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Reduction restrictions of Darboux and Laplace transformations for the Goursat equation

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We study Darboux and Laplace transformations of the solutions and potentials of the Goursat equation which is equivalent to one of the Lax pair equations for the 2D-MKdV hierarchy. The reduction restrictions for these transformations are considered. The derived reduction equations are generalizations of the Liouville and sinh-Gordon equation. The integrability of these equations by the ST method is proved. The binary Darboux transformation for the Goursat equation is suggested. We find exact rational nonsingular solutions of the 2D-MKdV equations via the Moutard transformation for the Goursat equation. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1427761]

I. INTRODUCTION

A covariance of general Lax pairs leads to abundant but generally useless integrable systems while reductions of them may have straight applications in mathematical physics.[1] In $2+1$ or higher dimensions there exists the problem of an elaboration of similar or another approach to classification or at least of a choice of invariant subsets by some key rule.[2] Such a rule is directly connected with discrete covariance of Lax equations[3] that appears in the classic Laplace scheme developed by Darboux, Moutard, and Le Roux[4] and investigated recently from different points of view (e.g., Refs. 5 and 6). We consider subclasses of Laplace (Darboux)-covariant "potentials," i.e., introduce a notion of "reduction equation" that follows directly from the constraint form invariance. It means that Darboux (Laplace) transforms together with the appropriate partial solutions of a basic equation determine some discrete symmetry and chains of solutions. Let us demonstrate this by an example.

It is known that the Laplace transformation (LT) of the equation

$$\psi_{xy}+a\psi_y+b\psi=0 \tag{1}$$

has the form

$$a\to a_{-1}=a-\partial_x\ln(b-a_y),\quad b\to b_{-1}=b-a_y,\quad \psi\to\psi_{-1}=\psi_x+a\psi, \tag{2}$$

$$a\to a_1=a+\partial_x\ln b,\quad b\to b_1=b+\partial_y(a+\partial_x\ln b),\quad \psi\to\psi_1=\frac{\psi_y}{b}, \tag{3}$$

and plays a significant role in the theory of soliton equation development.

The Goursat equation (GE) has the form[7]

$$\zeta_{xy}=2\sqrt{\lambda\zeta_x\zeta_y},$$

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where $\zeta = \zeta(x,y)$ and $\lambda = \lambda(x,y)$. We call λ a potential function. This equation can be linearized by the substitution $\psi = \sqrt{\zeta_x}$ and $\chi = \sqrt{\zeta_y}$. We get

$$\psi_y = \sqrt{\lambda}\,\chi, \quad \chi_x = \sqrt{\lambda}\,\psi$$

or

$$\psi_{xy} = \tfrac{1}{2}(\ln\lambda)_x\psi_y + \lambda\,\psi, \tag{4}$$

and the similar equation for the χ but we will not need one.

Equation (4) is the particular case of Eq. (1) with two potentials $a = a(x,y)$ and $b = b(x,y)$. This equation has two types of local discrete symmetries:

(1) Laplace transformations (2) and (3), mentioned above, and

(2) Darboux transformations (DT):

$$a \to a_1 = a - \partial_x \ln(a+\tau), \quad b \to b_1 = b + \tau_y, \quad \psi \to \psi_1 = \psi_x - \tau\psi, \tag{5}$$

$$a \to {}_1a = -(\tau + b\rho), \quad b \to {}_1b = b - (b\rho)_y, \quad \psi \to {}_1\psi = \rho\psi_y - \psi, \tag{6}$$

where $\tau = \phi_x/\phi$, $\rho = \phi/\phi_y$, ψ and ϕ are particular solutions of (1) by preassigned a and b, and we call ϕ **the support function** of the DT.

The aim of this work is to study the validity of LT and DT for the GE. It is clear that after single DT or LT the reduction restriction

$$a = -\partial_x \ln b \tag{7}$$

will be true only for the special class of potentials and we will specify it in Sec. II.

Our interest in the GE is connected with the two applications of this equation in geometry and in the solitons theory, respectively.

(1) Let x be the complex coordinate, $y = -\overline{x}$, $\sqrt{\lambda}$ is the real-valued function, and ψ and χ from (1) are complex-valued functions. Then one defines three real-valued functions X_i, $i = 1,2,3$, which are the coordinates of surface in R^3:[8]

$$X_1 + iX_2 = 2i\int_\Gamma (\overline{\psi^2}dy' - \overline{\chi^2}dx'),$$

$$X_1 - iX_2 = -2i\int_\Gamma (\psi^2 dy' - \chi^2 dx'), \tag{8}$$

$$X_3 = -2\int_\Gamma (\overline{\psi}\chi dy' + \overline{\chi}\psi dx'),$$

where Γ is an arbitrary path of integration in the complex plane. The corresponding first fundamental form, the Gaussian curvature K, and the mean curvature H yield

$$ds^2 = 4U^2 dx dy, \quad K = \frac{1}{U^2}\partial_x\partial_y \ln U, \quad H = \frac{\sqrt{\lambda}}{U},$$

where

$$U = |\psi|^2 + |\chi|^2,$$

and any analytic surface in R^3 can be globally represented by (8) (see Ref. 9).

(2) The system of the 2D-MKdV equations looks like:

$$4\lambda^2(\lambda_t - A\lambda_x + B\lambda_y - \lambda_{3x} - \lambda_{3y}) + 4\lambda^3[(2\lambda + B)_y + (2\lambda - A)_x]$$
$$+6\lambda(\lambda_y\lambda_{yy} + \lambda_x\lambda_{xx}) - 3(\lambda_x^3 + \lambda_y^3) = 0, \tag{9a}$$

$$B_x = 3\lambda_y - \lambda_x, \quad A_y = \lambda_y - 3\lambda_x.$$

Here $\lambda = \lambda(x,y,t)$, $A = A(x,y,t)$, $B = B(x,y,t)$. If we introduce the function $u = \sqrt{\lambda}$ then we can rewrite (9) in the more customary form (see Ref. 10):

$$\begin{aligned} &u_t + 2u^2(u_x + u_y) + \tfrac{1}{2}(B_y - A_x)u + Bu_y - Au_x - u_{3y} - u_{3x} = 0 \\ &B_x = (3\partial_y - \partial_x)u^2, \quad A_y = (\partial_y - 3\partial_x)u^2. \end{aligned} \tag{9b}$$

The reduction conditions

$$A = -B = -2u^2, \quad u_y = u_x,$$

lead us to the MKdV equation,

$$u_t + 12u^2u_x - 2u_{3x} = 0,$$

so we call (9a) the 2D-MKdV equations.

The 2D-MKdV equations (9) is the compatibility condition of the linear system (so-called $[L, A]$ pair) which contains Eq. (4) and

$$\psi_t = \psi_{3x} + \psi_{3y} - \frac{3}{2}\frac{\lambda_y}{\lambda}\psi_{yy} + \left[\frac{3}{4}\left(\frac{\lambda_y}{\lambda}\right)^2 - \lambda - B\right]\psi_y + (A - \lambda)\psi_x + \frac{1}{2}(A_x - \lambda_x)\psi.$$

We will study (9a) in the last section (Sec. IV).

Remark 1: In Ref. 11 A. I. Zenchuk studied the discrete transformation (2), (3), (5), and (6) of solutions and potentials in the general case of the linear second order partial differential equation with two independent variables. The simplest $(k=2)$ closed chains of these transformations are considered and the author obtain a novel **integrable** equation:

$$\tfrac{1}{2}S_{xy} - e^S - e^{-S}[C_1 - C_2\partial_x^{-1}(e^{-S})_y] = 0, \tag{10}$$

where $C_2 > 0$.

In the present work we use reduction restriction (7) as a (weak) condition of closing. In Sec. II we will obtain a new integrable equation [see (19)] which looks like (10) and it is a somewhat two-dimensional generalization of the sinh-Gordon equation. In Sec. III we suggest the binary DT for a construction of explicit solutions of the GE. These transformations allow one to obtain new solutions of the GE without solving some reduction equation. We also discuss the transformation for Laplace invariants.

II. THE REDUCTION EQUATIONS

The reduction restriction (7) is valid only for special types of potentials. These functions are solutions of the special equations which we call **reduction equations**. In this section we will obtain these equations for the LT and DT.

(I) Let us consider the Laplace transformations (2). The invariance of the reduction constraint means

$$\lambda_{-1} = \lambda - \frac{1}{2}\partial_x\partial_y \ln\lambda = \frac{C}{2\lambda}. \tag{11}$$

It is obvious that Eq. (11) is valid for the LT (3) because the one is inverse to the transformation (2).

It easy to show that the reduction equation for this transformation is the well-known sinh-Gordon equation:

$$\partial_x \partial_y \ln \lambda = 2\lambda - \frac{C}{\lambda}, \tag{12}$$

where $C = \mathrm{const}$, and the new potential λ_{-1} is a solution of (12) too.

Let us remark that in the case of $C=0$ we obtain $\lambda_{-1}=0$ and the Liouville equation instead of (12). In this case the GE may be integrated and

$$\lambda = \frac{f' g'}{(f+g)^2}, \quad \zeta = -\frac{1}{C^2} \partial_y \ln(f+g) + V,$$

where $f=f(x)$ and $g=g(y)$ are arbitrary differentiable functions, $C=\mathrm{const}$, $V=V(y)$ is the function such that

$$V' = \left[\frac{1}{2C} (\ln g')' \right]^2 = \frac{1}{4C^2} \left(\frac{g''}{g'} \right)^2,$$

and

$$\psi = \frac{\sqrt{f' g'}}{C(f+g)}, \quad \chi = \frac{1}{2C} \partial_y \ln \left(-\partial_y \frac{1}{f+g} \right).$$

Proposition 1: Let M and L be two Laplace invariants of Eq. (4). It means that

$$M = \tfrac{1}{2} \partial_x \partial_y \ln \lambda - \lambda, \quad L = -\lambda.$$

Using the reduction equation (11) we get

$$M = -\frac{C}{2\lambda}, \quad L = -\lambda$$

and

$$M_{-1} = M_1 = L, \quad L_{-1} = L_1 = M.$$

(II) Let us consider the DT (5). Inserting both transforms into the reduction condition (7) yields

$$\lambda_{-1} = \lambda - \tau_y = \lambda \left(\tau - \frac{\lambda_x}{2\lambda} \right). \tag{13}$$

Denote now $\alpha = \ln \phi$, $\Lambda = \ln \lambda$. Since

$$\lambda - \tau_y = (-\tfrac{1}{2}\Lambda_x + \alpha_x)\alpha_y,$$

and $\tau = \alpha_x$ one gets from the transform (13) the condition for Λ:

$$(\alpha_x - \tfrac{1}{2}\Lambda_x)[\alpha_y - \exp(\lambda)(\alpha_x - \tfrac{1}{2}\Lambda_x)] = 0. \tag{14}$$

The setting zero for the first parentheses yields

$$\Lambda_{xy} = 2 \exp(\Lambda),$$

and $\alpha=\Lambda/2-c(y)$, where $c(y)$ is arbitrary function. But in this case we get $\lambda_1=0$, and the Liouville equation is in the realm of the reduction equation.

Setting equal to zero the square brackets in (14) one arrives at the relevant equation

$$(\exp(-2\alpha)\lambda)_x=(\exp(-2\alpha))_y, \tag{15}$$

therefore

$$\theta_x=\psi^2=\frac{1}{F_x+C_2}, \quad \lambda=\frac{F_y+C_1}{F_x+C_2},$$

where $F=F(x,y)$ is any differentiable function and $C_{1,2}=\mathrm{const}$. Substituting (15) into (4) we get

$$\begin{aligned}2(C_2+F_x)C_1^2&+[(F_{yxx}+4F_y)C_2+F_xF_{yxx}+4F_yF_x-F_{xx}F_{yx}]C_1+(F_{yxx}F_y-\tfrac{1}{2}F_{yx}^2+2F_y^2)C_2\\&+2F_y^2F_x-\tfrac{1}{2}F_{yx}^2F_x-F_yF_{xx}F_{yx}+F_xF_yF_{yxx}=0.\end{aligned} \tag{16}$$

We define new fields:

$$F_x=P-C_2, \quad F_y=Q-C_1.$$

Then (16) can be split into the system

$$2Q_xQP_x-(2Q_{xx}Q-Q_x^2+4Q^2)P=0, \quad P_y=Q_x. \tag{17}$$

After integration of the first equation we get

$$P=\frac{CQ_x}{\sqrt{Q}}\exp(G), \quad G_x=2\frac{Q}{Q_x},$$

where C is the third constant of integration. It is necessary that the second equation in (17) will be true. Let

$$Q=n^2, \quad G=\ln m,$$

where $m=m(x,y)$ and $n=n(x,y)$. The reduction equation takes the simple form

$$(n^2)_x=2C(mn_x)_y, \quad m_xn_x=mn. \tag{18}$$

This system can be rewritten in more convenient form. Let

$$n_x=n\exp(S), \quad m_x=m\exp(-S),$$

$S=S(x,y)$. After substituting into (18) we get

$$S_y=\frac{1}{C}\frac{n}{m}-\partial_y\ln(mn),$$

therefore

$$S_{xy}=4\sinh S\,\partial_y\partial_x^{-1}\cosh S. \tag{19}$$

Equation (19) is the reduction equation for the DT (5). It looks like Eq. (10) and it is the generalization of $d=2$ sinh-Gordon equation. We will present the Lax pair analog for Eq. (19) by the following proposition:

Proposition 2: Let us introduce the $[L,A]$ pair for Eq. (19) in the form

$$K\psi=0, \quad K_1 D\psi=0$$

where

$$K=\partial_x\partial_y-\frac{1}{2}\frac{\lambda_x}{\lambda}\partial_y-\lambda, \quad K_1=\partial_x\partial_y-\frac{1}{2}\frac{\lambda_{1,x}}{\lambda_1}\partial_y-\lambda_1, \quad D=\partial_x-\tau,$$

the variables λ and λ_1 are defined by the equalities

$$\lambda=\frac{(S_x+2\cosh S)_y}{4\sinh S}\exp(-S), \quad \lambda_1=\frac{(S_x+2\cosh S)_y}{4\sinh S}\exp(S), \tag{20}$$

and

$$\tau_y\equiv\lambda-\lambda_1.$$

It is possible to check the statement by direct substitution. Thus the reduction equations for the DT (5) has either the form of Eq. (19) or the Liouville equation.

We can study the reduction equations for the DT (6) analogously. As a result we get

$$\lambda=C_1\phi_y\exp(F), \quad {}_1\lambda=-\frac{C_1C_2\phi^2}{\phi_y}\exp(F), \tag{21}$$

where ϕ is the support function of the DT (6) and the reduction equation can be written like the system:

$$\phi_{xy}=\phi_y[F_x+2C_1\phi\exp(F)], \quad F_y\phi_y-C_2\phi.$$

Proposition 3: By the construction (20) for the DT (5) we get

$$M=-\lambda_1, \quad L=-\lambda,$$

and

$$M_1=M\exp(-2S), \quad L_1=L\exp(2S).$$

Quite similar for the DT (6) the use of (21) gives

$$M=-\frac{C_2(-\phi_x+\phi F_x+C_1\phi^2\exp(F))}{\phi_y}, \quad L=-C_1\phi_y\exp(F),$$

and

$${}_1M=-\frac{\phi_y^2}{C_2\phi^2}M, \quad {}_1L=-\frac{C_2\phi^2}{\phi_y^2}L.$$

The multiple of the Laplace invariants ML is invariant in both cases.

III. BINARY DT

In Ref. 12 Ganzha studied the analog of the Moutard transformation for the Goursat equation. This transformation is valid without a reduction restriction and reduction equations. In this section we obtained binary Darboux transformation for the GE with the same property.

We introduce new variables ξ and η:

$$\partial_y=\partial_\eta-\partial_\xi, \quad \partial_x=\partial_\eta+\partial_\xi,$$

and rewrite (4) in the matrix form

$$\Psi_\eta=\sigma_3\Psi_\xi+U\Psi, \tag{22}$$

where

$$\Psi=\begin{pmatrix}\psi_1 & \psi_2\\ \chi_1 & \chi_2\end{pmatrix},\quad U=\sqrt{\lambda}\sigma_1, \tag{23}$$

$\psi_k=\psi_k(\xi,\eta)$, $\chi_k=\chi_k(\xi,\eta)$ with the $k=1,2$ particular solutions of (1) with some $\lambda(\xi,\ \eta)$, and $\sigma_{1,3}$ are the Pauli matrices. Let Ψ_1 some solution of Eq. (22) and $\Psi\neq\Psi_1$. We define a matrix function $\tau\equiv\Psi_{1,\xi}\Psi_1^{-1}$. Equation (22) is covariant with respect to DT:

$$\Phi[1]=\Phi_\xi-\tau\Phi,\quad U[1]=U+[\sigma_3,\tau]. \tag{24}$$

Remark 2: It is not difficult to check that the DT (24) is the superposition formula for the two simpler Darboux transformations given by formulas (5) and (6).

Remark 3: Equation (22) is the spectral problem for the Davey–Stewartson (DS) equations.[13] The LTs produce an explicitly invertible Bäcklund autotransformation for the DS equations. In Ref. 14 we showed that these transformations allow one to construct solutions to the DS equations that fall off in all directions in the plane according to exponential and algebraic law.

Let us consider a closed one-form

$$d\Omega=d\xi\Phi\Psi+d\eta\Phi\sigma_3\Psi,\quad \Omega=\int d\Omega,$$

where a 2×2 matrix function Φ solves the equation

$$\Phi_\eta=\Phi_\xi\sigma_3-\Phi U. \tag{25}$$

We shall apply the DT for (22). One can verify by immediate substitution that (25) is covariant with respect to the transform if

$$\Phi[+1]=\Omega(\Phi,\Psi_1)\Psi_1^{-1}.$$

Now we can alternatively affect U by the following transformation:

$$U[+1,-1]=U+[\sigma_3,\Psi_1\Omega^{-1}\Phi].$$

The particular solution of Eq. (25) has the form

$$\Phi_1=\begin{pmatrix}s_1\psi_1+s_2\psi_2 & -s_1\chi_1-s_2\chi_2\\ s_3\psi_1+s_4\psi_2 & -s_3\chi_1-s_4\chi_2\end{pmatrix}, \tag{26}$$

where $s_k=\text{const}$ $(k=1,...,4)$. It is convenient to choose one in the form

$$\Phi_1=\Psi_1^T\sigma_3, \tag{27}$$

where Ψ_1^T is the transposed matrix Ψ_1 [(27) is the particular case of (26)].

In this case

$$U[+1,-1]=U-2A_F, \tag{28}$$

where A_F is the off-diagonal part of the matrix A:

$$A=\Psi_1\Omega^{-1}\Psi_1^T,$$

$\Omega = \Omega(\Phi_1, \Psi_1)$, and

$$A_F^T = A_F = f\sigma_1, \tag{29}$$

where $f = f(\xi, \eta)$ is a some function.

Using (23), (28), and (29) we can see that $U[+1,-1]$ has the same form as the initial matrix U:

$$U[+1,-1] \equiv \begin{pmatrix} 0 & \sqrt{\lambda[+1,-1]} \\ \sqrt{\lambda[+1,-1]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\lambda} - 2f \\ \sqrt{\lambda} - 2f & 0 \end{pmatrix},$$

thus the reduction restriction is valid without the reduction equations.

The new function $\Phi[+1,-1]$ has the form

$$\Phi[+1,-1] = \Phi - \Omega(\Phi, \Psi_1)(\Omega(\Psi_1, \Psi_1))^{-1}\Phi_1, \tag{30}$$

where Φ is arbitrary solution of Eq. (25).

Using binary DT (28) and (30) we can construct a new solution of the GE from understanding particular solutions of one.

As a result we get the following theorem (in the old variables x and y):

Theorem: Let

$$\psi_{k,y} = \sqrt{\lambda}\chi_k, \quad \chi_{k,x} = \sqrt{\lambda}\psi_k, \quad \alpha_{k,y} = -\sqrt{\lambda}\beta_k, \quad \beta_{k,x} = -1\sqrt{\lambda}\alpha_k,$$

where $k = 1,2$. Then new functions

$$\alpha_1' = \alpha_1 - \frac{A_1\psi_1 + A_2\psi_2}{D}, \quad \beta_1' = \beta_1 + \frac{A_1\chi_1 + A_2\chi_2}{D}$$

are solutions of the equations

$$\alpha'_{1,y} = \sqrt{\lambda'}\beta_1', \quad \beta'_{1,x} = \sqrt{\lambda'}\alpha_1',$$

where

$$\sqrt{\lambda'} = -\sqrt{\lambda} + \frac{\psi_1\chi_1\Omega_{22} + \psi_2\chi_2\Omega_{11} - (\psi_1\chi_2 + \psi_2\chi_1)\Omega_{12}}{D},$$

and

$$\Omega_{11} = \int dx\,\psi_1^2 + dy\,\chi_1^2, \quad \Omega_{12} = \Omega_{21} = \int dx\,\psi_1\psi_2 + dy\,\chi_1\chi_2,$$

$$\Omega_{22} = \int dx\,\psi_2^2 + dy\,\chi_2^2, \quad D = \Omega_{11}\Omega_{22} - \Omega_{12}^2,$$

$$\Lambda_{11} = \int dx\,\alpha_1\psi_1 + dy\,\beta_1\chi_1, \quad \Lambda_{12} = \int dx\,\alpha_1\psi_2 + dy\,\beta_1\chi_2,$$

$$\Lambda_{21} = \int dx\,\alpha_2\psi_1 + dy\,\beta_2\chi_1, \quad \Lambda_{22} = \int dx\,\alpha_2\psi_2 + \beta_2\chi_2,$$

$$A_1 = \Lambda_{11}\Omega_{22} - \Lambda_{12}\Omega_{12}, \quad A_2 = \Lambda_{12}\Omega_{11} - \Lambda_{11}\Omega_{12}.$$

Here $\int = \int_\Gamma$ where Γ is an arbitrary path of integration in the plane. It is easy to obtain the expressions for the functions α_2' and β_2', but we will not do it.

Thus the binary DT allows one to construct explicit solutions of the GE without the solving of some reduction equation.

IV. THE MOUTARD TRANSFORMATION FOR THE 2D-MKdV EQUATIONS

The Lax ($[L,A]$) pair for the 2D-MKdV equations (9a) has the form

$$\psi_{xy}=\frac{u_x}{u}\psi_y+u^2\psi,$$
$$\psi_t=\psi_{3x}+\psi_{3y}-3\frac{u_y}{u}\psi_{yy}+\left[3\left(\frac{u_y}{u}\right)^2-u^2-B\right]\psi_y+(A-u^2)\psi_x+\frac{1}{2}(A-u^2)_x\psi. \tag{31}$$

In Ref. 12 Ganzha studied the one of analog of the Moutard transformation for the Goursat equation. To use this transformation for obtaining exact solutions of (9a) we must complete a definition of the Moutard transformation. It is easy to do. Let ϕ be the second solution of (31) (the support function). Then we have a closed one-form,

$$d\theta=dx\,\theta_1+dy\,\theta_2+dt\,\theta_3,\quad \theta\equiv\int d\theta,$$

where

$$\theta_1=\phi^2,\quad \theta_2=\left(\frac{\phi_y}{u}\right)^2,$$

$$\theta_3=(A-u^2)\phi^2-\phi_y^2-\phi_x^2+2\phi\phi_{xx}+\frac{(2\phi_{3y}\phi_y-\phi_{yy}^2-B\phi_y^2)u^2-2u\phi_y(u_y\phi_y)_y+3(u_y\phi_y)^2}{u^4}.$$

We define the **generalized** Moutard transformation in the following way:

$$u\to\tilde{u}=u-\sqrt{(\ln\theta)_x(\ln\theta)_y},\quad A\to\tilde{A}=A-(\partial_x\partial_y-3\partial_x^2)\ln\theta,$$
$$B\to\tilde{B}=B+(\partial_x\partial_y-3\partial_y^2)\ln\theta,\quad \psi\to\tilde{\psi}=\frac{\phi Q}{\theta}, \tag{32}$$

where

$$Q\equiv\int dQ,$$

$$dQ=dx\,Q_1+dy\,Q_2+dt\,Q_3,$$

and ($w=\psi/\phi$)

$$Q_1=\theta w_x,\quad Q_2=-\frac{\theta^3(1/\theta)_{xy}w_y}{\theta_{xy}},$$

$$Q_3=\theta w_{3x}+c_1w_{3y}+c_2w_{xx}+c_3w_{yy}+c_4w_x+c_5w_y,$$

with

$$c_1=-\frac{\theta_{xy}}{2u^2}+\theta,\quad c_2=\frac{3}{2}\theta(\ln\theta_x)_x-\theta_x,\quad c_4=\left(\frac{3\phi_{xx}}{\phi}+A-u^2\right)\theta-\frac{\theta_{xx}}{2},$$

$$c_3=\frac{u_y\theta_{xy}}{2u^3}+\frac{\phi\phi_{yy}}{u^2}-\frac{3u_y\theta}{u}+3\left(\frac{\theta}{2}(\ln\theta_x)_y-\theta_y\right),$$

$$c_5=-\frac{3u_y^2\theta_{xy}}{2u^4}+\frac{1}{u^3}(\theta_{xy}u_{yy}+u_y\phi\phi_{yy})+\frac{1}{u^2}\left(3\theta u_y^2-\phi\phi_{3y}+\frac{1}{2}\left[B-\frac{\phi_{yy}}{\phi}\right]\theta_{xy}\right)$$
$$+\left(\frac{3\phi_{yy}}{\phi}-B\right)+\frac{u_y}{u}\left(2\theta_y-\frac{3\theta\theta_{xy}}{\theta_x}\right)+\frac{\theta_{xy}}{2}-u^2\theta.$$

The one-form dQ is closed,

$$Q_{1,y}=Q_{2,x},\quad Q_{1,t}=Q_{3,x},\quad Q_{2,t}=Q_{3,y}.$$

It is easy to verify that the $[L,A]$ pair (31) is covariant with respect to the generalized Moutard transformation (32).

Now we use these transformations to construct exact solutions of the 2D-MKdV equations (9a). First we would mention the known localized solutions from Ref. 10. Let us choose $u=\text{const}$, $A=B=0$. We will consider two examples.

(1) If we take the solution of (31) as $\phi=\sinh\xi$, where

$$\xi=ax+\frac{u^2}{a}y+\frac{(u^2-a^2)(u^4-a^4)}{a^3}t, \tag{33}$$

with the real $a=\text{const}$, then using (32) we get new solutions of the 2D-MKdV equations,

$$\tilde{u}=\frac{u[2\eta-a^3\sinh(2\xi)]}{2\eta+a^3\sinh(2\xi)},\quad \tilde{A}=\frac{16a^3\sinh\xi[3a^5\sinh\xi-(u^2-3a^2)\eta\cosh\xi]}{(2\eta+a^3\sinh(2\xi))^2},$$

$$\tilde{B}=\frac{16av^2\cosh\xi[3a^3u^2\cosh\xi-(3u^2-a^2)\eta\sinh\xi]}{(2\eta+a^3\sinh(2\xi))^2},$$

where

$$\eta=a^2(u^2y-a^2x)+(u^2-a^2)(3u^4+3a^4+2a^2u^2)t. \tag{34}$$

(2) To construct the algebraic solutions of (9a) we choose the solutions of (31) as

$$\phi=(-1)^n\int_\alpha^\beta dk\,\zeta(k)\exp(\xi(k))\frac{d^n}{dk^n}\delta(k-k_0),$$

with $\xi(k)$ from (33), $a=a(k)$, $\beta>k_0>\alpha>0$, and $\zeta(k)$ is some arbitrary differentiable function. For $n=1$, $\zeta=1$, we get

$$\tilde{u}=\frac{u(a^6-2\eta^2-2a^3\eta)}{2\eta^2+2a^3\eta+a^6},\quad \tilde{A}=-\frac{8a^6(u^2+3a^2)\eta(\eta+a^3)}{(2\eta^2+2a^3\eta+a^6)^2},$$
$$\tilde{B}=\frac{8u^2a^4(3u^2+a^2)\eta(\eta+a^3)}{(2\eta^2+2a^3\eta+a^6)^2}, \tag{35}$$

with the η from (34) and $a=a(k_0)$. Equation (35) is a simple nonsingular algebraic solution of the 2D-MKdV.

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