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Quadratic stochastic operators on Banach lattices

Michał Badocha[1] · **Wojciech Bartoszek**[1]

Abstract We study the convergence of iterates of quadratic stochastic operators that are mean monotonic. They are defined on the convex set of probability measures concentrated on a weakly compact order interval $S = [0, f]$ of a fixed Banach lattice F. We study their regularity and identify the limits of trajectories either as the "infimum" or "supremum" of the support of initial distributions.

Keywords Quadratic stochastic operator · Nonlinear Markov process · Weak convergence · Phenotypic evolution

Mathematics Subject Classification Primary 60J35; Secondary 47H20 · 92D25

1 Introduction

The theory of quadratic stochastic operators (q.s.o.) is rooted in works of Bernstein (cf. [8,9]). Their importance was rediscovered in 1950s by S. Ulam. His seminal report [28] prompted for many theoretical publications (cf. [4–7,11,12,16,20,21,29]), monographs (cf. [13,23]) and computer supported mathematical projects (cf. [3,14, 15]) aiming to develop a unified theory illustrating (asymptotic) properties of q.s.o.. This subject has been intensively studied in mathematics and biology for almost three decades. In spite of that important questions have remained unsolved and the task is

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far from being completed. The asymptotic behavior of real or abstract mathematical models of biological systems, consisting with three or more biotypes, is not fully understood. We anticipate that q.s.o. will play in future an important role in genetics, population dynamics, social sciences as well as in other areas. Mathematically they perfectly suit to model the evolution of statistical configurations of all kinds of biotypes (genome, phenotypes e.t.c.) both for finite or continuous populations. The reader is referred to [13,17] for a comprehensive and updated review of the topic. The list of very recent articles and online presentations is long. Let us only mention [18,19, 25,26], where the notion of quadratic stochastic operators is put into abstract vector (Banach) spaces with specific order or norm structures. In [24] the author studies nonhomogeneous Markov chains on ordered Banach spaces which are strongly linked to some picture of quadratic stochastic operators. Namely, as it has been proved in [7], generated by a quadratic stochastic operator $\mathbb{V}$ the so-called associated Markov chains may be efficiently used to determine the behavior of $\mathbb{V}^n$.

There has been much interest in recent years in self-organizing search methods in the q.s.o. field. Recently Ganikhodjaev, Saburov and Muhitdinov (see [15]) have generalized the notion of q.s.o. to bilinear forms on σ-additive measures on [0, 1]. In particular, points of the unit interval [0, 1] serve to code (continuum valued) traits attributed to each individual from a considered population. As usual children inherit their traits from (two) parents, who mate randomly. The paper [15] drops the Mendelian paradigm and introduces trends (however, only in a one-dimensional direction), which are steady in time. The authors proved in [15] regularity in the case when there is no mutation and an inherited trait comes (randomly) from one of its parents. In this paper we propose further extensions. In particular, we obtain regularity under essentially weaker constraints. Moreover, our methods applied to theorems and examples from [15] simplify existing proofs.

Instead of one particular trait we propose to characterize an individual by infinite dimensional set of parameters, encoded by vectors from an order interval $S = [0, f]$ in a fixed real Banach lattice $(F, \|\cdot\|, \leq)$. Let us very briefly recall basic notions necessary to formulate our results (regarding the theory of Banach lattices and other facts from functional analysis the reader is referred to [2] or [27]). A Banach space $(F, \|\cdot\|)$ equipped with a partial order $\leq$ is a Banach lattice if the lattice operations (the infimum $x \wedge y$ and the supremum $x \vee y$) are well defined in F, satisfy axioms of Riesz spaces and are compatible with the norm topology (cf. [2], pp. 4, 5, 181 or [27], pp. 47–52). In particular, $\|y\| \leq \|x\|$ whenever $|y| \leq |x|$ in F. The modulus in F is defined as $|x| = x \vee (-x)$, and therefore $\||x|\| = \|x\|$ for all $x \in F$. The positive cone $F_+ = \{x \in F : 0 \leq x\}$ is a (weakly) closed subset of F (cf. [27], Proposition 5.2). Let us mention that for all $x, y \in F$ we have $x \vee y = \frac{x+y+|x-y|}{2}$, and $x \wedge y = \frac{x+y-|x-y|}{2}$. The order interval, with endpoints $a \leq b \in F$, is defined as $I_{a,b} = [a, b] = \{x \in F : a \leq x \leq b\}$. If $a = 0$ and $f \in F_+$, we simply write I_f in place of $I_{0,f} = [0, f]$. In the case $F = \mathbb{R}$, with ordinary modulus $|\cdot|$, order intervals are classical segments $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. It is well known that order intervals in Banach lattices are always bounded, convex and (weakly) closed. However, they are not weakly compact in general. Classical spaces $(L^p(\mu), \|\cdot\|_p)$ and $(C(K), \|\cdot\|_{\sup})$ are important examples of Banach lattices. It follows from the Banach–

Alaoglu theorem that order intervals are weakly compact if F is reflexive. In particular, if $F = L^p(\mu)$, where $1 < p < \infty$ (using different arguments also in $L^1(\mu)$), then order intervals are weakly compact. However, this does not hold for $C(K)$ or $L^\infty(\mu)$. We recall that the weak topology on a Banach space $(F, \|\cdot\|)$ is the smallest topology $\mathcal{T}$ so that all norm continuous linear functionals $\xi \in F'$ are $\mathcal{T}$ continuous. The weak topology is denoted by $\mathcal{T}_w$ and it is a locally convex topology generated by the basis consisting of neighborhoods $\mathcal{U}_{\xi_1,\dots,\xi_n,\varepsilon} = \{u \in F : |\xi_j(u)| < \varepsilon, \ j = 1, \dots, n\}$ of the zero vector $0 \in F$, where $n \in \mathbb{N}$, $\xi_1, \dots, \xi_n \in F'$, and $\varepsilon > 0$. The net x_α converges weakly to x in F if and only if for every fixed $\xi \in F'$ we have $\lim_\alpha \xi(x_\alpha) = \xi(x)$. Weak compactness plays in the category of Banach lattices an important role (cf. [2], chapter 4.2).

In this paper we shall deal with order intervals (phase spaces) $S \subset F$, which are assumed to be compact and metrizable for the weak topology (in particular, by the Mazur theorem they are norm separable). Hence the weak and norm Borel structures on S coincide. We denote the Borel σ field in S by $\mathcal{B}$. As usual δ_x stands for the Dirac measure at $x \in S$. Compactness of S implies (cf. §6 in [10]) that the convex set $\mathcal{P}(S)$ of all probability measures μ, on the measurable space $(S, \mathcal{B})$, is compact (metrizable) for the weak measure convergence. We recall that a sequence of probability measures $\mu_n \in \mathcal{P}(S)$ converges to $\mu \in \mathcal{P}(S)$ in the weak convergence of measures if $\lim_{n\to\infty} \int_S f d\mu_n \to \int_S f d\mu$ holds for all $\mathcal{T}_w$ continuous (bounded) functions $f : S \to \mathbb{R}$. Then we write $\mu_n \Rightarrow \mu$. In order not to overexploit the term "weak" (especially in two different contexts) we shall use the notion of the Fortet–Mourier norm. We recall that given a finite Borel (signed) measure μ on S, the Fortet–Mourier norm is defined as $\|\mu\|_{\text{FM}} = \sup\{|\int f d\mu| : 0 \le f \le 1 \text{ and } \text{Lip} f \le 1\}$ (cf. [22], p. 48). As the reader may guess, given a continuous function $f : S \to \mathbb{R}$

$$\text{Lip} f = \sup\left\{\frac{|f(x) - f(y)|}{\varrho_S(x, y)} : x \ne y \in S\right\},$$

where ϱ_S is a fixed metric on S compatible with the relative weak topology $\mathcal{T}_w|_S$ on the set S (the choice of a specific metric ϱ_S is not crucial). It is well known (see [22], p. 49, theorem 1.46) that on the space $\mathcal{P}(S)$ the convergence $\mu_n \Rightarrow \mu$ holds if and only if $\|\mu_n - \mu\|_{\text{FM}} \to 0$.

By $\text{bar}(\mu)$ we denote the barycenter of μ, as long as it exists. We understand it as the Pettis integral $\int_F z d\mu(z)$; i.e. $\xi(\text{bar}(\mu)) = \xi(\int_F z d\mu(z)) = \int_F \xi(z) d\mu(z)$ for all $\xi \in F'$. Clearly $\text{bar}(\mu)$ is well defined if the topological support $\text{supp}(\mu)$ is weakly compact (cf. [1], pp. 10–12).

We denote $i(\mu) = \inf \text{supp}(\mu)$, $s(\mu) = \sup \text{supp}(\mu)$ and $I(\mu) = [i(\mu), s(\mu)]$ as long as all is well defined. For this, let us suppose that F has an order continuous norm (cf. [2], pp. 185–186) and $\text{supp}(\mu)$ is contained in an order interval $I_{a,b} = [a, b] \subseteq F_+$. Clearly, $r_n = x_1 \wedge \cdots \wedge x_n \in I_{a,b}$, if $x_j \in \text{supp}(\mu)$, $n = 1, 2, \dots$. Hence $\|r_n\| \ge \|a\|$ is bounded from below. Using induction method and the property of norm order continuity (cf. [2], Theorem 4.9, p. 186) we may construct a sequence $(x_j)_{j\ge1}$ such that $\lim_{n\to\infty} r_n = r \in [a, b]$ exists and the norm $\|r\|$ is as small as possible (we can take a dense sequence $x_j \in \text{supp}(\mu)$ actually). Notice that $r \le x$ for all $x \in \text{supp}(\mu)$. If $v \le x$ for all $x \in \text{supp}(\mu)$, then $v \le x_j$ for all j, hence $v \le r_n$ for all n and finally

$v \le r$. In particular, $r = i(\mu)$ does exist (however not necessarily $r \in \text{supp}(\mu)$). Similarly we obtain the existence of $s(\mu)$.

2 Basics on quadratic stochastic operators

Given a weakly compact metrizable set $S \subseteq F$, let $\mathbb{P} = \{P(x, y, \cdot)\}_{x,y \in S}$ be a family of set functions (defined on the measurable space $(S, \mathcal{B})$, where $\mathcal{B}$ stands for the Borel σ-field) satisfying:

(2.1) $P(x, y, \cdot) = P(y, x, \cdot) \in \mathcal{P}(S)$ for all $x, y \in S$,
(2.2) $S \times S \ni (x, y) \to P(x, y, A) \in [0, 1]$ is measurable for every fixed $A \in \mathcal{B}$.

Then the system $\mathbb{P} = \{P(x, y, \cdot)\}_{x,y \in S}$ is called a *quadratic transition probability function*. If moreover it satisfies

(2.3) $S \times S \ni (x, y) \to P(x, y, \cdot) \in \mathcal{P}(S)$ is continuous, where $S \times S$ is endowed with the weak $\times$ weak topology and in $\mathcal{P}(S)$ we have the Fortet–Mourier norm $\| \cdot \|_{\text{FM}}$,

then such a quadratic transition function is called *Feller* [clearly the condition (2.3) implies (2.2)].

Let us mention briefly, that quadratic transition functions can be used to define perhaps the simplest class of nonlinear Markov processes $\{\xi_k\}_{k \ge 0}$. We will not dwell on this subject but rather focus only on the evolution and weak limits of distributions $P(\xi_k \in \cdot)$. However, the problem of behavior of trajectories $\xi_k(\omega)$, or the rate of convergence, remains untouched.

Any measurable quadratic transition probability function $\mathbb{P}$ may be uplifted to a bilinear mapping of $\mathcal{P}(S)$. Namely, define

$$\mathcal{P}(S) \times \mathcal{P}(S) \ni (\mu, \nu) \to Q(\mu, \nu)(\cdot) = \int_S \int_S P(x, y, \cdot) d\mu(x) d\nu(y) \in \mathcal{P}(S).$$

Definition 2.1 A *quadratic stochastic operator* (generated by the family $\mathbb{P}$) is defined as $\mathbb{V}(\mu) = Q(\mu, \mu)$. It is called *Feller* if the quadratic transition probability function $\mathbb{P}$ is *Feller* [i.e. when (2.3) holds].

It is not hard to prove (cf. [4,7]) that $\|\mathbb{V}(\mu) - \mathbb{V}(\nu)\|_{\text{TV}} \le 2\|\mu - \nu\|_{\text{TV}}$ for all $\mu, \nu \in \mathcal{P}(S)$, so $\mathbb{V}$ is a continuous transformation for the total variation norm $\| \cdot \|_{\text{TV}}$ topology. Moreover, if it is Feller, then $\mathbb{V}$ is continuous on $\mathcal{P}(S)$ for the Fortet–Mourier norm; i.e.

$$\lim_{n \to \infty} \mathbb{V}(\mu_n) = \mathbb{V}\left(\lim_{n \to \infty} \mu_n\right) \quad \text{in the norm } \| \cdot \|_{\text{FM}},$$

as long as $\mu_n \in \mathcal{P}(S)$ converges in the norm $\| \cdot \|_{\text{FM}}$ Indeed, let $\|\mu_n - \mu\|_{\text{FM}} \to 0$ and $g: S \to \mathbb{R}$ be weakly continuous. Then

$$\int_S g(z) \mathbb{V}(\mu_n)(dz) = \int_S \int_S \int_S g(z) P(x, y, dz) d\mu_n(x) d\mu_n(y).$$

By (2.3) the function $S \times S \ni (x, y) \to \int_S g(z) P(x, y, dz)$ is weakly continuous. It is well known that convergence in the Fortet–Mourier norm is preserved if we bring it to direct products. In other words if $\|\mu_n - \mu\|_{\mathrm{FM}} \to 0$ and $\|\nu_n - \nu\|_{\mathrm{FM}} \to 0$ then $\|\mu_n \otimes \nu_n - \mu \otimes \nu\|_{\mathrm{FM}} \to 0$. Thus

$$\lim_{n \to \infty} \int_S \int_S \int_S g(z) P(x, y, dz) d\mu_n(x) d\mu_n(y) = \int_S \int_S \int_S g(z) P(x, y, dz) d\mu(x) d\mu(y)$$
$$= \int_S g(z) \int_S \int_S P(x, y, dz) d\mu(x) d\mu(y) = \int_S g(z) \mathbb{V}(\mu)(dz).$$

The natural notion of regularity, in the context of q.s.o., was introduced very early (see [17,23]). We adopt it for the Fortet–Mourier topology.

Definition 2.2 We say that a q.s.o. $\mathbb{V} \colon \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is *weakly regular* if

$$\lim_{n \to \infty} \|\mathbb{V}^n(\mu) - L(\mu)\|_{\mathrm{FM}} = 0$$

for every $\mu \in \mathcal{P}(S)$, where the limit measure $L(\mu) \in \mathcal{P}(S)$. If there exists a unique measure $\mu_* \in \mathcal{P}(S)$ such that

$$\lim_{n \to \infty} \|\mathbb{V}^n(\mu) - \mu_*\|_{\mathrm{FM}} = 0$$

for all $\mu \in \mathcal{P}(S)$, then a q.s.o. $\mathbb{V}$ is called *weak mixing*.

Depending on the (topological) point of view different modes of convergence are distinguished. To connect our results with other contemporary studies we recall (cf. [4,7]) the following notions:

Definition 2.3 We say that a q.s.o. $\mathbb{V} \colon \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is *strongly regular* if

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for every $\mu \in \mathcal{P}(S)$, where the limit measure $L(\mu) \in \mathcal{P}(S)$. If there exists a unique measure $\mu_* \in \mathcal{P}(S)$ such that

$$\lim_{n \to \infty} \|\mathbb{V}^n(\mu) - \mu_*\|_{\mathrm{TV}} = 0$$

for all $\mu \in \mathcal{P}(S)$, then a q.s.o. $\mathbb{V}$ is called *strong mixing*.

The uniform convergence give us:

Definition 2.4 We say that a q.s.o. $\mathbb{V} \colon \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is *uniformly regular* if

$$\lim_{n \to \infty} \sup_{\mu \in \mathcal{P}(S)} \|\mathbb{V}^n(\mu) - L(\mu)\|_{\mathrm{TV}} = 0.$$

If there exists a unique measure $\mu_* \in \mathcal{P}(S)$ such that

$$\lim_{n \to \infty} \sup_{\mu \in \mathcal{P}(S)} \|\mathbb{V}^n(\mu) - \mu_*\|_{\mathrm{TV}} = 0,$$

then a q.s.o. $\mathbb{V}$ is called *norm mixing*.

Strong and norm mixing q.s.o. and their geometric structure have been recently described in [4,5,7]. The weak mixing is studied in [6]. However, Bartoszek et al. [6] is restricted to kernel quadratic stochastic operators. In particular, $P(x, y, \cdot)$ are assumed to be absolutely continuous with respect to the Lebesgue measure and additionally $\int z P(x, y, dz) = \frac{x+y}{2}$ for all x, y (roughly speaking—the offspring is on average the mean of the parents). Such operators are called centred kernel q.s.o. On the other hand, the models considered in [6] generally apply to unbounded phase spaces. Their weak limit of $\mathbb{V}^n(f)$ may be (depending on f) both discrete (Dirac δ) or other probability measures, depending to what extent the CLT works. Mathematical methods used in their proofs come from the theory of characteristic functions, making it difficult to obtain closed form statements about the limits. In our paper we focus solely on weak regularity, assuming bounded domains and we look what happens if the evolution has a trend. In particular, our quadratic transition probabilities are not centred. We may consider our approach as generalizations of [15] and complementary to [6].

If for all $x, y \in S$ the measure $P(x, y, \cdot)$ is a convex combination of δ_x, δ_y, then the q.s.o. $\mathbb{V}$ is called Volterra. Clearly the Volterra q.s.o. can only model the mutation free evolution. The Mendelian situation occurs when $P(x, y, \cdot) = 0.5\delta_x + 0.5\delta_y$. We notice that in the last case $\mathbb{V}(\mu) = \mu$ for all $\mu \in \mathcal{P}(S)$.

Following [15], when $S = [0, 1]$ and $F = \mathbb{R}$, Volterra q.s.o. are called Lebesgue q.s.o. It has been proved (see Theorem 4.6 in [15]) that any Lebesgue q.s.o. such that $P(x, y, \cdot) = p\delta_{x \wedge y}(\cdot) + (1 - p)\delta_{x \vee y}(\cdot)$, with $P(x, x, \cdot) = \delta_x(\cdot)$ is weakly regular, and the iterates $\mathbb{V}^n(\mu)$ converge to $\delta_{i(\mu)}$, if $0.5 < p \le 1$ or respectively to $\delta_{s(\mu)}$ if $0 \le p < 0.5$ (when $p = 0.5$ then $\mathbb{V}$ is the identity transformation).

Our approach brings further extensions, with two folded generalizations. Firstly, in the case of Lebesgue operators $\mathbb{V}$ on $S = [0, 1]$, we only assume that $P(x, y, \cdot) = \alpha(x, y)\delta_{x \wedge y}(\cdot) + (1 - \alpha(x, y))\delta_{x \vee y}(\cdot)$, where $\alpha(x, y) > 0.5$ for all $x \ne y \in [0, 1]$ (or $\alpha(x, y) < 0.5$ for all $x \ne y \in [0, 1]$ respectively). We will show that this condition may be relaxed further (see our Corollaries 5.1 and 5.2). In particular, we do not require the Volterra condition. Secondly, we shall extend Ganikhodjaev's, Saburov's and Muhitdinov's results from [15] to general phase spaces, assuming that $S = I$ is a weakly compact order interval. Let us recall (cf. [2], Theorem 4.9) that order intervals in Banach lattices with order continuous norm are weakly compact. And finally, in general Banach lattices the notion of Lebesgue q.s.o. may split in different directions. For this let us only mention that $\alpha(x, y)\delta_{x \wedge y}(\cdot) + (1 - \alpha(x, y))\delta_{x \vee y}(\cdot)$ is not necessarily a convex combination of δ_x, δ_y.

3 Monotonicity of means and regularity

Theorem 3.1 *Let $I = [0, f] \subset F$ be a weakly metrizable order interval in a Banach lattice $(F, \|\cdot\|, \leq)$ with order continuous norm and let $\{P(x, y, \cdot)\}_{x, y \in I}$ be a quadratic transition probability function satisfying:*

(3.1) $\int_I z P(x, y, dz) \leq \frac{x+y}{2}$, *for all $x, y \in I$.*

Then for every $\mu \in \mathcal{P}(I)$ the barycenters of $\mathbb{V}^n(\mu)$ converge in the norm to $c(\mu) \in I$. In particular $\lim_{n \to \infty} \int_I \xi(z) \mathbb{V}^n(\mu)(dz) = \xi(c(\mu))$ for every $\xi \in F'$.

If additionally the transition probabilities $P(x, y, \cdot)$ are Feller and have the property that

(3.2) *the equality in (3.1) implies $x = y$,*

then $\lim_{n \to \infty} \mathbb{V}^n(\mu) = \delta_{c(\mu)}$ in the $\|\cdot\|_{\text{FM}}$ norm.
If moreover,

(3.3) $\operatorname{supp} P(x, y, \cdot) \subseteq [x \wedge y, x \vee y]$ *for all $x, y \in I$.*

then $c(\mu) \in I(\mu)$.

Proof It is sufficient to evaluate the limits $\lim_{n \to \infty} \int_I \xi(z) V^n(\mu)(dz)$ only for positive $\xi \in F'_+$, as the dual space $F' = F'_+ - F'_-$ is a Banach lattice too (cf. [2], Proposition 5.5). For every natural n

$$
\begin{aligned}
\xi\left(\text{bar}(\mathbb{V}^{n+1}(\mu))\right) &= \int_{[0,f]} \xi(z) \mathbb{V}^{n+1}(\mu)(dz) \\
&= \int_{[0,f]} \xi(z) \int_{[0,f]} \int_{[0,f]} P(x, y, dz) \mathbb{V}^n(\mu)(dx) \mathbb{V}^n(\mu)(dy) \\
&= \int_{[0,f]} \int_{[0,f]} \left(\xi\left(\int_{[0,f]} z P(x, y, dz)\right)\right) \mathbb{V}^n(\mu)(dx) \mathbb{V}^n(\mu)(dy) \\
&\leq \int_{[0,f]} \int_{[0,f]} \xi\left(\frac{x+y}{2}\right) \mathbb{V}^n(\mu)(dx) \mathbb{V}^n(\mu)(dy) \\
&= \int_{[0,f]} \int_{[0,f]} \xi\left(\frac{x}{2}\right) \mathbb{V}^n(\mu)(dx) \mathbb{V}^n(\mu)(dy) \\
&\quad + \int_{[0,f]} \int_{[0,f]} \xi\left(\frac{y}{2}\right) \mathbb{V}^n(\mu)(dx) \mathbb{V}^n(\mu)(dy) \\
&= \int_{[0,f]} \xi(z) \mathbb{V}^n(\mu)(dz) = \xi\left(\text{bar}\left(\mathbb{V}^n(\mu)\right)\right).
\end{aligned}
$$

Hence (cf. [2], Proposition 5.5), $0 \leq \text{bar}(\mathbb{V}^{n+1}(\mu)) \leq \text{bar}(\mathbb{V}^n(\mu))$ for all $n \in \mathbb{N}$, as the order in F' is induced by $F'_+ = \{\xi \in F' : \xi(x) \geq 0$ for all positive $x \in F_+\}$. The barycenters of $\mathbb{V}^n(\mu)$ form a nonincreasing sequence in I. It follows (cf. [2], Theorem 4.9, p. 186) that $\text{bar}(\mathbb{V}^n(\mu))$ converges (in the norm) to a vector $c(\mu) \in [0, f]$.

Let $n_j \nearrow \infty$ be any increasing sequence such that $\mathbb{V}^{n_j}(\mu)$ converges weakly to some $\nu \in \mathcal{P}(S)$. By the Feller assumption

$$\mathbb{V}(\nu) = \mathbb{V}\left(\lim_{j \to \infty} \mathbb{V}^{n_j}(\mu)\right) = \lim_{j \to \infty} \mathbb{V}^{n_j+1}(\mu).$$

It follows that

$$c(\mu) = \lim_j \ \mathrm{bar}(\mathbb{V}^{n_j+1}(\mu)) = \ \mathrm{bar}(\mathbb{V}(\nu)) \leq \ \mathrm{bar}(\nu) = \lim_j \ \mathrm{bar}(\mathbb{V}^{n_j}(\mu)) = c(\mu).$$

We obtain $\mathrm{bar}(\mathbb{V}(\nu)) = \mathrm{bar}(\nu)$. Hence for any positive $\xi \in F'$ we get

$$\begin{aligned}
0 &= \int_I \xi(x)\nu(dx) - \int_I \xi(x)\mathbb{V}(\nu)(dx) \\
&= \int_I \int_I \xi\left(\frac{x+y}{2}\right)\nu(dx)\nu(dy) - \int_I \int_I \int_I \xi(z)P(x, y, dz)\nu(dx)\nu(dy) \\
&= \int_I \int_I \xi\left(\frac{x+y}{2} - \int_I zP(x, y, dz)\right)\nu(dx)\nu(dy).
\end{aligned}$$

By positivity of ξ and condition (3.1) we get $\xi(\frac{x+y}{2} - \int_I zP(x, y, dz)) = 0$ for $\nu \times \nu$ almost all (x, y). Now, applying the additional assumption (3.2), we get $x = y$ for $\nu \times \nu$ almost all (x, y). Hence $\nu = \delta_{c(\mu)}$. Since n_j was arbitrary (with only restriction that $\mathbb{V}^{n_j}(\mu)$ converges), thus $\lim_{n \to \infty} \mathbb{V}^n(\mu) = \delta_{c(\mu)}$.

Finally applying (3.3) we notice that $\mathrm{supp}\mathbb{V}(\mu) \subseteq I(\mu)$. Iterating this inclusion we obtain

$$\overline{\bigcup_{n=0}^{\infty} \mathrm{supp}(\mathbb{V}^n(\mu)} \subseteq I(\mu).$$

We obtain $c(\mu) \in I(\mu)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The next result concerns a specific class of quadratic transition probabilities. They generalize the notion of Lebesgue quadratic stochastic operators introduced in [15].

Definition 3.2 Let $S = [0, f] \subset F$ be an order interval in a Banach lattice F. A quadratic transition probability family $\{P(x, y, \cdot)\}_{x,y \in S}$ is called order cramped if

$$P(x, y, \cdot) = \alpha(x, y)\delta_{x \wedge y} + \beta(x, y)\delta_{x \vee y} + \tfrac{1-\alpha(x,y)-\beta(x,y)}{2}\delta_x + \tfrac{1-\alpha(x,y)-\beta(x,y)}{2}\delta_y, \tag{3.4}$$

where Borel functions $\alpha, \beta \colon [0, f] \times [0, f] \to [0, 1]$ are symmetric (i.e. $\alpha(x, y) = \alpha(y, x)$ and $\beta(x, y) = \beta(y, x)$), and satisfy $\alpha(x, y) + \beta(x, y) \leq 1$. The corresponding quadratic stochastic operator $\mathbb{V}$ also is called order cramped. If $\alpha(x, y) + \beta(x, y) = 1$ then (following [15]) we call such operators Lebesgue.

Obviously $\text{supp}(P(x, y, \cdot)) \subseteq [x \wedge y, x \vee y]$. If $F = \mathbb{R}$ then we obtain a Volterra q.s.o.. Finally if α, β are weakly continuous on $S \times S$, then we get a Feller order cramped q.s.o..

Theorem 3.3 *Let* $\mathbb{V}$ *be an order cramped quadratic stochastic operator defined on an order interval* $I = [0, f]$ *of a real Banach lattice* F *with order continuous norm. If* I *is weakly metrizable, the functions* α *and* β *are weakly continuous and* $\alpha(x, y) > \beta(x, y)$ *if* $x \neq y$, *then the corresponding q.s.o.* $\mathbb{V}$ *is weakly regular, and for all* $\mu \in \mathcal{P}(\mu)$

$$\lim_{n \to \infty} \left\| \mathbb{V}^n(\mu) - \delta_{c(\mu)} \right\|_{\text{FM}} = 0,$$

where $c(\mu) \in I(\mu)$.

Proof It suffices to verify the condition (3.1) of Theorem 3.1. For this

$$\int_I z P(x, y, dz) = \alpha(x, y)(x \wedge y) + \beta(x, y)(x \vee y) + \frac{1 - \alpha(x, y) - \beta(x, y)}{2}(x + y)$$

$$= \alpha(x, y)\frac{x + y - |x - y|}{2} + \beta(x, y)\frac{x + y + |x - y|}{2}$$

$$- (\alpha(x, y) + \beta(x, y))\frac{x + y}{2} + \frac{x + y}{2}$$

$$= \frac{x + y}{2} - \frac{\alpha(x, y) - \beta(x, y)}{2}|x - y| \leq \frac{x + y}{2},$$

with equality only if $x = y$. By Theorem 3.1, for all $\mu \in \mathcal{P}(I)$ we obtain the convergence $\lim_{n \to \infty} \mathbb{V}^n(\mu) = \delta_{c(\mu)}$ in the $\| \cdot \|_{\text{FM}}$ norm. We notice that, if $x, y \in \text{supp}(\mu)$, then $\text{supp}(P(x, y, \cdot)) \subseteq I(\mu)$. Hence $\text{supp}(\mathbb{V}(\mu)) \subseteq I(\mu)$ and by induction $\text{supp}(\mathbb{V}^n(\mu)) \subseteq I(\mu)$ for all $n \in \mathbb{N}$. It follows that $c(\mu) \in I(\mu)$. □

4 Finite dimensional case

In the finite dimensional case, when $F = \mathbb{R}^d$ and the q.s.o. is Lebesgue, the last result may be strengthen. We assume that in $\mathbb{R}^d$ we have the standard order, so the positive cone is $\mathbb{R}^d_+ = \{(x_1, \ldots, x_d) : x_j \geq 0, j = 1, \ldots, d\}$. The lattice norm $\| \cdot \|$ may be taken arbitrary as all norms on finite dimensional vector spaces are equivalent (or apply Corollary 4.4 from [2]).

Theorem 4.1 *Let* $I = [0, f]$ *be a an order interval in the Banach lattice* $F = \mathbb{R}^d$ *and* $P(x, y, \cdot) = \alpha(x, y)\delta_{x \wedge y} + \beta(x, y)\delta_{x \vee y}$, *where* $\alpha(x, y) = \alpha(y, x)$, $\beta(x, y) = \beta(y, x)$ *are continuous functions such that* $\alpha(x, y) > \beta(x, y)$ *if* $x \neq y$ *and satisfy* $\alpha(x, y) + \beta(x, y) = 1$. *Then the corresponding Lebesgue q.s.o. generated by* $\mathbb{P}$ *is weakly regular and for every* $\mu \in \mathcal{P}(I)$

$$\lim_{n \to \infty} \|\mathbb{V}^n(\mu) - \delta_{i(\mu)}\|_{\text{FM}} = 0.$$

Proof By Theorem 3.3 the sequence $\mathbb{V}^n(\mu)$ converges in $\|\cdot\|_{FM}$ to $\delta_{c(\mu)}$, where $c(\mu) \in [i(\mu), s(\mu)]$. Define $W_u = [i(\mu), u]$, where $i(\mu) < u \leq s(\mu)$ and set $D_u = I \setminus W_u$. Then

$$
\begin{aligned}
\mathbb{V}(\mu)(W_u) &= \int_{W_u} \int_{W_u} P(x, y, W_u) d\mu(x) d\mu(y) + 2 \int_{W_u} \int_{D_u} P(x, y, W_u) d\mu(x) d\mu(y) \\
&\quad + \int_{D_u} \int_{D_u} P(x, y, W_u) d\mu(x) d\mu(y) \\
&\geq \mu(W_u)^2 + 2 \int_{W_u} \int_{D_u} \alpha(x, y) d\mu(x) d\mu(y) \\
&\geq \mu(W_u)^2 + \mu(W_u)\mu(D_u) = \mu(W_u).
\end{aligned}
$$

Clearly $i(\mathbb{V}(\mu)) = i(\mu)$. Therefore, we can iterate the above estimation to obtain

$$
\mu(W_u) \leq \mathbb{V}(\mu)(W_u) \leq \cdots \leq \mathbb{V}^n(\mu)(W_u) \leq \cdots
$$

We have already noticed that $i(\mu) = \lim_{n \to \infty} r_n$, where $r_n = x_1 \wedge \cdots \wedge x_n$, for some sequence $x_j \in \text{supp}(\mu)$. It follows from the definition of Lebesgue quadratic transition function that $r_n \in \text{supp}(\mathbb{V}^n(\mu))$. Now for arbitrarily fixed $\varepsilon > 0$ we find $k \geq 0$ such that $\|i(\mu) - r_k\| < \varepsilon$ and define $u_k = (1 - \varepsilon)r_k + \varepsilon s(\mu)$. We shall show that $\mathbb{V}^k(\mu)(W_{u_k}) > 0$. Let $J = \{j : r_{k,j} = s(\mu)_j\} \subseteq \{1, \ldots, d\}$. Then for every $j \in J$ we have $u_{k,j} = s(\mu)_j$ as well. Define $\varrho_* = \min\{s(\mu)_j - r_{k,j} : j \notin J\}$. Clearly the set $\mathcal{U}_\varrho(r_k) = \{x \in I : \|x - r_k\|_\infty < \varrho\}$ is an open neighborhood of r_k (both for $\|\cdot\|$ and $\|\cdot\|_\infty$ norms) in the relative topology on I. Notice that, whenever $\varrho < \varepsilon\varrho_*$, then $\mathcal{U}_\varrho(r_k) \subseteq [i(\mu), u_k] = W_{u_k}$. In fact, for all $j \in J$ if $x \in \mathcal{U}_\varrho(r_k)$, then $x_j \leq s(\mu)_j = u_{k,j}$. On the other hand, if $j \notin J$, then

$$
\begin{aligned}
x_j &< r_{k,j} + \varrho = u_{k,j} + \varepsilon r_{k,j} - \varepsilon s(\mu)_j + \varrho = u_{k,j} - \varepsilon(s(\mu)_j - r_{k,j}) + \varrho \\
&\leq u_{k,j} - \varepsilon\varrho_* + \varrho < u_{k,j}.
\end{aligned}
$$

Hence $0 < \mathbb{V}^k(\mu)(\mathcal{U}_\varrho(r_k)) \leq \mathbb{V}^k(\mu)(W_{u_k})$. By monotonicity we obtain

$$
\lim_{n \to \infty} \mathbb{V}^n(\mu)\left([i(\mu), u_k]\right) \geq \mathbb{V}^k(\mu)(W_{u_k}) > 0.
$$

This implies $c(\mu) \in [i(\mu), u_k]$ and by the triangle inequality

$$
\begin{aligned}
\|c(\mu) - i(\mu)\| &\leq \|u_k - i(\mu)\| \leq \|u_k - r_k\| \\
&\quad + \|r_k - i(\mu)\| < \varepsilon\|f\| + \varepsilon = \varepsilon(1 + \|f\|).
\end{aligned}
$$

We get $c(\mu) = i(\mu)$ as $\varepsilon > 0$ may be as small as we wish.   □

In the finite dimensional case the Lebesgue quadratic operators may be generalized further. If $J \subseteq \{1, 2, \ldots, d\}$ we introduce 2-argument operations $\wedge_{J1}, \vee_{J1}, \wedge_{J2}$ and $\vee_{J2}$. Namely, if $x, y \in \mathbb{R}^d$ we define

$$(x \wedge_{J1} y)_k = \begin{cases} (x \wedge y)_k & \text{if } k \in J \\ x_k & \text{if } k \notin J. \end{cases} \tag{4.1}$$

$$(x \vee_{J1} y)_k = \begin{cases} (x \vee y)_k & \text{if } k \in J \\ x_k & \text{if } k \notin J. \end{cases} \tag{4.2}$$

$$(x \wedge_{J2} y)_k = \begin{cases} (x \wedge y)_k & \text{if } k \in J \\ y_k & \text{if } k \notin J. \end{cases} \tag{4.3}$$

$$(x \vee_{J2} y)_k = \begin{cases} (x \vee y)_k & \text{if } k \in J \\ y_k & \text{if } k \notin J. \end{cases} \tag{4.4}$$

We notice that if $J = \emptyset$ then $x \wedge_{\emptyset 1} y = x = x \vee_{\emptyset 1} y$, $x \wedge_{\emptyset 2} y = y = x \vee_{\emptyset 2} y$. On the other hand, if $J = \{1, 2, \ldots d\}$ then $x \wedge_{\{1,2,\ldots,d\}1} y = x \wedge y = x \wedge_{\{1,2,\ldots,d\}2} y$ and $x \vee_{\{1,2,\ldots,d\}1} y = x \vee y = x \vee_{\{1,2,\ldots,d\}2} y$.

Let $\alpha_J(x, y) = \alpha_J(y, x)$ and $\beta_J(x, y) = \beta_J(y, x)$ be continuous functions on the order interval $[0, f]^2 \subseteq \mathbb{R}^d \times \mathbb{R}^d$, where the index J runs over all subsets of $\{1, 2, \ldots, d\}$. If $\sum_J (\alpha_J(x, y) + \beta_J(x, y)) = \frac{1}{2}$, then

$$\begin{aligned} P(x, y, \cdot) = \sum_J \big(& \alpha_J(x, y)\delta_{x\wedge_{J1}y} + \alpha_J(x, y)\delta_{x\wedge_{J2}y} \\ & + \beta_J(x, y)\delta_{x\vee_{J1}y} + \beta_J(x, y)\delta_{x\vee_{J2}y} \big) \end{aligned}$$

defines properly a quadratic probability transition function. From the biological point of view these operators may model mutations in configurations described by coordinate subsets J.

Proposition 4.2 *Let $\mathbb{P}$ be a quadratic probability transition function defined above acting on the finite dimensional interval $I \subset \mathbb{R}^d$, where $\mathbb{R}^d$ is a Banach lattice with ordinary coordinate order. If for all $J \subseteq \{1, 2, \ldots, d\}$ we have $\alpha_J(x, y) > \beta_J(x, y)$ if $x \neq y$, then for every $\mu \in \mathcal{P}(I)$ we have*

$$\lim_{n\to\infty} \|\mathbb{V}^n(\mu) - \delta_{c(\mu)}\|_{\text{FM}} = 0,$$

where $c(\mu) \in [i(\mu), s(\mu)]$.

Proof We will verify conditions (3.1), (3.2) and (3.3) from Theorem 3.1. (3.3) follows directly from the definition of $P(x, y, \cdot)$. Given a vector $x \in I$ and $J \subseteq \{1, 2, \ldots, d\}$, let us define a vector $x|_J \in \mathbb{R}^d$ by $(x|_J)_k = x_k$ if $k \in J$, and $(x|_J)_k = 0$ if $k \in J^c = \{1, 2, \ldots, d\}\backslash J$. We have

$$\begin{aligned} & \int_I z P(x, y, dz) \\ & = \sum_J \big(\alpha_j(x, y) (x \wedge_{J1} y + x \wedge_{J2} y) + \beta_J(x, y) (x \vee_{J1} y + x \vee_{J2} y) \big) \end{aligned}$$

$$= \sum_J \left(\alpha_J(x, y)(x \wedge y)|_J + \alpha_J(x, y)x|_{J^c} + \alpha_J(x, y)(x \wedge y)|_J + \alpha_J(x, y)y|_{J^c} \right.$$

$$\left. + \beta_J(x, y)(x \vee y)|_J + \beta_J(x, y)x|_{J^c} + \beta_J(x, y)(x \vee y)|_J + \beta_J(x, y)y|_{J^c} \right)$$

$$= \sum_J 2\alpha_J(x, y) \left((x \wedge y)|_J - \left(\frac{x+y}{2} \right)|_J \right)$$

$$+ 2\beta_J(x, y) \left((x \vee y)|_J - \left(\frac{x+y}{2} \right)|_J \right) + \alpha_J(x, y)(x+y) + \beta_J(x, y)(x+y)$$

$$= \sum_J \left(2(\alpha_J(x, y) - \beta_J(x, y)) \left((x \wedge y)_J - \left(\frac{x+y}{2} \right)|_J \right) \right.$$

$$\left. + 2\beta_J(x, y) \left((x \wedge y)|_J - \left(\frac{x+y}{2} \right)|_J + (x \vee y)|_J - \left(\frac{x+y}{2} \right)|_J \right) \right)$$

$$+ \sum_J (\alpha_J(x, y) + \beta_J(x, y))(x + y)$$

$$= \sum_J \left(2(\alpha_J(x, y) - \beta_J(x, y)) \left((x \wedge y)_J - \left(\frac{x+y}{2} \right)|_J \right) + 0 \right)$$

$$+ \frac{1}{2}(x + y) \leq \frac{x+y}{2},$$

as $2(\alpha_J(x, y) - \beta_J(x, y))((x \wedge y)_J - (\frac{x+y}{2})|_J \leq 0$ for all J. Hence (3.1) is satisfied. If $x \neq y$ then $(x \wedge y)_J - (\frac{x+y}{2})|_J \lneq 0$ for some $J \subseteq \{1, 2, \ldots, d\}$. Hence (3.2) holds. Applying Theorem 3.1 we end the proof. $\square$

5 Final conclusions and examples

The following results are strictly related to [15]. Under extra assumptions we identify $\lim_{n \to \infty} \mathbb{V}^n(\mu)$ as $\delta_{i(\mu)}$ (or $\delta_{s(\mu)}$ respectively). We start with discussing classical Lebesgue q.s.o. introduced in [15]; i.e. consider $F = \mathbb{R}$ and $I = [0, 1]$. The transition functions are defined as

$$P(x, y, \cdot) = \alpha(x, y)\delta_{x \wedge y} + \beta(x, y)\delta_{x \vee y}, \tag{22}$$

where the functions α, β are symmetric on $I \times I$ and Borel measurable (continuity is not required), and $\alpha + \beta = 1$. In [15] both $\alpha(x, y) \equiv \alpha =$const and $\beta(x, y) \equiv \beta =$ const.

Theorem 5.1 *Let $\mathbb{V}$ be the Lebesgue q.s.o. defined by $P(x, y, \cdot)$ as above, where $\alpha(x, y) > \beta(x, y)$ if $x \neq y$. If for some $\varepsilon > 0$ and $0 < \varrho < \frac{1}{2}$ the measure $\mu \in \mathcal{P}([0, 1])$ satisfies*

$$\alpha(x, y) > \frac{1}{2} + \varrho \text{ if } x \wedge y \in [i(\mu), i(\mu) + \varepsilon], \tag{3}$$

then $\lim_{n \to \infty} \|\mathbb{V}^n(\mu) - \delta_{i(\mu)}\|_{\text{FM}} = 0$.

Proof For a fixed $0 < \tau < \varepsilon$ we set

$$\gamma_n = \mathbb{V}^n(\mu)\left([i(\mu), i(\mu) + \tau]\right).$$

We get

$$
\begin{aligned}
\gamma_{n+1} &= \mathbb{V}^{n+1}(\mu)\left([i(\mu), i(\mu) + \tau]\right) \\
&\geq \left(\mathbb{V}^n(\mu)\left([i(\mu), i(\mu) + \tau]\right)\right)^2 \\
&\quad + 2\left(\frac{1}{2} + \varrho\right)\mathbb{V}^n(\mu)\left([i(\mu), i(\mu) + \tau]\right)\mathbb{V}^n(\mu)\left((i(\mu) + \tau, s(\mu)]\right) \\
&\geq \gamma_n^2 + 2\left(\frac{1}{2} + \varrho\right)\gamma_n(1 - \gamma_n).
\end{aligned}
$$

Clearly $\gamma_0 = \mu([i(\mu), i(\mu) + \tau]) > 0$. Notice that $\varphi^{\circ n}(\mu([i(\mu), i(\mu) + \tau])) \leq \gamma_n \leq 1$, where the quadratic function $\varphi\colon [0, 1] \to [0, 1]$ is defined by the formula $\varphi(t) = t^2 + 2(\frac{1}{2} + \varrho)t(1 - t)$. Obviously the iterates $\lim_{n\to\infty} \varphi^{\circ n}(t) = 1$ for all $0 < t \leq 1$. Thus $\lim_{n\to\infty} \gamma_n = 1$. It follows that $\lim_{n\to\infty} \mathbb{V}^n(\mu)([i(\mu), i(\mu) + \tau]) = 1$, where $\tau > 0$ may be as small as we wish. We have proved $\lim_{n\to\infty} \mathbb{V}^n(\mu) = \delta_{i(\mu)}$. $\qquad\square$

By symmetry the following version of weak regularity holds;

Theorem 5.2 *Let $\mathbb{V}$ be the Lebesgue q.s.o. defined by $P(x, y, \cdot)$ as above, where $\alpha(x, y) < \beta(x, y)$ if $x \neq y$. If for some $\varepsilon > 0$ and $0 < \varrho < \frac{1}{2}$ the measure $\mu \in \mathcal{P}([0, 1])$ satisfies*

$$\beta(x, y) > \frac{1}{2} + \varrho \ \text{ if } \ x \vee y \in [s(\mu) - \varepsilon, s(\mu)], \tag{4}$$

then $\lim_{n\to\infty} \|\mathbb{V}^n(\mu) - \delta_{s(\mu)}\|_{\text{FM}} = 0$.

Even in the one-dimensional case, when $F = \mathbb{R}$ and $I = [0, 1]$, the question on weak regularity of q.s.o. is nontrivial. In particular, it is a challenge to find a pattern of general limit measures. In the example below we see that the condition on the global mean monotonicity may be relaxed.

Example 5.3 Let $I = [0, 1] \subset \mathbb{R}$. Given $0 < u < v < 1$ and Borel functions $p\colon [0, u) \to (\frac{1}{2} + \varrho, 1]$, $q\colon (v, 1] \to [0, \frac{1}{2} - \varrho)$, where $0 < \varrho < \frac{1}{4}$, let us define a symmetric function:

$$
\alpha(x, y) = \begin{cases}
p(x), & \text{for } 0 \leq x < u, x \leq y \leq 1 \\
\frac{1}{2}, & \text{for } u \leq x \leq v, x \leq y \leq 1 \\
q(x), & \text{for } v < x \leq 1, x \leq y \leq 1.
\end{cases}
$$

Let $\mathbb{V}$ be the Lebesgue q.s.o. corresponding to α (i.e. $P(x, y, \cdot) = \alpha(x, y)\delta_{x\wedge y} + (1 - \alpha(x, y))\delta_{x\vee y}$). We recall that intervals $[a, b] \subseteq [0, 1]$ are invariant for Lebesgue q.s.o. $\mathbb{V}$ (i.e. if $\text{supp}(\mu) \subseteq [a, b]$ then $\text{supp}(\mathbb{V}(\mu)) \subseteq [a, b]$). Hence $\{P(x, y, \cdot)\}$

restricted to $[a, b]$ generates a q.s.o. which coincides with $\mathbb{V}|_{\mathcal{P}([a,b])}$. The same trick which is used in the proof of Theorem 5.1 gives the convergence of $\gamma_n = \mathbb{V}^n(\mu)([i(\mu), i(\mu) + \tau)) \to 1$, for every probability Borel measure μ on $[0, 1]$ such that $\mu([0, u)) > 0$. It follows that $\lim_{n\to\infty} \|\mathbb{V}^n(\mu) - \delta_{i(\mu)}\|_{FM} = 0$. On the other hand if $\mu((v, 1]) = 1$, then by the Theorem 5.2 we obtain $\lim_{n\to\infty} \|\mathbb{V}^n(\mu) - \delta_{s(\mu)}\|_{FM} = 0$.

More generally, if $\mu([u, 1]) = 1$, then $\lim_{n\to\infty} \mathbb{V}^n(\mu) = \mu|_{[u,v]} + \mu((v, 1])\delta_{s(\mu)}$ in the $\|\cdot\|_{FM}$ norm. Indeed, for every Borel $A \subseteq [u, v]$ we have

$$
\begin{aligned}
\mathbb{V}(\mu)(A) &= \int_A \int_A P(x, y, A) d\mu(x) d\mu(y) + 2 \int_A \int_{A^c} P(x, y, A) d\mu(x) d\mu(y) \\
&\quad + \int_{A^c} \int_{A^c} P(x, y, A) d\mu(x) d\mu(y) \\
&= \mu^2(A) + \mu(A)\mu(A^c) = \mu(A).
\end{aligned}
$$

By induction, for every natural n and every Borel $A \subseteq [u, v]$, we have $\mathbb{V}^n(\mu)(A) = \mu(A)$. Hence the sequence $\mathbb{V}^n(\mu)$ restricted to the interval $[u, v]$ is constant; i.e. $\mathbb{V}^n(\mu)|_{[u,v]} = \mu|_{[u,v]}$. If $\mu([u, 1]) = 1$ and $\mu((v, 1]) > 0$ let $\mu_1 = \mu|_{[u,v]}$, $\mu_2 = \mu|_{(v,1]}$ and consider $A = [s(\mu) - \tau, s(\mu)]$, where $0 < \tau < s(\mu) - v$. Then

$$
\begin{aligned}
\mathbb{V}(\mu)(A) &= Q(\mu_1 + \mu_2, \mu_1 + \mu_2)(A) = Q(\mu_1, \mu_1)(A) + 2Q(\mu_1, \mu_2)(A) \\
&\quad + Q(\mu_2, \mu_2)(A) \\
&= 0 + 2 \int \int P(x, y, A) d\mu_1(x) d\mu_2(y) + \int \int P(x, y, A) d\mu_2(x) d\mu_2(y) \\
&= \mu([u, v])\mu(A) + 2 \int_{(v, s(\mu) - \tau)} \int_A P(x, y, A) d\mu_2(x) d\mu_2(y) \\
&\quad + \int_A \int_A P(x, y, A) d\mu_2(x) d\mu_2(y) \\
&\geq \mu([u, v])\mu(A) + \mu^2(A) + 2\left(\tfrac{1}{2} + \varrho\right)\mu((v, s(\mu) - \tau))\mu(A) \\
&= 2\left(\tfrac{1}{2} + \varrho\right)(1 - \mu([u, v]) - \mu((s(\mu) - \tau), s([\mu]))) \mu(A) + \mu^2(A) \\
&\quad + \mu([u, v])\mu(A).
\end{aligned}
$$

Now denote $\gamma_n = \mathbb{V}^n(\mu)([s(\mu) - \tau, s(\mu)])$, $n = 0, 1, \ldots$ and $C = \mu([u, v])$. Since $\mathbb{V}^n([u, v]) = \mu([u, v]) = C$ for every natural n, thus by the above estimation (replace there μ by $\mathbb{V}^n(\mu)$) we get a recursion

$$
1 - C \geq \gamma_{n+1} \geq 2\left(\tfrac{1}{2} + \varrho\right)(1 - C - \gamma_n)\gamma_n + \gamma_n^2 + C\gamma_n.
$$

Now consider an elementary quadratic function $\varphi(x) = 2(\tfrac{1}{2} + \varrho)(1 - C - x)x + x^2 + Cx$ and find that for all $0 < x < 1 - C$ we have $0 < x < \varphi(x) < 1 - C$. Hence the iterates $\varphi^{\circ n}(x)$ increase to $1 - C$, i.e. $\lim_{n\to\infty} \mathbb{V}^n([s(\mu) - \tau, s(\mu)]) = 1 - \mu([u, v]) = 1 - C$ for all small enough $\tau > 0$. It follows that for every probability measure μ, concentrated on $[u, 1]$, we have

$$\lim_{n\to\infty} \mathbb{V}^n(\mu) = \mu|_{[u,v]} + (1 - \mu([u,v]))\,\delta_{s(\mu)}$$

in the $\|\cdot\|_{\mathrm{FM}}$ norm.

Problem Describe weakly regular Lebesgue quadratic operators in the one-dimensional case, for general symmetric (continuous) $\alpha(x, y)$, and identify limit measures.

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