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Rank two bipartite bound entangled states do not exist

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Abstract

We explore the relation between the rank of a bipartite density matrix and the existence of bound entanglement. We show a relation between the rank, marginal ranks, and distillability of a mixed state and use this to prove that any rank n bound entangled state must have support on no more than an $n \times n$ Hilbert space. A direct consequence of this result is that there are no bipartite bound entangled states of rank two. We also show that a separability condition in terms of a quantum entropy inequality is associated with the above results. We explore the idea of how many pure states are needed in a mixture to cancel the distillable entanglement of a Schmidt rank n pure state and provide a lower bound of $n-1$. We also prove that a mixture of a non-zero amount of any pure entangled state with a pure product state is distillable. © 2002 Published by Elsevier Science B.V.

Perhaps the central topic in quantum information theory has been the study of entanglement, the non-classical correlations between separated parts of a quantum system. In the early days of quantum theory Einstein, Podolsky and Rosen discuss the paradoxical "spooky action at a distance" of entangled particle pairs to express their disbelief that quantum theory could provide a complete picture of reality [11]. Later, Bell used entanglement to prove that quantum mechanics is inconsistent with local reality [1]. An understanding of entanglement seems to be at the heart of theories of quantum computation and quantum cryptography [2], as it has been at the heart of quantum mechanics itself.

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In the case of bipartite pure states, entanglement is rather well understood, in the sense that there is a good measure of entanglement, namely the entropy of the reduced density matrix of one party. For a pure state $|\psi\rangle$ in a Hilbert space belonging to two parties (traditionally named Alice and Bob) $\mathscr{H}_A \otimes \mathscr{H}_B$ we define

$$E(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_A|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B|\psi\rangle\langle\psi|), \tag{1}$$

where S is the von Neumann entropy function of a density matrix given by $S(\rho) = -\mathrm{Tr}(\rho\log\rho)$. Different entangled bipartite pure states of the same entanglement E can be asymptotically converted amongst one another while conserving entanglement [3]. In the case of three or more parties sharing entanglement, the situation is more complicated (cf. [20,7]), as it is for mixed states even for only two parties [4,6].

There are several measures of entanglement for bipartite mixed states. The entanglement of formation E_f is defined as

$$E_f(\rho) = \min \sum_{i=1}^{k} p_i E(|\psi_i\rangle), \tag{2}$$

where the minimization is over all ensembles of pure states ψ_i and non-negative real numbers p_i summing up to 1 such that

$$\sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i| = \rho. \tag{3}$$

The distillable entanglement $D(\rho)$ is defined as the asymptotic amount of pure entanglement that can be gotten out of the mixed state ρ using only local quantum operations and classical communication (these were defined originally in [4,6]). For pure states these two measures are equal and equal to the entropy of the reduced density matrix of each party as in Eq. (1). For mixed states, we have the property $D \leqslant E_f$, reflecting the intuitive idea that one cannot distill more entanglement out of a state than was used in preparing it. Indeed, it has been shown [11] that there exist states for which $D=0$ but $E_f>0$. Such states are known as the *bound entangled* states.

The entanglement of formation and the related relative entropy measure of entanglement [22] share the problem that it is unknown whether or not they are *additive*. Is the entanglement of formation of two copies of a state $\rho^{\otimes 2} = \rho\otimes\rho$ always exactly twice that of one copy? More generally, is the entanglement of formation of n copies of a state always exactly n times the entanglement of formation of just one copy? This has turned out to be a rather difficult problem to solve, but a conjecture about the bound entangled states of [16] may give us a clue. Bound entangled states have finite entanglement of formation, but might it be that in the asymptotic limit as n approaches infinity we have $E_f(\rho_{\mathrm{bound}}^{\otimes n})/n \to 0$, which would explain why no entanglement can be distilled from them? [1]

[1] Since the initial submission of this paper, Vidal and Cirac have shown [23] that bound entangled states do exist that have a non-zero asymptotic entanglement of formation (which is equal to the entanglement cost [12]). Still, the motivation remains to look for low rank bound entangled states as a means of possible finding states with subadditive entanglement of formation or even asymptotically zero but single-copy finite entanglement of formation.

One way of exploring this question is to numerically calculate the entanglement of formation for a bound entangled state ρ, and then for $\rho^{\otimes 2}$, $\rho^{\otimes 3}$, etc., and look for subadditivity. We have done this for some small examples, but unfortunately, the difficulty of the calculation scales exponentially with the number of copies of ρ, and our only results so far have been negative. On the other hand, the rate of the exponential growth is strongly dependent on the rank of the density matrix.

When trying to find the minimum ensemble (as in Eq. (2)) for a density matrix of rank R it is known that at most $k = R^2$ pure states ψ_i are required [21] and each pure state is of dimension R. Thus, for a density matrix of the form $\rho^{\otimes n}$ with ρ of rank r, $\rho^{\otimes n}$ will have rank $R = r^n$ and the minimization will in general require r^{2n} vectors of dimension r^n. For this reason, it is desirable to find bound entangled states of low rank. The bound entangled states of smallest known rank have rank four [5].

Recently, results have been obtained suggesting the sensitivity of bound entanglement to rank. In [24] it was shown by means of numerical analysis that randomly generated bound entangled states in 2×4 typically have a participation ratio (a quantity related to the rank) $\tilde{R} \equiv 1/\mathrm{Tr}(\varrho^2)$ between 5 and 6. On the other hand, for $2 \times n$ it was proved [17] that all states which remain positive semi-definite under partial transposition [2] [15] (PPT states) of rank n and full rank of the reduced density matrix of the second party are separable.

In this paper, we prove that if a bipartite density matrix's rank is less than the rank of the marginal density matrix of either party, then the density matrix has distillable entanglement (is not bound). We use this to prove the negative result that there do not exist bound entangled states of rank two, and to put some restrictions on what a rank three bound entangled state must be like. Note that this implies that there are no unextendible product bases (UPBs) [5,8] in $m \times n$ with $mn - 2$ members because the complementary state to such a UPB would be rank 2 and bound entangled.

We also show that a mixture of a pure product state with a non-vanishing amount of any pure entangled state is distillable and hence not bound entangled. Finally, we conjecture that any irreducible bound entangled state in $n \times n$ has a rank greater than n and shows that this would imply that no rank three bound entangled state exists.

We will first prove a powerful theorem relating the distillability, rank and marginal ranks of a mixed state. Then we will use this result to prove that there exists no bound entangled state of rank two.

Consider a bipartite density matrix ρ whose two parts belong to Alice and to Bob. We denote its *marginal or local density matrices* by $\rho^{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}(\rho)$ and $\rho^{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}}(\rho)$, respectively, obtained by tracing out Bob and Alice. Its *marginal rank* on Alice's side $\mathscr{R}(\rho^{\mathrm{A}})$ is the rank of ρ^{A} and it is similar for Bob. For pure states, the marginal ranks are equal and are also called the Schmidt rank of the state [18]. We say that a state ρ in $m \times n$ is *irreducible* if and only if $\mathscr{R}(\rho^{\mathrm{A}}) = m$ and $\mathscr{R}(\rho^{\mathrm{B}}) = n$. Intuitively, this means that the density matrix fully utilizes each of the local Hilbert spaces of Alice and Bob.

[2] The partial transpose is defined by

$$\mathrm{PT}(\rho)_{ij,kl} = \rho_{il,kj}, \tag{4}$$

where the i and k indices are associated with Hilbert space $\mathscr{H}_{\mathrm{A}}$ and the j and l indices with $\mathscr{H}_{\mathrm{B}}$.

In our proof we use the reduction criterion of separability and distillability [13]: If a state ρ is separable, then $\mathbf{1} \otimes \rho^{\mathrm{B}} - \rho \geqslant 0$ and $\rho^{\mathrm{A}} \otimes \mathbf{1} - \rho \geqslant 0$. (A Hermitian matrix H is positive semi-definite, or $H \geqslant 0$ for short, if and only if $\langle \psi | H | \psi \rangle \geqslant 0 \ \forall_{\psi}$, or equivalently H has no negative eigenvalues.) If this criterion is violated, then ρ is distillable. This provides a necessary condition for separability and a sufficient condition for distillability.

Theorem 1. *If* $\mathscr{R}(\rho) < \max[\mathscr{R}(\rho^{\mathrm{A}}), \mathscr{R}(\rho^{\mathrm{B}})]$, *then* ρ *is distillable*.

Proof. Without loss of generality let $R = \mathscr{R}(\rho_{\mathrm{A}}) > \mathscr{R}(\rho) = r$. By local filtering Alice takes ρ to ρ_{f} such that,

$$\rho_{\mathrm{f}}^{\mathrm{A}} = \frac{1}{R} \mathbf{1}_R. \tag{5}$$

This can always be done with non-zero probability by applying the local filter $W = \sum_{i=1}^{R} (1/\mu_i R) |\mu_i\rangle\langle\mu_i|$, where μ_i are the non-zero eigenvalues and $|\mu_i\rangle$ are the corresponding eigenvectors of ρ^{A} [13]. Thus the eigenvalues of $\rho_{\mathrm{f}}^{\mathrm{A}}$ are $1/R$. Since ρ_{f} has a unit trace and is of rank r, its largest eigenvalue $\lambda_{\max}$ cannot be less than $1/r$ which in turn is larger than $1/R$. Choosing $|\psi\rangle$ to be the eigenvector of ρ_{f} corresponding to the eigenvalue $\lambda_{\max}$ we have

$$\langle \psi | \rho_{\mathrm{f}}^{\mathrm{A}} \otimes \mathbf{1} - \rho_{\mathrm{f}} | \psi \rangle = \langle \psi | \rho_{\mathrm{f}}^{\mathrm{A}} \otimes \mathbf{1} | \psi \rangle - \lambda_{\max} \leqslant \frac{1}{R} - \frac{1}{r} < 0. \tag{6}$$

Thus the reduction criterion for separability is violated and hence ρ_{f} is distillable. Since ρ_{f} can be obtained from ρ by a local quantum operation (filtering), this implies that ρ is distillable. □

Remark. It is also easy to show by the same reasoning that if $\mathscr{R}(\rho) = R$ then ρ is distillable except in the case where ρ_{f} is proportional to the identity. This is the case where $\lambda_{\max}$ is smallest: $\lambda_{\max} = 1/R = 1/r$.

Turning around the inequality (Theorem 1) we have that if a mixed state ρ is separable or bound entangled (i.e. not distillable), then

$$\mathscr{R}(\rho) \geqslant \max[\mathscr{R}(\rho^{\mathrm{A}}), \mathscr{R}(\rho^{\mathrm{B}})]. \tag{7}$$

Using the monotonicity of logarithms, one can rephrase the theorem as an entropy inequality: For separable or bound entangled states the entropy $S_0(\rho) \equiv \log(\mathscr{R}(\rho))$ must satisfy the inequality

$$S_0(\rho) \geqslant S_0(\rho^{\mathrm{A}}) \quad \text{and} \quad S_0(\rho) \geqslant S_0(\rho^{\mathrm{B}}). \tag{8}$$

The entropy S_0 (called the Hartley entropy) is a special case of quantum Renýi entropies

$$S_\alpha \equiv (1-\alpha)^{-1} \log(\operatorname{Tr} \rho^\alpha). \tag{9}$$

The counterparts of the separability condition (8) have already been proved for the case of $\alpha = 2$ and the cases where α limits to 1 and ∞ (see [14]). It is interesting to note that the ratio of violation of (8) in the limit $\alpha \downarrow 1$ (the von Neumann entropy) gives in the case of 2×2 Werner states the yield of the hashing method of distillation of entanglement [6].

Let us now look at some implications of this result for separable and for bound entangled states.

Corollary 1. *A rank n separable or bound entangled state in a Hilbert space $\mathscr{H}$ has support in at most an $n\times n$ subspace of $\mathscr{H}$. Further, there is no rank two bound entangled state.*

Proof. The first statement follows directly from Eq. (7). The second statement is a consequence of the first and the fact that in 2×2 every entangled state is distillable [15]. □

An open question is whether there exists a rank three bound entangled state. We can put some constraints on the form of such a state: If a rank three bound entangled state exists, then Corollary 1 shows that it must have support on no more than a 3×3 subspace. It must also be that such a state is irreducible because no bound entanglement can exist in 2×2 or 2×3 [15]. Then, as in the remark above, ρ_{f} must be proportional to the identity on a three-dimensional subspace in 3×3.

Given the fact that a rank n bound entangled state must live in an $n\times n$ subspace, one may ask whether there are irreducible bound entangled states of rank n in $n\times n$. Of course there is no rank two bound entangled state in 2×2, and all the known examples of bound entanglement are not of this type. The closest one can get to this among known examples is the UPB state of rank four in 3×3 [5]. Thus we conjecture:

Conjecture 1. *There are no irreducible bound entangled states of rank n in $n\times n$, i.e., any rank n bound entangled state can be expressed in a bipartite space of dimension $n\times(n-1)$ or $(n-1)\times n$.*

If this conjecture holds, then there are no rank three bound entangled states at all.

Theorem 1 also has consequences for the "cancellation of distillable entanglement". Suppose that one has a Schmidt rank n pure state $|\Psi\rangle=\sum_{i=1}^{n}\sqrt{\mu_i}|\psi_i^{\mathrm{A}}\rangle\otimes|\psi_i^{\mathrm{B}}\rangle$. How many other arbitrary pure states $|\phi_j\rangle$ are needed in a mixture $\rho=p_0|\Psi\rangle\langle\Psi|+\sum_{j=1}^{k} p_j|\phi_j\rangle\langle\phi_j|$ before ρ stops being distillable? We have the following corollary.

Corollary 2. *If $\rho=p_0|\Psi\rangle\langle\Psi|+\sum_{j=1}^{n-2}p_j|\phi_j\rangle\langle\phi_j|$ with $p_0>0$ and $|\Psi\rangle$ of Schmidt rank n, then ρ is distillable.*

Proof. The proof follows from Theorem 1 and the fact that $\mathscr{R}(\rho)\leqslant n-1$ and $\mathscr{R}(\rho^{\mathrm{A}})\geqslant n$ since $|\Psi\rangle$ is Schmidt rank n. Thus, $\mathscr{R}(\rho)\leqslant n-1<n\leqslant\mathscr{R}(\rho^{\mathrm{A}})$ and the theorem applies. □

For $n=2$ the above corollary gives the empty result that a Schmidt rank two state mixed with zero other states is distillable. We will show a slight extension of this namely:

Theorem 2. *A mixture of a non-zero amount of any entangled pure state with any pure product state is always distillable.*

Proof. Consider the entangled and product pure states in the Hilbert space of $\mathscr{H}_\mathrm{A} \otimes \mathscr{H}_\mathrm{B}$. Since the state ρ made by mixing them has rank two, it is always distillable unless ρ has support on at most a 2×2 subspace, by Corollary 1. Thus, without loss of generality we may choose the product state to be $|0\rangle|0\rangle$ and the entangled state to be $|\psi\rangle = a|0\rangle|0\rangle + b|0\rangle|1\rangle + c|1\rangle|0\rangle + d|1\rangle|1\rangle$.

In $\mathscr{H}_2 \otimes \mathscr{H}_2$ a density matrix ρ is entangled [19] and distillable [14] if and only if the partial transpose of ρ is not positive semi-definite.

We consider the mixture

$$\rho = p|00\rangle\langle 00| + |\psi\rangle\langle\psi|. \tag{10}$$

This mixture would be normalized by a denominator of $1 + p$, but this will not affect positivity and so we omit it.

The partial transpose $\mathrm{PT}(\rho)$ is

$$\rho' = \mathrm{PT}(\rho) = \begin{pmatrix} p + |a|^2 & a^*b & ac^* & bc^* \\ ab^* & |b|^2 & ad^* & bd^* \\ a^*c & a^*d & |c|^2 & c^*d \\ b^*c & b^*d & cd^* & |d|^2 \end{pmatrix}. \tag{11}$$

If a matrix has a negative determinant, then the matrix is not positive semi-definite. Expanding the determinant of ρ' using Cramer's rule on the top row we can write that

$$\begin{aligned} \det(\rho') &= p\det(C_{11}) + \det(\rho' - p|00\rangle\langle 00|) \\ &= p\det(C_{11}) + \det(\mathrm{PT}(|\psi\rangle\langle\psi|)), \end{aligned} \tag{12}$$

where C_{11} is the 3×3 matrix formed by leaving out the first row and first column of ρ'. The second term is always negative since $|\psi\rangle$ is entangled (this is easily seen by doing the partial transpose in the Schmidt basis and noting the fact that the eigenvalues and hence the determinant of the partial transpose are invariant based on which the partial transposition is done). Writing out the first term as

$$-p|d|^2(|a|^2|d|^2 - ab^*cd^* - a^*bc^*d + |b|^2|c|^2) = -p|d|^2|(ad - bc)|^2, \tag{13}$$

we see that it is always less than or equal to zero. Thus, the determinant of ρ' is negative, implying ρ' is not positive semi-definite and ρ is distillable. □

Remark. In contrast to this theorem, a mixture of two entangled pure state *can* be separable, for example, the equal mixture of $(1/\sqrt{2})(|00\rangle + |11\rangle)$ and $(1/\sqrt{2})(|00\rangle - |11\rangle)$.

We have explored the relation between the rank of a density matrix and the existence of bound entanglement. In particular we find that a density matrix with rank smaller than either of the marginal ranks is distillable. This gave us the result that any rank n bound entangled state must belong to an $n \times n$ subspace. A direct consequence of this result is that there are no bound entangled states of rank two.

We also pointed out that there is a separability criterion in terms of a quantum entropy inequality which is naturally associated with our results. Further, we have explored the idea of how many pure states are needed to cancel the distillable entanglement of a Schmidt rank n pure state and we provided a lower bound of $n-1$. We also showed the related result that a mixture of a non-zero amount of any entangled pure state with a pure product state is distillable. Thus mixing with a single pure product state cannot prevent the distillability of an entangled pure state.

It should be noted that our results on bound entanglement hold even for bound entangled states whose partial transpositions are not positive semi-definite, should such states exist (for evidence they may see [9,10]). This can be seen because our proofs are based directly on distillability.

It is an open question whether bound entangled states of rank three exist. Another related question is whether states with rank equal to the marginal ranks can be bound entangled or whether the rank needs to be strictly greater.

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