathcal{M}_*\Sigma^n$ one associates a proper local map $(f\restriction_{f^{-1}(R^n)}, f^{-1}(R^n))$. Conversely, if $(f,U) \in \mathcal{P}(n)$, then the map $f^+ : \Sigma^n \to \Sigma^n$ given by

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in U, \\ * & \text{otherwise,} \end{cases}$$

is continuous. Using this observation we see that the map

$$\mu : \mathcal{P}(n) \to \mathcal{M}_*\Sigma^n,$$
$$\mu\big((f,U)\big) = f^+$$

is a bijection. Since $\mathcal{M}_*\Sigma^n$ is equipped with the supremum metric, $\mathcal{P}(n)$ also has the metric structure induced by the pullback.

2. Main result

The diagram (1.1) induces the following commutative diagram of maps between sets of otopy classes (all the maps are induced by inclusions).

$$\begin{array}{ccc} \mathcal{P}_\nabla[n] & \xrightarrow{\mathfrak{a}} & \mathcal{P}[n] \\ {\scriptstyle \mathfrak{b}}\downarrow & & \downarrow{\scriptstyle \mathfrak{c}} \\ \mathcal{F}_\nabla[n] & \xrightarrow{\mathfrak{d}} & \mathcal{F}[n] \end{array} \tag{2.1}$$

Let us formulate the main result of this paper.

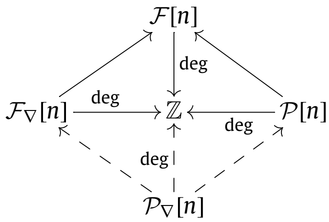
Theorem A. *All the maps in the diagram* (2.1) *are bijections.*

Remark 3. It is worth pointing out that our result includes a version of Parusiński's Theorem: the maps $\mathfrak{a}$ and $\mathfrak{d}$ are bijections. However, our proof makes no appeal to the original proof of Parusiński.

It is clear from the topological degree theory [e.g. 5, Ch. II and Ch. IV] that all the maps in the following commutative diagram are bijections.

$$\begin{array}{ccccc} \mathcal{M}_*[\Sigma^n] & \longleftarrow & \mathcal{P}[n] & \longrightarrow & \mathcal{F}[n] \\ & {\scriptstyle \deg}\searrow & \downarrow{\scriptstyle \deg} & \swarrow{\scriptstyle \deg} & \\ & & \mathbb{Z} & & \end{array}$$

Consequently, the map $\mathfrak{c}$ in the diagram (2.1) is bijective. But in [1] we proved that $\deg : \mathcal{F}_\nabla[n] \to \mathbb{Z}$ is a bijection, so it is clear from diagram (2.2)



(2.2)

that all the solid arrows represent bijections. Therefore the true difficulty in proving Theorem A lies in the following:

Theorem B. deg : $\mathcal{P}_\nabla[n] \to \mathbb{Z}$ *is bijective.*

We will see that, in fact, only injectivity causes a problem.

3. Canonical, elementary and standard maps

We shall consider the open unit ball $B := \{x \in \mathbb{R}^n \mid \sum_i x_i^2 < 1\}$ and the open unit cube $C := \{x \in \mathbb{R}^n \mid \max_i |x_i| < 1\}$. Let $|c_i| = 1$ for $i = 1, \dots, n$. If $\varphi(x) = \frac{\sum_{i=1}^n c_i x_i^2}{1-\sum_{i=1}^n x_i^2}$, then $\nabla\varphi : B \to \mathbb{R}^n$ is called *ball-canonical* and if $\varphi(x) = \sum_{i=1}^n \frac{c_i x_i^2}{1-x_i^2}$, then $\nabla\varphi : C \to \mathbb{R}^n$ is called *cube-canonical*. We say that two maps ball-canonical and cube-canonical are *of the same type* if both of them have the same sequence of coefficients $\{c_i\}_1^n$. Moreover, if $\varphi(x) = \sum_{i=1}^n \frac{x_i^2}{1-x_i^2}$, then $\nabla\varphi : C \to \mathbb{R}^n$ is called *plus-elementary* and if $\varphi(x) = \frac{-x_1^2}{1-x_1^2} + \sum_{i=2}^n \frac{x_i^2}{1-x_i^2}$, then $\nabla\varphi$ is called *minus-elementary*. The center of domains of all above maps is the origin of $\mathbb{R}^n$, but in the sequel we will use the same terminology for translations of the these maps to any point. Note that such translations are evidently proper gradient otopies. Finally, for $l \in \mathbb{N}$ a finite disjoint union of l plus-elementary (resp. minus-elementary) maps is called *l-standard* (resp. $(-l)$-*standard*).

4. Proof of Theorem B

The proof of Theorem B is based on the following lemmas, which will be proved in the next three sections. In fact, it is easy to see that Theorem B is an immediate consequence of Corollary 4.5.

Lemma 4.1. *Any proper gradient local map is pg-otopic to a finite disjoint union of ball-canonical maps.*

Lemma 4.2. *Any ball-canonical map is pg-otopic to cube-canonical one of the same type.*

Lemma 4.3. *Any two cube-canonical maps with different coefficients at only one position annihilate i.e. their union is pg-otopic to the empty map.*

Corollary 4.4. *Any cube-canonical map is pg-otopic to some elementary map.*

Proof. Let φ be cube-canonical and $1 < k \leqslant n$. By Lemma 4.3 $\nabla\varphi$ is pg-otopic to $\nabla\varphi \sqcup \nabla\psi \sqcup \nabla\xi$, where $\nabla\psi$ and $\nabla\xi$ are also cube-canonical, ψ has only the first coefficient and ξ has only the first and k-th coefficient different from those of φ. Since $\nabla\varphi$ and $\nabla\psi$ also annihilate, $\nabla\varphi$ is pg-otopic to $\nabla\xi$. That means that using a proper gradient otopy we are able to replace a given cube-canonical map with one having coefficients changed on exactly two positions: first and k-th. Repeating the above procedure it is possible in the end to obtain an elementary map. □

Corollary 4.5. *For each $l \in \mathbb{Z}$ any proper gradient local map of degree l is pg-otopic to an l-standard map.*

Proof. Let $\nabla\varphi \in \mathcal{P}_\nabla(n)$ with $\deg \nabla\varphi = l$. By Lemma 4.1 $\nabla\varphi$ is pg-otopic to a finite union of ball-canonical maps. From Lemma 4.2 we can replace all of them with cube-canonical maps. Similarly, by Corollary 4.4 we can replace cube-canonical maps with elementary maps. Finally by Lemma 4.3 pairs of elementary maps of different signs annihilate, so we obtain an l-standard map. □

5. Proof of Lemma 4.1

Let $(f, U) \in \mathcal{P}_\nabla(n)$ with $f = \nabla\varphi$. The proof will be divided into 5 steps. Each of them represents a proper gradient otopy. In the first we replace the initial potential by a Morse function. Next, in the Step 2, we replace the Morse function by a locally quadratic potential. Step 3 shows how to blow up the domain U to obtain big enough neighborhoods of critical points. In Step 4 we restrict the domain to the disjoint union of unit balls. Finally, Step 5 transforms matrices in the formulas to the diagonal form.

Step 1. We deform the potential φ to a Morse function φ_M. By density of the Morse functions and openness of the proper vector fields (see [8]) we can choose a Morse function φ_M such that the straight-line homotopy of potentials $(1-t)\varphi + t\varphi_M$ induces a proper gradient homotopy.

Step 2. Let us denote by $B_r(p)$ the open r-ball around p and let $B_r := B_r(0)$.

We deform the potential φ_M to some potential ψ satisfying

- $\mathrm{Crit}(\varphi_M) = \mathrm{Crit}(\psi)$,
- there is $\epsilon > 0$ such that for each $p \in \mathrm{Crit}(\psi)$ the map ψ is a nondegenerate quadratic form around p i.e.

$$\psi\restriction_{B_\epsilon(p)}(x) = (x-p)^T A_p (x-p)$$

  for some nondegenerate symmetric matrix A_p,
- $B_\epsilon(p) \cap B_\epsilon(q) = \emptyset$ for $p \neq q$,
- $\mathrm{cl}\, B_\epsilon(p) \subset U$

via homotopy being proper on gradients.

Let $x = 0$ be a critical point of φ_M. We have

$$\varphi_M(x) = \frac{1}{2} x^T H_0\varphi_M x + R(x),$$

where $R(x)$ is C^2-function such that $R(x) = o(\|x\|^2)$. Note that it implies that $\|\nabla R(x)\| = o(\|x\|)$.

There exists a C^∞-function $\eta : [0,\infty) \to [0,1]$ satisfying

- $\eta(x) := \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{if } x \geqslant 2, \end{cases}$
- $|\eta'(x)| < 2$ for all $x \in [0,\infty)$.

Define $\lambda : \mathbb{R}^n \to \mathbb{R}$ by the formula $\lambda(x) := \eta(\|x\|/\epsilon)$. Observe that $\|\nabla\lambda(x)\| < 2/\epsilon$ for $x \in \mathbb{R}^n$. Set

$$\chi(x,t) := \frac{1}{2} x^T H_0\varphi_M x + \big(1 - t\lambda(x)\big) R(x)$$

for $(x,t) \in U \times I$. Obviously, the homotopy χ does not change φ_M outside $B_{2\epsilon}$. Finally, set $\psi(x) := \chi(x,1)$. We will show that there is $\epsilon > 0$ such that $x = 0$ is the only zero of $\nabla\psi$ in $B_{2\epsilon}$. Let $c := \min_{\|x\|=1} \|H_0\varphi_m\|$. We choose $\epsilon > 0$ such that in $B_{2\epsilon}$,

- $|R(x)| < \frac{c}{16}\|x\|^2$,
- $\|\nabla R(x)\| < \frac{c}{4}\|x\|$.

Then for $x \in B_{2\epsilon}$,

$$\begin{aligned} \big\|\nabla\psi(x)\big\| &= \big\|H_0\varphi_M(x) + \nabla\big(1-\lambda(x)\big)\cdot R(x) + \big(1-\lambda(x)\big)\cdot\nabla R(x)\big\| \\ &\geqslant c\cdot\|x\| - \frac{2}{\epsilon}\frac{c}{8}\epsilon\|x\| - \frac{c}{4}\|x\| = \frac{c}{2}\|x\| \end{aligned}$$

which is our claim.

Above we have defined χ in case of one critical point ($x = 0$). Of course, we can define χ in the similar way near each point of $\mathrm{Crit}(\varphi_M)$.

Step 3. If ϵ from Step 2 is less than 1 we define

- $\Omega_t := \frac{1}{(1-t)+t\epsilon} U$,
- $\alpha(x,t) := \psi[((1-t)+t\epsilon)x]$.

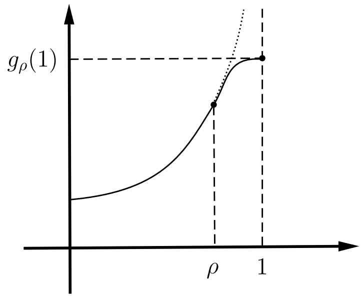


Fig. 2. Function g_ρ.

Observe that for $p \in \mathrm{Crit}(\psi)$,

$$\alpha_1(x) = \psi(\epsilon x) = \left(x - \frac{p}{\epsilon}\right)^T A'_p \left(x - \frac{p}{\epsilon}\right)$$

on $B_1(p/\epsilon)$, where $A'_p = \epsilon^2 A_p$.

Step 4. We construct a proper gradient otopy $\nabla_x \beta$ between $\nabla \alpha_1 = \nabla \beta_0$ and $\nabla \beta_1$, where β_1 is defined not on the whole U, but only on unit balls around critical points of α_1. Let $g_1 : [0,1) \to [1,\infty)$ be given by $g_1(x) = \frac{1}{1-x^2}$. Consider the family of auxiliary functions $g_\rho : [0,1] \to [1,\infty)$ indexed by the real parameter $\rho \in [0,1)$ and uniquely determined by the conditions (see Fig. 2)

- $g_\rho\restriction_{[0,\rho]} \equiv g_1\restriction_{[0,\rho]}$,
- g_ρ is a quadratic polynomial on $[\rho, 1]$,
- g_ρ is C^1-function such that $g'_\rho(1) = 0$.

We define otopy $(\nabla_x \beta, \Omega)$ by

- $\Omega_t := \begin{cases} U & \text{if } t \in [0,1), \\ \bigcup_{p \in \mathrm{Crit}(\alpha_1)} B_1(p) & \text{if } t = 1, \end{cases}$
- $\beta(x,t) := \begin{cases} g_t(\|x-p\|) \cdot \alpha_1(x) & \text{if } t \in [0,1] \text{ and } x \in B_1(p), \\ g_t(1) \cdot \alpha_1(x) & \text{otherwise.} \end{cases}$

Observe that

- $\beta_0 = \alpha_1$,
- $\beta_1(x) = \frac{1}{1-\|x-p\|^2}(x-p)^T A'_p (x-p)$ for $x \in B_1(p)$,
- $(\nabla_x \beta, \Omega)$ is proper by Corollary A.2 applied to each ball $B_1(p)$ and the family of functions g_ρ.

Step 5. It is enough to consider one fixed ball centered at $p = 0$. Let us define otopy $(\nabla_x \gamma, \Omega)$ by

- $\Omega := B_1 \times I$,
- $\gamma(x,t) := \frac{1}{1-\|x\|^2}(x)^T A_t(x)$, where A_t is a path in $GL_n(\mathbb{R}) \cap \mathrm{Symm}(n)$ connecting A to some diagonal matrix with ± 1 on the diagonal.

Observe that $(\nabla_x \gamma, \Omega)$ is proper by Corollary A.3. $\square$

6. Proof of Lemma 4.2

Suppose that $\nabla \varphi$ is ball-canonical (see Section 3). There is no loss of generality in assuming that the potential $\varphi : B \to \mathbb{R}$ has the form

$$\varphi(x,y) := \frac{\sum_{i=1}^m x_i^2 - \sum_{j=1}^l y_j^2}{1 - \sum_{i=1}^m x_i^2 - \sum_{j=1}^l y_j^2},$$

where $m + l = n$. Set

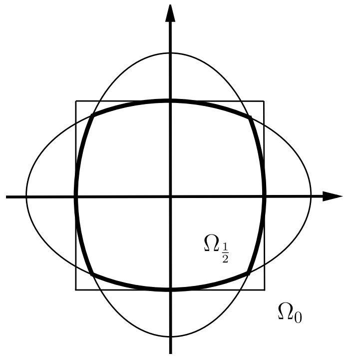


Fig. 3. Transformation of the domain Ω_t.

$$S(x,y,t) := 1 - t\sum_{i=1}^{m} x_i^2 - t\sum_{j=1}^{l} y_j^2,$$
$$F_i(x,y,t) := S(x,y,t) - (1-t)x_i^2 \quad \text{for } 1 \leqslant i \leqslant m,$$
$$G_j(x,y,t) := S(x,y,t) - (1-t)y_j^2 \quad \text{for } 1 \leqslant j \leqslant l.$$

Define

- $\Omega := \{(x,y,t) \mid t \in I,\ F_i > 0,\ G_j > 0 \text{ for all } i, j\}$,
- $\chi(x,y,t) := \sum_{i=1}^{m} \frac{x_i^2}{F_i} - \sum_{j=1}^{l} \frac{y_j^2}{G_j}$.

Observe that

$$\chi_0(x,y) = \sum_{i=1}^{m} \frac{x_i^2}{1-x_i^2} - \sum_{j=1}^{l} \frac{y_j^2}{1-y_j^2} \quad \text{and} \quad \chi_1 = \varphi.$$

The otopy $(\nabla_x\chi, \Omega)$ connects the cube-canonical χ_0 with the ball-canonical χ_1.

Example. Let $m = l = 1$. Then

- $\Omega_t = \{(x,y) \mid 1 - x^2 - ty^2 > 0,\ 1 - tx^2 - y^2 > 0\}$,
- $\chi(x,y,t) = \frac{x^2}{1-x^2-ty^2} - \frac{y^2}{1-tx^2-y^2}$.

So $\chi_0(x,y) = \frac{x^2}{1-x^2} - \frac{y^2}{1-y^2}$ (cube-canonical) and $\chi_1(x,y) = \frac{x^2-y^2}{1-x^2-y^2}$ (ball-canonical), see Fig. 3.

What is left is to show that $(\nabla_x\chi, \Omega)$ is proper i.e. if the sequence $\{(w_k, t_k)\} \subset \Omega$ has no accumulation point in Ω, then $\|\nabla_x\chi(w_k, t_k)\| \to \infty$. So assume that $\{(w_k, t_k)\} \subset \Omega$ has no accumulation point in Ω. In particular,

$$\min_{i,j}\{F_i(w_k,t_k), G_j(w_k,t_k)\} \to 0 \quad \text{with } k \to \infty.$$

Note that the proof will be completed by showing the following claim.

Claim. *If* $\min_{i,j}\{F_i, G_j\} \leqslant 1/2$*, then*

$$\|\nabla_x\chi(w_k,t_k)\| \geqslant \frac{1}{2(m+1)(l+1)} \cdot \frac{1}{\min_{i,j}\{F_i, G_j\}}.$$

We first observe that if $\min_{i,j}\{F_i, G_j\} \leqslant 1/2$, then

$$\forall_j\ y_j^2 < \frac{1}{2(m+1)l} \quad \Longrightarrow \quad x_r^2 := \max_i\{x_i^2\} \geqslant \frac{1}{2(m+1)}. \tag{6.1}$$

Conversely, suppose that

$$\forall_j\ y_j^2 < \frac{1}{2(m+1)l} \quad \text{and} \quad \forall_i\ x_i^2 < \frac{1}{2(m+1)}.$$

Then $\sum_i x_i^2 + \sum_j y_j^2 < 1/2$ and hence, by the definition of F_i and G_j, $F_i > 1/2$ and $G_j > 1/2$ for all i, j, contrary to our assumption.

Similarly,

$$\forall_i\ x_i^2 < \frac{1}{2(l+1)m} \quad \Longrightarrow \quad y_s^2 := \max_j\{y_j^2\} \geqslant \frac{1}{2(l+1)}. \tag{6.2}$$

Let us compute

$$\frac{\partial\chi}{\partial x_r} = 2x_r\left(\sum_i \frac{tx_i^2}{F_i^2} - \sum_j \frac{ty_j^2}{G_j^2} + \frac{S}{F_r^2}\right) := 2x_r A_r,$$

$$\frac{\partial\chi}{\partial y_s} = 2y_s\left(\sum_i \frac{tx_i^2}{F_i^2} - \sum_j \frac{ty_j^2}{G_j^2} - \frac{S}{G_s^2}\right) := 2y_s B_s.$$

Now we only need to consider the following three cases (by (6.1) and (6.2) at least one of them holds). Since $|x_i|, |y_j| \leqslant 1$, we will use the inequalities $|x_i| \geqslant x_i^2$ and $|y_j| \geqslant y_j^2$.

Case 1. $\boxed{\forall_j\ y_j^2 < \frac{1}{2(m+1)l}}$ By (6.1) $x_r^2 := \max_i\{x_i^2\} \geqslant \frac{1}{2(m+1)}$ and so $x_r^2 > \sum_j y_j^2$, which implies $F_r := \min_{i,j}\{F_i, G_j\}$. This gives

$$\sum_i \frac{tx_i^2}{F_i^2} - \sum_j \frac{ty_j^2}{G_j^2} \geqslant \sum_i \frac{tx_i^2}{F_i^2} - \frac{tx_r^2}{F_r^2} \geqslant 0$$

and hence finally

$$\left|\frac{\partial\chi}{\partial x_r}\right| = |2x_r||A_r| \geqslant \frac{1}{m+1} \cdot \frac{S}{F_r^2} \geqslant \frac{1}{m+1} \cdot \frac{1}{\min_{i,j}\{F_i, G_j\}}.$$

Case 2. $\boxed{\forall_i\ x_i^2 < \frac{1}{2(l+1)m}}$ Analogous reasoning shows that

$$\left|\frac{\partial\chi}{\partial y_s}\right| \geqslant \frac{1}{l+1} \cdot \frac{1}{\min_{i,j}\{F_i, G_j\}}.$$

Case 3. $\boxed{\exists_{r,s}\ x_r^2 := \max_i\{x_i^2\} \geqslant \frac{1}{2(l+1)m} \text{ and } y_s^2 := \max_j\{y_j^2\} \geqslant \frac{1}{2(m+1)l}}$ By definition,

$$A_r - B_s = \frac{S}{F_r^2} + \frac{S}{G_s^2} \geqslant \frac{1}{F_r} + \frac{1}{G_s} \geqslant \frac{1}{\min_{i,j}\{F_i, G_j\}}.$$

Hence $\max\{|A_r|, |B_s|\} \geqslant \frac{1}{2\min_{i,j}\{F_i,G_j\}}$ and, in consequence,

$$\max\left\{\left|\frac{\partial\chi}{\partial x_r}\right|, \left|\frac{\partial\chi}{\partial y_s}\right|\right\} = \max\{2|x_r||A_r|, 2|y_s||B_s|\}$$
$$\geqslant \frac{1}{2(m+1)(l+1)} \cdot \frac{1}{\min_{i,j}\{F_i, G_j\}}.$$

Combining all the above cases we immediately get our claim. □

7. Proof of Lemma 4.3

To simplify notation we restrict ourselves to showing Lemma 4.3 for two elementary maps of different signs. Furthermore, the otopy parameter t will run the interval $[-1, 1]$ instead of the usual $[0, 1]$ to get nice formulas. Let us define the domain

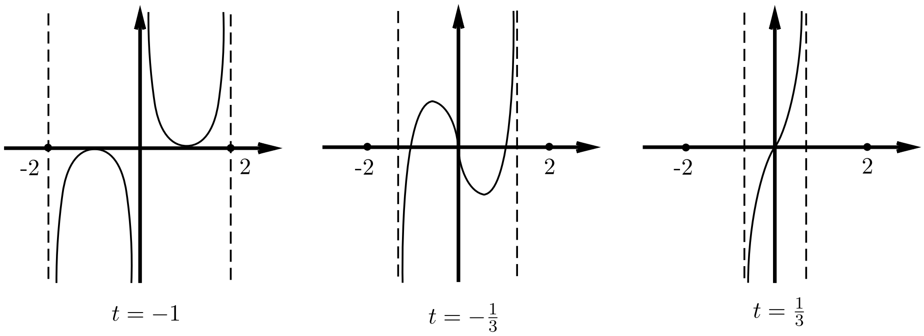


Fig. 4. Annihilation of elementary maps on potentials.

- $\Omega_t := \begin{cases} \{(x,t) \mid 0 < |x_1| < 2,\ |x_i| < 1 \text{ for } i > 1\} & \text{if } t = -1, \\ \{(x,t) \mid |x_1| < 1-t,\ |x_i| < 1 \text{ for } i > 1\} & \text{if } t \in (-1,1], \end{cases}$
- $\Omega := \bigcup_{t\in[-1,1]} \Omega_t$.

and the otopy potential (see Fig. 4),

$$\chi(x,t) := \begin{cases} \frac{-(x_1-t)^2}{1-(x_1-t)^2} + \sum\limits_{i>1} \frac{x_i^2}{1-x_i^2} + \operatorname{sgn}(t+1)\frac{t^2}{1-t^2} & \text{for } -1+t < x_1 \leqslant 0, \\ \frac{(x_1-t)^2}{1-(x_1-t)^2} + \sum\limits_{i>1} \frac{x_i^2}{1-x_i^2} - \operatorname{sgn}(t+1)\frac{t^2}{1-t^2} & \text{for } 0 \leqslant x_1 < 1-t. \end{cases}$$

Note that $\Omega_1 = \emptyset$. It is easy to see that $(\nabla_x \chi, \Omega)$ is a proper gradient otopy. □

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Appendix A

For a matrix A, let $\mu(A) := \min_{\|x\|=1} \|Ax\|$. Note that if $A \in GL_n(\mathbb{R})$, then $\mu(A) > 0$. Let us denote by $\mathrm{Symm}(n)$ the set of $n \times n$ real symmetric matrices.

Proposition A.1. *If*

- $A \in \mathrm{Symm}(n)$,
- $g \in C^1([0,1), \mathbb{R}_+)$ *with* $g' \geqslant 0$,
- $\varphi : B_1 \to \mathbb{R}$ *is given by* $\varphi(x) := g(\|x\|) \cdot x^T Ax$,

then $\|\nabla\varphi(x)\| \geqslant 2g(\|x\|)\mu(A)\|x\|$.

Proof. Since

$$\nabla\varphi(x) = g'(\|x\|) \cdot \frac{x^T Ax}{\|x\|} \cdot x + g(\|x\|) \cdot 2Ax := w_1 + w_2,$$

we have

$$w_1 \cdot w_2 = 2g'(\|x\|)g(\|x\|)\frac{(x^T Ax)^2}{\|x\|} \geqslant 0.$$

Hence

$$\|\nabla\varphi(x)\| \geqslant \|w_2\| \geqslant 2g(\|x\|)\mu(A)\|x\|. \qquad \square$$

Corollary A.2. *Let* $A \in GL_n(\mathbb{R}) \cap \mathrm{Symm}(n)$. *Assume that we have a family of functions* $\{g_\rho\}_{\rho\in I}$ *such that*

- $g_\rho \in C^1([0,1), \mathbb{R}_+)$,

- $g'_\rho \geqslant 0$,
- $\lim_{\substack{\rho\to 1\\ x\to 1}} g_\rho(x) = \infty$.

Let $\varphi_\rho : B_1 \to \mathbb{R}$ be given by $\varphi_\rho(x) := g_\rho(\|x\|) \cdot x^T Ax$. Then

$$\lim_{\substack{\rho\to 1\\ \|x\|\to 1}} \left\|\nabla\varphi_\rho(x)\right\| = \infty.$$

Corollary A.3. *If*

- $A : I \to GL_n(\mathbb{R}) \cap \mathrm{Symm}(n)$ *is continuous,*
- $\Omega := B_1 \times I$,
- $\chi : \Omega \to \mathbb{R}$ *is given by* $\chi(x,t) := \frac{1}{1-\|x\|^2} x^T A_t x$,

then $(\nabla_x\chi, \Omega)$ is a proper gradient otopy.

Proof. Assume that a sequence $\{(x_n, t_n)\} \subset \Omega$ has no accumulation points in Ω (i.e. $\|x_n\| \to 1$). Let $\mu := \min_{t\in I} \mu(A_t)$. Of course, $\mu > 0$. There is no loss of generality in assuming that $\|x_n\| \geqslant 1/2$. By Proposition A.1

$$\left\|\nabla_x\chi(x_n, t_n)\right\| \geqslant \frac{1}{1-\|x_n\|^2}\mu(A_{t_n}) \geqslant \frac{1}{1-\|x_n\|^2}\mu \to \infty,$$

which completes the proof. □

References

[1] P. Bartłomiejczyk, P. Nowak-Przygodzki, Gradient otopies of gradient local maps, Fund. Math. 214 (1) (2011) 89–100.
[2] P. Bartłomiejczyk, K. Gęba, M. Izydorek, Otopy classes of equivariant local maps, J. Fixed Point Theory and Appl. 25 (2010) 195–203.
[3] J.C. Becker, D.H. Gottlieb, Vector fields and transfers, Manuscripta Math. 72 (1991) 111–130.
[4] J.C. Becker, D.H. Gottlieb, Spaces of local vector fields, Contemp. Math. 227 (1999) 21–28.
[5] G.E. Bredon, Topology and Geometry, Springer-Verlag, New York, 1993.
[6] E.N. Dancer, K. Gęba, S. Rybicki, Classification of homotopy classes of gradient equivariant maps, Fund. Math. 185 (2005) 1–18.
[7] D.H. Gottlieb, G. Samaranayake, The index of discontinuous vector fields, New York J. Math. 1 (1994) 130–148.
[8] M.W. Hirsch, Differential Topology, Springer-Verlag, New York, 1976.
[9] A. Parusiński, Gradient homotopies of gradient vector fields, Studia Math. XCVI (1990) 73–80.