Quantum channel capacities: Multiparty communication

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We analyze different aspects of multiparty communication over quantum memoryless channels and generalize some of the key results known from bipartite channels to the multiparty scenario. In particular, we introduce multiparty versions of subspace and entanglement transmission fidelities. We also provide alternative, local, versions of fidelities and show their equivalence to the global ones in context of capacity regions defined. An equivalence of two different capacity notions with respect to two types of fidelities is proven. In analogy to the bipartite case it is shown, via sufficiency of isometric encoding theorem, that additional classical forward side channel does not increase capacity region of any quantum channel with k senders and m receivers which represents a compact unit of general quantum networks theory. The result proves that recently provided capacity region of a multiple access channel [M. Horodecki *et al.*, Nature **436**, 673 (2005); J. Yard *et al.*, e-print quant-ph/0501045], is optimal also in a scenario of an additional support of forward classical communication.

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I. INTRODUCTION

The theory of quantum channels is nowadays a very important domain of quantum information theory. The bipartite case of one sender and one receiver has been extensively analyzed in literature. In particular, for noiseless channel the coding theorem has been established in Refs. [1,2]. The problem of noisy channels has been defined on the basis of minimal fidelity subspace transmission [3]. An alternative definition of quantum information transmission based on quantum entanglement has been introduced in Ref. [4] and shown [5] to coincide with that of Ref. [3]. Moreover, it has been shown that forward classical communication from a sender to a receiver does not help [3,5]. Research towards channel capacity formula [6,8,7] has been finalized by the asymptotic coherent information formula in two ways: there was a proof through conjectured hashing inequality [9] that has been proven [10] and a direct proof [11]. Summary of the state of art of a bipartite zero(one)-way quantum channel capacity can be found in Ref. [12]. Further, other capacity notions measuring an ability to transmit classical information via quantum channel [13] or its entanglement assisted analogue [14] have been analyzed. The first notion leads to a well known additivity conjecture (see Ref. [15]) linking four conjectured relations from quantum information theory. In the mean time much more has been known about relations between different versions of channel capacities [16] (see Refs. [17,18]).

Some time ago quantum channels with more than one sender or receiver have attracted more attention. Sending classical information via multiple access quantum channel have been considered first in Ref. [19]. An issue of sending quantum information in general multiparty scenario has been raised in Ref. [20] where, in particular, it has been shown that quantum broadcast channel capacity with two-way classical communication is nonadditive. This is linked to superactivation of the multipartite bound entanglement phenomenon [21]. Very recently a capacity region for multiple access channels have been provided via quantum-state merging technique [22,23] and direct technique using some links between classical-quantum and quantum-quantum transfer [24].

Here we consider the most general scenario like in Ref. [20] and generalize the most important results from a bipartite case. We achieve it by developing alternative, local, versions of quantum fidelities and then adopting with some refinements and modifications techniques from Refs. [5,8]. In particular, we consider two versions of quantum information transfer: subspace transmission [3] and entanglement transmission from Ref. [5]. We generalize the result of the latter showing that capacity regions defined with respect to both fidelities coincide under so-called QAEP assumption about the ensembles describing senders. We also show that if one drops the latter assumption then quantum capacity regions of multiple access channel and k-user channel do not change if one removes encodings which generalizes results of Ref. [8] (cf. Ref. [5]).

Finally we generalize one of the results of Ref. [5] showing that in the general case of k senders and m receivers forward classical communication does not improve capacity regions (originally an argument in favor of this result in the case of a bipartite channel appeared in Ref. [3]). This result implies that the regions derived in Refs. [22,24] do not change if we allow parties taking part in communication to be supported by one-way classical channel.

The paper is organized as follows. In Sec. II we recall two definitions of bipartite channel capacities. Then we turn to multiparty case and introduce two types of fidelities and define corresponding capacities regions. We also provide alternative, local, versions of fidelities and show their equivalence to previous ones. In Sec. III we show that subspace transmission is equivalent to entanglement transmission under assumption of QAEP of the sources. In Sec. IV we prove, using in particular local versions of fidelities, that encoding in zero-way regime can be always replaced with partial isometry encoding and by its trace-preserving extension. Independently we consider special cases of a k-user and mul-

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tiple access channels and show that entanglement transmission capacity regions do not change if we abandon encoding operations. In Sec. V we show one of the central results of the paper, i.e., that forward classical communication does not change capacity regions of the general *km*-user channel, i.e., if any of the *k* senders wants to send quantum information to any of the *m* receivers. Section VI contains a discussion and conclusions.

II. PRELIMINARIES

A completely positive trace-preserving operation is called a quantum channel. We focus on a situation in which one group of parties wants to send information to the other group and particles they send are subject to an act of a channel. In the case of one input–one output we will be talking about single-user channel (SUC, sometimes called bipartite channel), one-input–*m*-outputs correspond to broadcast channel (BC), *k*-inputs–one-output is a multiple access channel (MAC), *k*-inputs–*m*-outputs channel will be called *km*-user communication channel (*km*-UC, in the case of equal *k* inputs and outputs just *k*-UC).

Remark on notation: We use Ψ to denote projector $|\Psi\rangle\langle\Psi|$.

A. Protocols

Let C denote classical information exchanged between two groups of parties. We say about zero-way communication when no information is exchanged, one-way forward or backward if only one group communicates with the other, or two-way when classical messages are exchanged in both directions. We use the following symbols, respectively, $C=\varnothing, \rightarrow, \leftarrow, \leftrightarrow$ in these cases. Let $\mathbf{A}=\{\mathbf{A}_i\}_{i=1}^k=\{\{A_i^\alpha\}_{\alpha=1}^{l_i}\}_{i=1}^k$, $\mathbf{B}=\{\mathbf{B}_i\}_{i=1}^m=\{\{B_i^\beta\}_{\beta=1}^{b_i}\}_{i=1}^m$ denote parties taking part in communication (ith sender, from the group of k senders, is holding l_i particles and ith receiver, from the group of m receivers, is getting b_i particles; the number of particles on both sides obviously agrees) and $\mathcal{H}_{\mathbf{A}}=\otimes_{i=1}^k\mathcal{H}_{\mathbf{A}_i}=\otimes_{i=1}^k\{\otimes_{\alpha=1}^{l_i}\mathcal{H}_{\mathbf{A}_i}^\alpha\}$, $\mathcal{H}_{\mathbf{B}}=\otimes_{i=1}^m\mathcal{H}_{\mathbf{B}_i}=\otimes_{i=1}^m\{\otimes_{\beta=1}^{b_i}\mathcal{H}_{\mathbf{B}_i}^\beta\}$ Hilbert spaces associated to their particles. Let $\varepsilon^{(n)}$ and $\mathcal{D}^{(n)}$ be trace-nonincreasing maps acting as follows $\varepsilon^{(n)}:\mathcal{B}(\mathcal{H}_{\mathbf{A}}^{\otimes n})\rightarrow\mathcal{B}(\mathcal{H}_{iCh}^{\otimes n})$ and $\mathcal{D}^{(n)}:\mathcal{B}(\mathcal{H}_{oCh}^{\otimes n})\rightarrow\mathcal{B}(\mathcal{H}_{\mathbf{B}}^{\otimes n})$, where "$i/oCh$" stands for the input or output of a channel (later in this paper we will call these maps encoding and decoding and set $\dim\mathcal{H}_{\mathbf{A}}=\dim\mathcal{H}_{\mathbf{B}}$). We will call a *quantum protocol* (shortly protocol) $\mathcal{P}^C$ supplemented by classical side information C a set of maps $\{\mathcal{P}_n^C\}$ mapping channels $\Lambda^{\otimes n}:\mathcal{B}(\mathcal{H}_{iCh}^{\otimes n})\rightarrow\mathcal{B}(\mathcal{H}_{oCh}^{\otimes n})$ into channels $\hat{\Lambda}^{(n)}:\mathcal{B}(\mathcal{H}_{\mathbf{A}}^{\otimes n})\rightarrow\mathcal{B}(\mathcal{H}_{\mathbf{B}}^{\otimes n})$. Due to a different usage of classical support we have different forms of these maps. That is, in the case of $C=\varnothing$ we have

$$\mathcal{P}_n^{\varnothing}(\Lambda^{\otimes n})=\mathcal{D}^{(n)}\Lambda^{\otimes n}\varepsilon^{(n)}, \tag{1}$$

for $C=\rightarrow$ the map is

$$\mathcal{P}_n^{\rightarrow}(\Lambda^{\otimes n})=\sum_j\mathcal{D}_j^{(n)}\Lambda^{\otimes n}\varepsilon_j^{(n)}, \tag{2}$$

where $\mathcal{D}_j^{(n)}$ and $\varepsilon_j^{(n)}$ operate on the same spaces as $\mathcal{D}^{(n)}$ and $\varepsilon^{(n)}$ and $\sum_j\varepsilon_j^{(n)}=\varepsilon^{(n)}$. The most sophisticated case is a two-way communication scenario $C=\leftrightarrow$ (sometimes called ping-pong protocol) [25]. Both groups of parties, **A** and **B**, perform POVMs in turn, where any particular POVM can depend on all of the results obtained in previous measurements. This corresponds to a sequence of families of operations $V_{k_1},V_{k_1k_2},V_{k_1k_2k_3},\ldots,V_{k_1k_2k_3k_4\cdots k_l}$ with trace preservation condition $\sum_{k_1k_2k_3k_4\cdots k_l}V_{k_1k_2k_3k_4\cdots k_l}^\dagger V_{k_1k_2k_3k_4\cdots k_l}=\mathbb{I}$ for all l. Thus, denoting $\mathbf{k}=\{k_1k_2k_3k_4\cdots k_l\}$, we have

$$\mathcal{P}_n^{\leftrightarrow}(\Lambda^{\otimes n})=\sum_{\mathbf{k}}{}^{\mathbf{B}}\Lambda_{\mathbf{k}}^{(n)}(\Lambda^{\otimes n}){}^{\mathbf{A}}\Lambda_{\mathbf{k}}^{(n)}, \tag{3}$$

where ${}^{\mathbf{A}}\Lambda_{\mathbf{k}}^{(n)}$ and ${}^{\mathbf{B}}\Lambda_{\mathbf{k}}^{(n)}$ act on the same spaces as $\varepsilon^{(n)}$ and $\mathcal{D}^{(n)}$, respectively. One can show that the latter is the most general form of LOCC including previous ones. However, for the sake of further convenience we treated these cases separately.

B. Quantum channel rates, fidelities, and capacities (single user case)

There are various notions of quantum capacity rates and transmissions (see, e.g., Ref. [12]). Here we shall recall two that historically played the most important role.

1. Subspace transmission

We start with the concept of subspace transmission which was introduced by Bennett *et al.* [3].

The idea is to send pure states from a Hilbert space $\mathcal{H}$ being a subspace of a channel input Hilbert space $\mathcal{H}_{iCh}$ with a pure state fidelity defined as

$$F_s'(\psi,\Lambda)=\langle\psi|\Lambda(\psi)|\psi\rangle, \tag{4}$$

which minimzed over $|\psi\rangle\in\mathcal{H}$ constitutes minimum pure state fidelity

$$F_s(\mathcal{H},\Lambda)=\min_{|\psi\rangle\in\mathcal{H}}F_s'(\psi,\Lambda). \tag{5}$$

We define a rate of such transmission with a protocol $\mathcal{P}^C$ as

$$R_s(\mathcal{P}^C,\Lambda)\equiv\lim_{n\rightarrow\infty}\frac{\log_2\dim\mathcal{H}^{(n)}}{n} \tag{6}$$

and say that it is achievable if for a given protocol we have a pure state fidelity tending to one in the limit of large n, i.e.,

$$F_s(\mathcal{H}^{(n)},\mathcal{P}_n^C,\Lambda^{\otimes n})\xrightarrow{n\rightarrow\infty}1. \tag{7}$$

In this case we call a protocol $\{\mathcal{P}_n^C\}$ reliable. It is important to note that here we consider only the protocols for which the limit (6) exists (in contrast to the original version where *limes supremum* was put in this place), but it can easily be shown not to be a restriction. Quantum channel capacity for a subspace transmission is defined as the supremum of all achievable rates produced by all considered protocols $\mathcal{P}^C$. We use Q_s^C to denote a capacity in this case

$$Q_s^C(\Lambda)=\sup_{\mathcal{P}^C}R_s(\mathcal{P}^C,\Lambda). \tag{8}$$

There is also an alternative way to cope with the problem of quantum information transmission—entanglement transmission.

2. Entanglement transmission

The idea of entanglement transmission was developed in Ref. [5] and is as follows. Alicia produces bipartite state Ψ_{AB} and sends its subsystem B down the channel Λ to Bobby. The quantity which measures the resemblance of the output bipartite state to the initial Ψ_{AB} is called an entanglement fidelity and is defined by the relation (we use here notation staying in agreement with the one used in Ref. [24])

$$F_e(\Psi_{AB},\Lambda) \equiv \langle \Psi_{AB}|\mathbb{I}^A \otimes \Lambda^B(\Psi_{AB})|\Psi_{AB}\rangle. \tag{9}$$

It can be shown that it depends on Ψ_{AB} only through ϱ_B and Λ. An entanglement transmission rate in a given protocol $\mathcal{P}^C$ for a source $\varrho^{(n)}$ can be defined formally as

$$R_e(\mathcal{P}^C,\Lambda) \equiv \lim_{n\to\infty} \frac{S(\varrho^{(n)})}{n}. \tag{10}$$

Here $\varrho^{(n)}$ is a reduced Alicia's block density matrix corresponding to a bipartite pure state $\Psi_{AB}^{(n)}$, part of which is transmitted down a new channel $\mathcal{P}_n^C(\Lambda^{\otimes n})$ involving both encoding and decoding procedures.

As previously, we say that the rate is achievable if

$$F_e(\Psi_{AB}^{(n)},\mathcal{P}_n^C,\Lambda^{\otimes n}) \xrightarrow{n\to\infty} 1. \tag{11}$$

Channel capacity is defined, analogously to the previous case, as

$$Q_e^C(\Lambda) = \sup_{\mathcal{P}^C} R_e(\mathcal{P}^C,\Lambda). \tag{12}$$

Barnum *et al.* [5] have shown that both definitions coincide and give the same number for sources which satisfy *quantum asymptotic equipartition property* (QAEP), i.e., for all $\varrho^{(n)}$ that have asymptotically uniform spectrum in a special sense (see the Appendix).

Recently it has also been shown that it is equal to the so-called coherent information rate [11].

C. Quantum channel rates, fidelities, and capacities (multiuser case)

Let us now turn to the multiuser case and consider MAC, BC, and km-UC (Fig. 1). We then have for a subspace transmission in the most general case of km-UC,

$$F_s(\otimes_{j=1}^k(\otimes_{\alpha=1}^{l_j}\mathcal{H}_j^\alpha),\Lambda) \equiv \min_{\otimes_{j=1}^k(\otimes_{\alpha=1}^{l_j}|\psi_j^\alpha\rangle)\in\otimes_{j=1}^k(\otimes_{\alpha=1}^{l_j}\mathcal{H}_j^\alpha)} (\otimes_{j=1}^k(\otimes_{\alpha=1}^{l_j}\langle\psi_j^\alpha|))\Lambda(\otimes_{j=1}^k(\otimes_{\alpha=1}^{l_j}\psi_j^\alpha))(\otimes_{j=1}^k(\otimes_{\alpha=1}^{l_j}|\psi_j^\alpha\rangle)) \tag{13}$$

and in the case of entanglement transmission,

$$F_e(\otimes_{j=1}^k(\otimes_{i=1}^{l_j}\Psi_{(AB)_j}^i),\Lambda) \equiv (\otimes_{j=1}^k(\otimes_{i=1}^{l_j}\langle\Psi_{(AB)_j}^i|))\mathbb{I}^{\mathbf{A}}\otimes\Lambda^{\mathbf{B}}(\otimes_{j=1}^k(\otimes_{i=1}^{l_j}\Psi_{(AB)_j}^i))(\otimes_{j=1}^k(\otimes_{i=1}^{l_j}|\Psi_{(AB)_j}^i\rangle)). \tag{14}$$

We will use a term *global fidelities* for the above quantities. In each scenario we can assign every ith transmission between ith (sub)sender (sender sending one of her subsystem) and proper receiver in a manner we have done in the single user case. Literally, we define the rates as (we use abbreviated notation here)

$$R_s^{(i)} \equiv \lim_{n\to\infty} \frac{\log_2 \dim \mathcal{H}_i^{(n)}}{n} \tag{15}$$

for a subspace transmission, where $\mathcal{H}_i^{(n)}$ is a subspace of ith input Hilbert space of $\mathcal{H}_{iCh}^{\otimes n}$ and for entanglement transmission,

$$R_e^{(i)} = \lim_{n\to\infty} \frac{S(\varrho_i^{(n)})}{n}, \tag{16}$$

where $\varrho_i^{(n)}$ can be represented as a quantum material produced by ith source $\Sigma^{(i)}$ but formally is just a reduced Alicia's state of the bipartite pure state $\Psi_{(AB)_i}^{(n)}$. As it will become clear in next subsection superscript i should be in fact considered as a double superscript.

Similarly to the single user case we say that rates are achievable if there exists protocol for a given type of a scenario (i.e., for instance it requires product encoding but joint

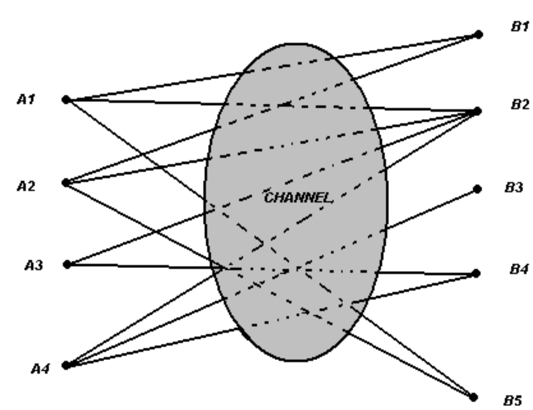

FIG. 1. General scheme of multiuser communication. An exemplary 45-UC.

decoding in the case of multiple access channels) such that senders can reliably, i.e., with global fidelity corresponding to these rates approaching one, send information to receivers. Formally we require (following Ref. [24]) the global fidelity to approach unity, but it can be easily seen that this is equivalent to the same requirement for a set of *local fidelities* (see the next section).

We define a quantum channel capacity to be a set of all L-tuples ($L=\Sigma_{i=1}^{k} l_i$) of achievable rates $[R^{(1)},R^{(2)},\ldots,R^{(L)}]$. Here they are one of three types: (i) from k senders to one receiver in the case of MAC, (ii) from k senders to m receivers in the case of km-UC, and (iii) from single sender to k receivers in the case of BC.

Other capacities will be analyzed in detail elsewhere [26].

D. Alternative expressions for fidelities

Instead of global fidelities, expressing resemblance of the whole input state to the whole output state, we can introduce local fidelities measuring how each density matrix sent through the channel was affected by its action. Namely, we have for subspace and entanglement transmission, respectively,

$$F_s^{(j,l)}(\mathcal{H}_j^l,\Lambda) = \min_{\otimes_{j=1}^{k}(\otimes_{\alpha=1}^{l_j}|\psi_j^{\alpha}\rangle)\in\otimes_{j=1}^{k}(\otimes_{\alpha=1}^{l_j}\mathcal{H}_j^{\alpha})} \langle\psi_j^l|\mathrm{tr}_{\,\mathbf{AB}\setminus(AB)_j^l}\Lambda(\otimes_{j=1}^{k}(\otimes_{\alpha=1}^{l_j}\psi_j^{\alpha}))|\psi_j^l\rangle, \tag{17}$$

$$F_e^{(j,l)}(\Psi_{(AB)_j^l},\Lambda) = \langle\Psi_{(AB)_j^l}|\mathrm{tr}_{\,\mathbf{AB}\setminus(AB)_j^l}\mathbb{I}^{\mathbf{A}}\otimes\Lambda^{\mathbf{B}}(\otimes_{j=1}^{k}(\otimes_{i=1}^{l_j}\Psi_{(AB)_j^i}))|\Psi_{(AB)_j^l}\rangle. \tag{18}$$

The following lemmas show that we can freely switch from global to local fidelities (and in the opposite way) as they both coincide in the case of high limit.

Lemma 1. For any k-partite density matrix $\varrho_{\mathbf{k}}$ and arbitrary states ϕ_i satisfying $(\otimes_{i=1}^{k}\langle\phi_i|)\varrho_{\mathbf{k}}(\otimes_{i=1}^{k}|\phi_i\rangle)>1-\epsilon$ ($\mathbf{k}=\{i\}_{i=1}^{k}$) we have (i) for all l $\langle\phi_l|\varrho_l|\phi_l\rangle>1-\epsilon$ and (ii) $(\otimes_{i=1}^{k}\langle\phi_i|)(\otimes_{i=1}^{k}\varrho_i)(\otimes_{i=1}^{k}|\phi_i\rangle)>1-k\epsilon$.

Proof. Proof is straightforward. Consider property (i). For $k=2$ we prove it by writing partial trace over subsystem 1 in the basis $|\phi_1^j\rangle$ (with j enumerating the basis vectors), taking $|\phi_1^0\rangle\equiv|\phi_1\rangle$, and considering respective fidelity. We then have

$$\langle\phi_2|\mathrm{tr}_{\,1}\varrho_{12}|\phi_2\rangle = \sum_{j\neq 0}\langle\phi_1^j|\otimes\langle\phi_2|\varrho_{12}|\phi_1^j\rangle\otimes|\phi_2\rangle + \langle\phi_1|\otimes\langle\phi_2|\varrho_{12}|\phi_1\rangle\otimes|\phi_2\rangle, \tag{19}$$

which immediately gives the desired bound because of the positivity of the first term and the assumption.

For any $k>2$ the proof goes along the same way, only instead of subsystem 2 we consider all the other subsystems. This gives property (i) for index $i=1$, which due to possible permutations is valid for all other indices.

Let us pass to the property (ii). It immediately follows by multiplication of inequalities from (i) and the fact that $(1-\epsilon)^k\geqslant 1-k\epsilon$. ■

Lemma 2. For any k-partite density matrix $\varrho_{\mathbf{k}}$ and arbitrary states ϕ_i satisfying $\langle\phi_i|\varrho_i|\phi_i\rangle>1-\epsilon_i$ we have $(\otimes_{i=1}^{k}\langle\phi_i|)\varrho_{\mathbf{k}}(\otimes_{i=1}^{k}|\phi_i\rangle)>1-\Sigma_{i=1}^{k}\epsilon_i$.

Proof. As previously it is sufficient to prove the observation for $k=2$ since its validity for higher k can be verified by induction.

First we introduce orthonormal bases $|\phi_1^j\rangle$, $|\phi_2^j\rangle$, next we define $\varrho_{op,rs}\equiv\langle\phi_1^o|\langle\phi_2^p|\varrho_{12}|\phi_1^r\rangle|\phi_2^s\rangle$. Then we have (because of the unit trace of ϱ_{12})

$$\varrho_{00,00}+\sum_{i\neq 0}\varrho_{i0,i0}+\sum_{j\neq 0}\varrho_{0j,0j}+\sum_{i\neq 0,j\neq 0}\varrho_{ij,ij}=1. \tag{20}$$

On the other hand, the condition $\langle\phi_1|\varrho_1|\phi_1\rangle>1-\epsilon_1$ takes the form

$$\varrho_{00,00}+\sum_{j\neq 0}\varrho_{0j,0j}>1-\epsilon_1, \tag{21}$$

which put into the previous equality gives $\Sigma_{i\neq 0}\varrho_{i0,i0}+\Sigma_{i\neq 0,j\neq 0}\varrho_{ij,ij}<\epsilon_1$, which implies $\Sigma_{i\neq 0}\varrho_{i0,i0}<\epsilon_1$. The latter put into the condition $\langle\phi_2|\varrho_2|\phi_2\rangle>1-\epsilon_2$ rewritten as $\varrho_{00,00}+\Sigma_{i\neq 0}\varrho_{i0,i0}>1-\epsilon_2$ leads to $\varrho_{00,00}>1-\epsilon_1-\epsilon_2$ which concludes the proof for $k=2$. As mentioned, for higher k the proof goes by induction. ■

Application of the above lemmas to fidelities is obvious, as we can combine both lemmas to get "if and only if" statement: *Global fidelities are high iff local ones are so.*

III. ENTANGLEMENT AND SUBSPACE TRANSMISSION: AN EQUIVALENCE

In this section we argue that quantum capacity of a quantum channel for entanglement transmission is equal to that of subspace transmission in multiuser communication scenarios for sources satisfying QAEP. What is worth to be stressed here is that the equivalence holds for *every* type of protocol involved, i.e., *every* type of classical side channel (zero-, one-, two-way). This is due to the fact that along the course of the proof we do not make any assumptions about the protocol.

A. Subspace transmission follows from entanglement transmission

In our considerations below we employ techniques used in Ref. [5] and use results of Sec. II D. Assume each of k

senders' every $\varrho_{B_l}^{\gamma(n)}$ (shortly $\varrho_l^{(n)}$) satisfy QAEP and can be sent reliably (i.e., with high local, consequently global, entanglement fidelity). We use the fact that we can restrict ourselves to $\varrho_l^{(n)}$ projected onto its typical subspace [5] and assume that the restricted source is sent with local entanglement fidelity at least $1-\eta_l$ (by the Lemma 2 global entanglement fidelity is at least $1-\Sigma_{l=1}^{L}\eta_l$, $L=\Sigma_{i=1}^{k}l_i$). Now we recursively remove the lowest pure state fidelity vectors $|\phi_l^{i(n)}\rangle$ (in consequence dimensions) from the K_l-dimensional support of each $\varrho_l^{(n)}$ in such a manner that for each m we keep an operator $\varrho_l^{(n)}-\Sigma_{i=1}^{m}q_l^i|\phi_l^{i(n)}\rangle\langle\phi_l^{i(n)}|$ positive (obviously a tensor product of them is also positive). We then have $\varrho_l^{(n)}=\Sigma_{i=1}^{K_l}q_l^i|\phi_l^{i(n)}\rangle\langle\phi_l^{i(n)}|$, which means that $\{q_l^i,|\phi_l^{i(n)}\rangle\}$ constitutes a pure state ensemble for $\varrho_l^{(n)}$. After our removal procedure we are left (for every $\varrho_l^{(n)}$) with a subspace with dimension $D_l=K_l-n_{(l)}$, where $n_{(l)}$ is the number of removed dimensions, from which each state has pure state fidelity

$$F_s^{(l)} \geqslant 1-\frac{\sum_{p=1}^{L}\eta_p}{\prod_{p=1}^{L}\alpha_p}, \quad \alpha_p \equiv \sum_{i=1}^{n_{(p)}} q_p^i. \tag{22}$$

We also get the bound for dimensions of the remaining subspaces. Namely, $D_l \geqslant (1-\alpha_l)2^{n(S_l-\epsilon_l)}$, which gives the rate for subspace transmission $\frac{\log_2 D_l}{n} \geqslant \frac{\log_2(1-\alpha_l)}{n}+S(\Sigma^{(l)})-\epsilon_l$. The latter means that all rates for subspace transmission at least as large as for entanglement transmission are achievable.

B. Entanglement transmission follows from subspace transmission

We use here a slightly modified version of the well-known theorem [5,8].

Theorem 1. If every pure product state $\otimes_{i=1}^{k}(\otimes_{n=1}^{l_i}|\Psi_i^n\rangle)$ from a space $S=\otimes_{i=1}^{k}(\otimes_{n=1}^{l_i}S_i^n)$ have pure state fidelity $F_s(S,\mathcal{E})>1-\eta$ then any density operator $\varrho=\otimes_{i=1}^{k}(\otimes_{n=1}^{l_i}\varrho_i^n)$, such that Ran $(\varrho_i^n)\subseteq S_i^n$, has entanglement fidelity $F_e(\varrho,\mathcal{E}) \geqslant 1-O(\eta)$.

Proof of the theorem (see Ref. [5]) uses two main ideas: (i) not only vectors from the bases have high pure state fidelity but arbitrary superpositions of them as well, and (ii) pure state fidelity averaged over phases is high if pure state fidelity is high. (Note that we consider here global fidelities which in virtue of lemmas from Sec. II C are equivalent to local ones in the context of capacities.) To recognize usefulness of the theorem take uniform density matrices on each lth subspace $\mathcal{H}_l^{(n)}$, i.e., $I_l^{(n)}/\dim\mathcal{H}_l^{(n)}$. We conclude from the theorem that this source can be sent reliably which means that all rates for entanglement transmission not less than for that of subspace transmission are achievable.

We obtained two opposite inequalities for rates of transmission which means equality of capacities. This concludes the proof of an equivalence of entanglement and subspace transmission.

IV. ENCODINGS

A. Sufficiency of isometric encodings

As one knows isometric encodings is sufficient to achieve SUC capacity [5]. We show that it is true for all classes of quantum channels considered in the paper (SUC, MAC, *km*-UC). Bearing in mind that the protocol is agreed before sending any information through the channel so that both encoding and decoding depend on the source we will explicitly construct isometry which will serve as an encoding. We start with recalling the theorem about sufficiency of isometric encodings in the case of a single user channel as its proper application will be a main tool in proving the upcoming central theorem of the section. To apply the lemma below for our purposes we take $\mathcal{A}=\mathcal{D}\circ\Lambda$, i.e., concatenation of a channel noise and a decoding. We have the following.

Lemma 3. [5] Given a trace-nonincreasing map $\mathcal{A}$ and a map ε trace preserving on the state $\varrho_B \equiv \mathrm{tr}_A\Psi_{AB}$, for which $F_e(\Psi_{AB},\mathcal{A}\circ\varepsilon)>1-\eta$, one can always find such partial isometry W that $F_e(\Psi_{AB},\mathcal{A}\circ W)>1-2\eta$.

Now we state the central result of the section as the theorem.

Theorem 2. Given a reliable protocol $P^C=\{\mathcal{D}^{(n)},\varepsilon^{(n)}\}$, $\varepsilon^{(n)}=\otimes_{i=1}^{k}\varepsilon_i^{(n)}$, there always exists an extendable to a trace-preserving map partial isometry $W^{(n)}=\otimes_{i=1}^{k}W_i^{(n)}$ such that a protocol $\widetilde{P^C}=\{\mathcal{D}^{(n)},W^{(n)}\}$ allows for reliable entanglement transmission with the same rate.

Proof. For clarity we will omit a superscript (n) in the proof. Using shorthand notation for $\mathcal{A}$ as previously (the difference is that in the setting considered now we have m-fold tensor product of decodings) reliable entanglement transmission condition in multiparty scenario takes the form

$$F_e(\otimes_{i=1}^{k}\Psi_{(AB)_i},\mathcal{A}\circ\otimes_{i=1}^{k}\varepsilon_i) > 1-\eta. \tag{23}$$

Assume now $\mathcal{A}$ and ε_i have Kraus representations as follows:

$$\mathcal{A}(\cdot)=\sum_{\alpha}A_\alpha(\cdot)A_\alpha^\dagger, \tag{24}$$

$$\varepsilon_i(\cdot)=\sum_{\beta_i}E_i^{\beta_i}(\cdot)E_i^{\beta_i\dagger}. \tag{25}$$

One verifies that independently of what purifications $\Psi_{(AB)_i}$ we choose we have [27]

$$F_e(\otimes_{i=1}^{k}\Psi_{(AB)_i},\mathcal{A}\circ\otimes_{i=1}^{k}\varepsilon_i)=\sum_{\alpha}\sum_{\beta_1\beta_2\cdots\beta_k}\left|\sum_{\gamma_1\gamma_2\cdots\gamma_k}(\otimes_{i=1}^{k}\langle\widetilde{\phi_{B_i}^{\gamma_i}}|)A_\alpha\otimes_{i=1}^{k}(E_i^{\beta_i}|\widetilde{\phi_{B_i}^{\gamma_i}}\rangle)\right|^2, \tag{26}$$

where ϕ's come from spectral decompositions of ϱ's, i.e.,

$$\varrho_{B_i} = \sum_{\gamma_i} \lambda_{\gamma_i} |\phi_{B_i}^{\gamma_i}\rangle\langle\phi_{B_i}^{\gamma_i}| \equiv \sum_{\gamma_i} |\widetilde{\phi_{B_i}^{\gamma_i}}\rangle\langle\widetilde{\phi_{B_i}^{\gamma_i}}|. \tag{27}$$

Defining some operators by partial inner product as follows:

$$A^{(1)}_{\alpha,\beta_2\beta_3\cdots\beta_k} \equiv \sum_{\gamma_2\gamma_3\cdots\gamma_k} (\otimes_{i=2}^k \langle\widetilde{\phi_{B_i}^{\gamma_i}}|) A_\alpha \otimes_{i=2}^k (E_i^{\beta_i}|\widetilde{\phi_{B_i}^{\gamma_i}}\rangle) \tag{28}$$

allows us to rewrite (26) as

$$F_e(\otimes_{i=1}^k \Psi_{(AB)_i}, \mathcal{A} \circ \otimes_{i=1}^k \varepsilon_i) = \sum_{\alpha,\beta_2\beta_3\cdots\beta_k} \sum_{\beta_1} \left| \sum_{\gamma_1} \langle\widetilde{\phi_{B_1}^{\gamma_1}}| A^{(1)}_{\alpha,\beta_1\beta_2\cdots\beta_k} E_1^{\beta_1} \alpha |\widetilde{\phi_{B_1}^{\gamma_1}}\rangle \right|^2 = F_e(\Psi_{(AB)_1}, \mathcal{A}_1 \circ \varepsilon_1) \tag{29}$$

with some channel $\mathcal{A}_1$ defined by its Kraus decomposition

$$\mathcal{A}_1(\cdot) = \sum_{\theta,\kappa_1\kappa_2\cdots\kappa_k} A^{(1)}_{\theta,\kappa_1\kappa_2\cdots\kappa_k}(\cdot) A^{(1)\dagger}_{\theta,\kappa_1\kappa_2\cdots\kappa_k}. \tag{30}$$

The above due to (23) is still high and by the Lemma 3 we can find such W_1 that

$$F_e(\Psi_{(AB)_1}, \mathcal{A}_1 \circ W_1) > 1 - 2\eta. \tag{31}$$

Now bearing in mind (28)–(30) we conclude that the latter can be written as

$$F_e(\otimes_{i=1}^k \Psi_{(AB)_i}, \mathcal{A} \circ W_1 \otimes (\otimes_{i=2}^k \varepsilon_i)) > 1 - 2\eta. \tag{32}$$

Now one can apply the trick with defining, by partial inner product, new channel $\mathcal{A}_2$ to the second system B_2 and arrive at the possibility of isometric encodings on that system. Further repeating analogous steps until one reaches the kth system gives us

$$F_e(\otimes_{i=1}^k \Psi_{(AB)_i}, \mathcal{A} \circ \otimes_{i=1}^k W_i) > 1 - 2^k\eta. \tag{33}$$

This proves the existence of partial isometries as good encodings. This with the aid of the fact that we still use the same source concludes the proof. ■

In general the isometry can be trace decreasing, however, as pointed out in Ref. [5] it can be embedded in the trace-preserving map without loss of fidelity. This makes it useful as a proper encoding.

B. Entanglement transmission capacity without encodings: MAC and k-UC

The authors of Ref. [8] have shown that introducing encoding to the coding scheme is not necessary to get the proper definition of a channel capacity for entanglement transmission of SUC. We show that it is also the case for MAC and k-UC in the setting one sender–one receiver. One must stress that we do not make the QAEP assumption about the sources. We focus on the entanglement transmission scenario.

The idea is to show that we can get rid of encodings on the Alicias' side if Bobbys perform an additional decoding operation. Here is the motivation (we shall operate on the definition of capacity for entanglement transmission). Suppose $\otimes_{i=1}^k|\phi_i^{(n)}\rangle$ is a purification of $\otimes_{i=1}^k\varrho_i^{(n)}$, which Alicias are supplied with. Performing encoding by them means adjoining environment in a standard state and unitary acting on the composed system. As the environments are just additional subsystems which are not sent over the channel they can be measured. This results with some probability (dependent on a result of a measurement) in different pure states $\otimes_{i=1}^k|\psi_i^{l(n)}\rangle$ on Alicias side. However, each Alicia have access only to a part of a resulting state, the other is a reference system which we assume to be out of control of any parties taking part in communication. The aim is to show that if they use $\otimes_{i=1}^k|\psi_i^{l(n)}\rangle$ as an input they can also achieve high entanglement fidelity as it was for the original $\otimes_{i=1}^k|\phi_i^{(n)}\rangle$ if additional (local) operations will be performed on Bobbys' side. It will remain to show that an entropy rate of a new source is close to that of the old one.

We start with the lemmas.

Lemma 4. [5] If $|\langle\phi_1|\psi\rangle|^2 > 1-\eta_1$ and $|\langle\phi_2|\psi\rangle|^2 > 1-\eta_2$ then $|\langle\phi_1|\phi_2\rangle|^2 > 1-\eta_1-\eta_2$ for normalized ϕ_i.

Lemma 5. [8] Given a density matrix satisfying

$$\langle\phi|\varrho|\phi\rangle > 1 - \epsilon \tag{34}$$

for some state ϕ we have (i)

$$\lambda_{\max} > 1 - \epsilon \tag{35}$$

and (ii)

$$|\langle\psi_{\max}|\phi\rangle|^2 > 1 - 2\epsilon, \tag{36}$$

where $\lambda_{\max}, \psi_{\max}$ are the largest eigenvalue and corresponding eigenstates, respectively.

The central element of a technique to prove the upcoming theorem is the subsequent Lemma 6 which is a subtle generalization of the corresponding lemma from Ref. [8].

Lemma 6. Given a density matrix ϱ in a Hilbert space $\mathcal{H}_{\mathbf{AB}} = \otimes_{i=1}^k \mathcal{H}_{(AB)_i} (\mathcal{H}_{(AB)_i} = \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i})$ satisfying a condition $(\otimes_{i=1}^k\langle\phi_i|)\varrho|(\otimes_{i=1}^k|\phi_i\rangle) > 1-\epsilon$ there exists a purification $|\Psi\rangle = \otimes_{i=1}^k|\Psi_i\rangle$ of $\otimes_{i=1}^k\varrho_i \equiv \otimes_{i=1}^k \mathrm{tr}_{\mathbf{AB}\setminus A_i}\varrho$ into Hilbert space $\mathcal{H}_{\mathbf{ABC}} = \otimes_{i=1}^k \mathcal{H}_{(ABC)_i} (\mathcal{H}_{(ABC)_i} = \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i} \otimes \mathcal{H}_{C_i})$ such that

$$\langle\Psi|(\varrho \otimes (\otimes_{i=1}^k 0^{C_i}))|\Psi\rangle > 1 - O(\epsilon), \tag{37}$$

where we can take $O(\epsilon) = (2k+4)\epsilon$.

Proof. We can write

$$\varrho \otimes (\otimes_{i=1}^{k} 0^{C_i}) = \lambda_{\max} |\phi_{\max}\rangle\langle\phi_{\max}| \otimes (\otimes_{i=1}^{k} 0^{C_i}) + (1-\lambda_{\max}) \varrho' \otimes (\otimes_{i=1}^{k} 0^{C_i}) \quad (38)$$

with $\lambda_{\max}$ the largest eigenvalue of ϱ and $|\phi_{\max}\rangle$ the corresponding eigenvector, and take purifications

$$|\Psi_i\rangle = \sqrt{\lambda_{\max}^{(i)}} |\phi_{\max}^{(i)}\rangle \otimes |0^{C_i}\rangle + \sqrt{1-\lambda_{\max}^{(i)}} \sum_{k=1}^{\dim \mathcal{H}_{A_i}} \sqrt{\mu_k} |\phi_k^{AB}\rangle \otimes |k^{C_i}\rangle \quad (39)$$

Then we have

$$\begin{aligned} \langle\Psi|\varrho \otimes (\otimes_{i=1}^{k} 0^{C_i})|\Psi\rangle &= (\otimes_{i=1}^{k}\langle\Psi_i|)\varrho \otimes (\otimes_{i=1}^{k} 0^{C_i})(\otimes_{i=1}^{k}|\Psi_i\rangle) \\ &= \lambda_{\max}|(\otimes_{i=1}^{k}\langle\Psi_i|)|\psi_{\max}\rangle \otimes (\otimes_{i=1}^{k}|0^{C_i}\rangle)|^2 \\ &= \lambda_{\max}\prod_{i=1}^{k}\lambda_{\max}^{(i)}|(\otimes_{i=1}^{k}\langle\phi_{\max}^{(i)}|)|\psi_{\max}\rangle|^2 \geq 1 - O(\epsilon), \end{aligned} \quad (40)$$

which concludes the proof. The inequality follows from Lemmas 1 and 6. ■

The preceding lemma is crucial for proving the main theorem of this section which is the following.

Theorem 3. For a given reliable protocol $\mathcal{P}=\{\mathcal{D}^{(n)}, \otimes_{i=1}^{k}\varepsilon_i^{(n)}\}$ for MAC and k-UC there always exists a reliable protocol $\widetilde{\mathcal{P}}=\{(\otimes_{i=1}^{k}\widetilde{\mathcal{D}}_i^{(n)})\circ\mathcal{D}^{(n)}, \mathbb{I}\}$ allowing for entanglement transmission with the same rate.

Proof. Assuming reliable transmission and taking $\otimes_{i=1}^{k}|\Psi_i^{(n)}\rangle$ purifying $\otimes_{i=1}^{k}\mathrm{tr}_{\mathbf{AB}\setminus A_i}\varrho_{\mathbf{AB}}^{\mathrm{out}(n)}$ we have by Lemma 6

$$\langle\Psi^{(n)}|\varrho_{\mathbf{AB}}^{\mathrm{out}(n)} \otimes (\otimes_{i=1}^{k} 0^{C_i})|\Psi^{(n)}\rangle > 1 - O(\epsilon), \quad (41)$$

where $\varrho_{\mathbf{AB}}^{\mathrm{out}(n)}$ is the density matrix after performing a protocol. One can see that

$$|\Psi^{(n)}\rangle = \mathbb{I}_{\mathbf{A}} \otimes (\otimes_{i=1}^{k} U_{(BC)_i}^{\dagger})|\psi_0^{I(n)}\rangle \equiv \mathcal{U}|\psi_0^{(n)}\rangle, \quad (42)$$

where $|\psi_0^{I(n)}\rangle \equiv (\otimes_{i=1}^{k}|\psi_i^{I(n)}\rangle)\otimes(\otimes_{i=1}^{k}|0^{C_i}\rangle)$ and U is unitary, as both $|\Psi^{(n)}\rangle$ and $|\psi_0^{I(n)}\rangle$ are purifications of the same state. Having substituted $\varrho_0^{\mathrm{out}(n)} = \varrho_{\mathbf{AB}}^{\mathrm{out}(n)} \otimes (\otimes_{i=1}^{k} 0^{C_i})$ we get

$$\langle\psi_0^{(n)}|\mathcal{U}\varrho_{\mathbf{0}}^{\mathrm{out}(n)}\mathcal{U}^{\dagger}|\psi_0^{(n)}\rangle \geq 1 - O(\epsilon). \quad (43)$$

We see that if we add the remaining terms with environment in other states to get the full trace we obtain

$$(\otimes_{i=1}^{k}\langle\psi_i^{(n)}|)\mathrm{tr}_{\,\mathbf{C}}\mathcal{U}\varrho_0^{\mathrm{out}(n)}\mathcal{U}^{\dagger}(\otimes_{i=1}^{k}|\psi_i^{(n)}\rangle) \geq 1 - O(\epsilon). \quad (44)$$

So replacing input $\otimes_{i=1}^{k}|\phi_i^{(n)}\rangle$ with $\otimes_{i=1}^{k}|\psi_i^{I(n)}\rangle$ and performing additional decoding operation $\widetilde{\mathcal{D}}$—appending k environments, rotating the whole state locally and tracing out environments—allowed us to transmit a given source reliably. We must stress that a structure of an additional operation is essential here: one can see from (42) that it must be like the coding operation. This is what makes the technique used above useful only in the case of k-UC (in the setting one sender–one receiver) and MAC. It is interesting that although in the case of MAC Bobby can perform global decoding it suffices to perform local operations to achieve reliable transmission. This is why previous considerations fail in the case of general broadcast channel (only in the case of broadcast channels for which product coding suffices we can apply our theorem). To conclude the proof it remains to show that an entropy of new sources producing $\varrho'^{(n)} \equiv \mathrm{tr}_{\mathbf{A}}(\otimes_{i=1}^{k}\psi_i^{I(n)})$ is close to the old one producing $\varrho^{(n)} \equiv \mathrm{tr}_{\mathbf{A}}(\otimes_{i=1}^{k}\phi_i^{(n)})$. The following lemma, which is a simple generalization of the lemma from Ref. [8], will be deciding.

Lemma 7. For a given pure state $|\phi\rangle \equiv \otimes_{i=1}^{k}|\phi_i\rangle$ and density matrix ϱ in Hilbert space $\mathcal{H}=\otimes_{i=1}^{k}\mathcal{H}_{(AB)_i}$ with $\langle\phi|\varrho|\phi\rangle \geq 1-\epsilon$ and $\epsilon < \frac{1}{72}$ we have

$$|S(\mathrm{tr}_{\,\mathbf{A}}\phi) - S(\mathrm{tr}_{\,\mathbf{A}}\varrho)| \leq 2\sqrt{2\epsilon}\log_2 \dim \mathcal{H}_{\mathbf{B}} + 2. \quad (45)$$

We have by the lemma and high entanglement fidelity assumption for $\otimes_{i=1}^{k}|\phi_i^{(n)}\rangle$,

$$|S(\mathrm{tr}_{\,\mathbf{B}}(\otimes_{i=1}^{k}\phi_i^{(n)})) - S(\mathrm{tr}_{\,\mathbf{B}}\varrho_{\mathbf{AB}}^{\mathrm{out}(n)})| \leq 2\sqrt{2\epsilon}\log_2 \dim \mathcal{H}_{\mathbf{A}}^{(n)} + 2 \quad (46)$$

for $\epsilon < \frac{1}{72}$. The following equalities hold: $\mathrm{tr}_{\mathbf{B}}\varrho_{\mathbf{AB}}^{\mathrm{out}(n)} = \mathrm{tr}_{\mathbf{B}}(\otimes_{i=1}^{k}\psi_i^{(n)})$, $S(\mathrm{tr}_{\mathbf{B}}(\otimes_{i=1}^{k}\psi_i^{(n)})) = S(\mathrm{tr}_{\mathbf{A}}(\otimes_{i=1}^{k}\psi_i^{(n)})) \equiv S(\varrho'^{(n)})$, $S(\mathrm{tr}_{\mathbf{B}}(\otimes_{i=1}^{k}\phi_i^{(n)})) = S(\mathrm{tr}_{\mathbf{A}}(\otimes_{i=1}^{k}\phi_i^{(n)})) \equiv S(\varrho^{(n)})$ which immediately implies

$$|S(\varrho^{(n)}) - S(\varrho'^{(n)})| \leq 2\sqrt{2\epsilon}\log_2 \dim \mathcal{H}_A^{(n)} + 2. \quad (47)$$

The latter means that in the limit of large n we achieve the same transmission rate for a new source. The reasoning holds for each subtransmission, i.e., each local entropy, which means that information is localized without changes.

However, we must stress again we are not able to show that the new source is QAEP if the original one was. Most probably it is not possible in general, i.e., encodings are necessary to preserve QAEP (besides the trivial case when input density matrix is almost maximally chaotic on channel input space).

V. FORWARD CLASSICAL COMMUNICATION DOES NOT IMPROVE CAPACITY REGIONS

Now we turn to the case when quantum transmission is supplemented by classical noiseless forward channel. It is a well-known fact that classical support does not increase capacity [3,5] of SUC. We show that it is useless in transmitting quantum information over all quantum channels considered in this paper. Our strategy will be to construct a reliable zero-way protocol from a reliable one-way protocol without changing a rate of a transmission. To this aim consider a set of protocols, indexed by j_i (which in fact is a multiindex) representing classical messages sent by Alicias, $\mathcal{P}_{\vec{j_i}} = \{\otimes_{i=1}^{m}\mathcal{D}_i^{j_i(n)}, \otimes_{i=1}^{k}\varepsilon_i^{j_i(n)}\}$ with $\otimes_{i=1}^{k}\varepsilon_i^{j_i(n)}$ summing over a set of messages to a trace-preserving operation and each $\otimes_{i=1}^{m}\mathcal{D}_i^{j_i(n)}$ trace preserving. As mentioned we have for the protocol $\Sigma_{j_i}F_e(\otimes_{i=1}^{k}(\otimes_{\alpha=1}^{l_i}\Psi_i^{\alpha(n)}), \otimes_{i=1}^{m}\mathcal{D}_i^{j_i(n)}\circ\mathcal{A}\otimes_{i=1}^{k}\varepsilon_i^{j_i(n)}) > 1-\eta$ which means that for one value $j_i \equiv j$ we have $F_e(\otimes_{i=1}^{k}(\otimes_{\alpha=1}^{l_i}\Psi_i^{\alpha(n)}), \otimes_{i=1}^{m}\mathcal{D}_i^{j(n)}\circ\mathcal{A}\otimes_{i=1}^{k}\varepsilon_i^{j(n)}/\mathrm{tr}(\otimes_{i=1}^{k}\varepsilon_i^{j(n)})(\otimes_{i=1}^{k}(\otimes_{\alpha=1}^{l_i}\varrho_i^{\alpha(n)}))) > 1-\eta$ which by Theorem 2 implies existence of a reliable protocol using extendable isometries as an encoding. This shows the possibility of a construction of a reliable zero-way protocol from a one-way protocol without changing a rate of a trnsmission, which shows uselessness of classical forward communication in quantum information transmission.

VI. DISCUSSION

We have considered general multiparty quantum channels and a capacity for a coherent quantum transfer in this scenario. We have defined capacities under subspace transmission and entanglement transmission and have shown that, like in a bipartite case, the two definitions do coincide. The alternative notions of fidelities for both scenario have also been considered and shown to be equivalent. We have also proven that in multiparty scenario forward classical communication does not help. This was achieved by generalization of a bipartite theorem on sufficiency of isometric encoding. The result proves optimality of recently derived by other authors zero-way capacity regions also for the one-way scenario. We have also considered a multiple access channel and a k-user channel separately and show that entanglement transmission capacity can be achieved without encoding. This result however does not seem to be true for broadcast channel and in cases when one assumes sources holding QAEP.

The results of the present paper can be applied to get simple capacity regions for quantum broadcast and k-user channels, but this will be considered elsewhere [26].

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APPENDIX

1. Quantum asymptotic equipartition property (QAEP)

Here we recall definitions of quantum analogs of typical subspace and equipartition property. Classical formulas are easily adopted to the quantum case and one has the following.

Definition 1. [5] We define ϵ-typical subspace of a n-block $\varrho^{(n)}$ produced by a quantum source Σ on a Hilbert space H to be the subspace $T_\epsilon^{(n)}$ of $H^{\otimes n}$ spanned by the eigenvectors $|\lambda\rangle$ of $\varrho^{(n)}$ with eigenvalues λ satisfying

$$2^{-n[S(\Sigma)+\epsilon]} \leqslant \lambda \leqslant 2^{-n[S(\Sigma)-\epsilon]}. \tag{A1}$$

Definition 2. [5] We say a quantum source Σ producing n-block material $\varrho^{(n)}$ satisfies QAEP iff for any positive ϵ and δ in the limit of large n the ϵ-typical subspace of $\varrho^{(n)}$ satisfies

$$\mathrm{tr}\,\Lambda^{(n)}\varrho^{(n)}\Lambda^{(n)} > 1-\delta, \tag{A2}$$

where $\Lambda^{(n)}$ denotes the projection onto $T_\epsilon^{(n)}$.

We can think about ϵ-typical subspace as a small set containing almost all probability. Obviously a tensor product of QAEP sources is also QAEP.

2. Proof of Eq. (22)

Consider global entanglement fidelity $F_e(\otimes_{i=1}^{k}(\otimes_{\alpha=1}^{l_i}\Psi_i^{\alpha(n)}), \mathcal{A}^{(n)}) \geqslant 1-\eta$ with $\eta = \Sigma_{l=1}^{L}\eta_l$. By convexity of entanglement fidelity in the input operator we have the following:

$$1-\eta \leqslant (1-\gamma)\prod_{l=1}^{L}\alpha_l + \prod_{l=1}^{L}(1-\alpha_l), \tag{A3}$$

where γ describes imperfection of global pure state transmission of vectors from the subspace after having removed dimensions (i.e., $F_s \geqslant 1-\gamma$). One can verify that

$$\prod_{l=1}^{L}\alpha_l + \prod_{l=1}^{L}(1-\alpha_l) \leqslant 1 \tag{A4}$$

as for k it is trivially true and multiplying of each component just lowers the number. With the aid of the above and results of Sec. II D we immediately conclude the bound for local pure state fidelity,

$$F_s^{(l)} \geqslant 1 - \frac{\sum_{l=1}^{L}\eta_l}{\prod_{l=1}^{L}\alpha_l}. \tag{A5}$$

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