Discussiones Mathematicae
Graph Theory 29 (2009) 615–627

RELATIONS BETWEEN THE DOMINATION PARAMETERS AND THE CHROMATIC INDEX OF A GRAPH

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Abstract

In this paper we show upper bounds for the sum and the product of the lower domination parameters and the chromatic index of a graph. We also present some families of graphs for which these upper bounds are achieved. Next, we give a lower bound for the sum of the upper domination parameters and the chromatic index. This lower bound is a function of the number of vertices of a graph and a new graph parameter which is defined here. In this case we also characterize graphs for which a respective equality holds.

Keywords: domination, domination parameters, chromatic index.

2000 Mathematics Subject Classification: 05C69, 05C15.

1. Introduction

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. Then we use the convention $V = V(G)$ and $E = E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ in G is denoted $N(v)$ and defined by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices the *open neighborhood* $N(S)$ is defined as the union of open nighborhoods $N(v)$ of vertices $v \in S$, the *closed nighborhood* is $N[S] = N(S) \cup S$, and $G[S]$ is the subgraph of G induced

by the vertices of S. The *degree* $d_G(v) = d(v)$ of a vertex v is the number of edges incident to v in G; clearly, this is equal to $|N(v)|$. The maximum degree, the minimum degree and the number of vertices of a graph G are denoted by $\Delta(G)$, $\delta(G)$ and $n(G)$, respectively. When there is no confusion we can use the the abbreviations $\Delta(G) = \Delta$, $\delta(G) = \delta$ and $n(G) = n$. A *corona* $H \circ K_1$ is the graph formed from H by adding a new vertex v' for each $v \in V(H)$ and the edge vv'. Two edges $e \neq f$ are *adjacent* if they have a vertex in common. A set of edges is *independent* if no two of its elements are adjacent. An independent set M of edges is called a *matching*. The *edge independence number* $\beta(G)$ is the size of the greatest matching in G. For any vertex $v \in V(G)$ of degree Δ, let $\beta_v(G) = \beta(G - N[v])$ and let $\beta_\Delta(G) = max\{\beta_v(G) : d(v) = \Delta\}$. A path between two vertices x and y is called a $x - y$ *path*. The *distance* $d_G(x, y)$ or $d(x, y)$ in G of two vertices x, y is the length of a shortest $x - y$ path in G. Two vertices u, v of G are *adjacent* if there is an edge $e = uv$ of G. A set of pairwise non-adjacent vertices is said to be *independent*. The *independence number* $\alpha(G)$ is the size of the greatest independent set of vertices in G. A set $S \subseteq V(G)$ is a *dominating* set in G if $N[S] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and the *upper domination number* $\Gamma(G)$ is the maximum cardinality of a minimal dominating set in G. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of a maximal independent set in G. A set $S \subseteq V(G)$ is said to be *irredundant* of G if for any vertex $x \in S$ is $N[x] - N[S - \{x\}] \neq \emptyset$. The *irredundance number* $ir(G)$ is the minimum cardinality taken over all maximal irredundant sets of vertices of G. The *upper irredundance number* of G, denoted by $IR(G)$, is the maximum cardinality of an irredundant set of G. The *lower domination parameters* of a graph G are $ir(G)$, $\gamma(G)$, $i(G)$ and the *upper domination parameters* are $\alpha(G)$, $\Gamma(G)$, $IR(G)$. It is known [4] that for any graph G, $ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)$.

The following results describe graphs for which the lower or upper domination parameters are equal.

Theorem 1.1 [3]. *If G is a graph containing no induced subgraph isomorphic to either $K_{1,3}$ or the $A - L$ graph, then $ir(G) = \gamma(G) = i(G)$.*

Theorem 1.2 [2]. *For any bipartite graph G, $\alpha(G) = \Gamma(G) = IR(G)$.*

A set $S \subseteq V(G)$ is a *packing set* of G if $N[x] \cap N[y] = \emptyset$ for all pairs of distinct vertices $x, y \in S$. The *packing number* $\rho(G)$ is the maximum cardinality of

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a packing set in G. Observe that every packing set is independent.

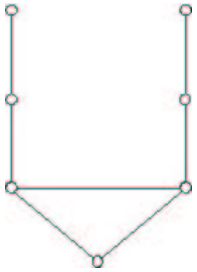

Figure 1. A-L graph.

An *edge colouring* of a graph G is a mapping $c : E(G) \longrightarrow \{1, 2, \ldots, k\}$ such that $c(e) \neq c(f)$ for all pairs of adjacent edges; numbers $1, 2, \ldots, k$ are called *colours*. The *chromatic index* $\chi'(G)$ is the smallest number of colours necessary to an edge colouring of a graph G.

The famous Vizing theorem [8] states values of the chromatic index.

Theorem 1.3. *Every graph G satisfies*

$$\Delta \leq \chi'(G) \leq \Delta + 1. \tag{1}$$

This theorem divides the finite graphs into two classes according to their chromatic index: graphs satisfying $\chi'(G) = \Delta$ are called *class* 1, those with $\chi'(G) = \Delta + 1$ are *class* 2. For instance, bipartite graphs belong to class 1.

Theorem 1.4 (König) [6]**.** *Every bipartite graph G satisfies $\chi'(G) = \Delta$.*

In this paper we give some bounds for the sum or the product of the domination parameters and the chromatic index. Moreover, we describe families of graphs for which these bounds are achieved. We use here integer bounds for any real number x. By $\lfloor x \rfloor$ we denote the greatest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer more than or equal to x. The following facts are obvious.

Fact 1.1. Let k be an integer and x be a real number. Then $k < x$ if and only if $k \leq \lceil x \rceil - 1$.

Fact 1.2. Let x and r be real numbers. If $x - r = \lfloor x \rfloor$ then $0 \leq r < 1$.

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2. The lower domination parameters

In this section are shown some upper bounds for the sum and the product of $\mu(G)$ and $\chi'(G)$, where $\mu = ir, \gamma$ or i. Next, we characterize classes of graphs for which appropriate equalities hold. The similar problems for the chromatic number are considered in [1], where there are the following results among others.

Theorem 2.1. *For any graph G,*

$$\mu(G) \leq n - \Delta - \beta_{\Delta}(G). \tag{2}$$

Theorem 2.2. *For every graph G,*

$$i(G) \leq n - \delta\rho(G). \tag{3}$$

Now, we consider the sum of the lower domination parameters and the chromatic index of graphs. From (1) and (2) the following results arise.

Corollary 2.1. *For any graph G,*

$$\mu(G) + \chi'(G) \leq n + 1 - \beta_{\Delta}(G). \tag{4}$$

Corollary 2.2. *Let G be a class 1 graph. Then*

$$\mu(G) + \chi'(G) \leq n - \beta_{\Delta}(G). \tag{5}$$

Next, we study which graphs fulfil (4) or (5) with equality, for $\mu = i$:

$$i(G) + \chi'(G) = n + 1 - \beta_{\Delta}(G) \tag{6}$$

and

$$i(G) + \chi'(G) = n - \beta_{\Delta}(G). \tag{7}$$

It follows from (1) and (2) that if (6) holds then G is a class 2 graph. It is clear that (6) is satisfied for every K_{2k+1}. However for $G = C_{2k+1}$ from (6) we obtain that $k = 1, 2$ or 3. Thus, we have a solution of (6) for complete graphs and cycles.

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Proposition 2.1. *Let $G = K_n$ or C_n. Then G satisfies* (6) *if and only if $G = K_{2k+1}$, C_5 or C_7.*

The proof of the next statement is similar.

Proposition 2.2. *Let $G = K_n$ or C_n. Then G satisfies* (7) *if and only if $G = K_{2k}$, C_4, C_9, C_{11} or C_{13}.*

Observe that in (6) and (7) for Proposition 2.1 and Proposition 2.2, respectively, we can replace i by μ, where $\mu = ir$, γ, i.

Now, let $p \geq 4$ be an integer and H be a graph with $|V(H)| = p$ and $\Delta(H) \leq p - 3$. Next, we define a graph G as $G = K_p \cup (H \circ K_1)$, where $V(K_p) \cap V(H) = \emptyset$. Note that $\Delta(H \circ K_1) \leq p - 2$ thus $\chi'(H \circ K_1) \leq p - 1$, and $\beta_\Delta(G) = p$; moreover, we can show $i(G) = p + 1$. If p is odd we have $i(G) + \chi'(G) = n + 1 - \beta_\Delta(G) = 2p + 1$ and (6) holds. For even values of p we obtain $i(G) + \chi'(G) = n - \beta_\Delta(G) = 2p$ and (7) is satisfied. For the above family of graphs is $\gamma = i$, so in (6) and (7) we can replace i by μ, where $\mu = \gamma$, i.

We can also find upper bounds for the product of the lower domination parameters and the chromatic index of a graph G. These bounds are functions of n, $\beta_\Delta(G)$ or $\rho(G)$. When there is no confusion we can use the abbreviations $\mu(G) = \mu$, $i(G) = i$, $\chi'(G) = \chi'$, $\beta_\Delta(G) = \beta$ and $\rho(G) = \rho$.

Now, we consider the product $\mu\chi'$ for class 1 graphs.

Theorem 2.3. *Let G be a class* 1 *graph. Then*

$$\mu(G)\chi'(G) \leq \left\lfloor \frac{(n - \beta_\Delta(G))^2}{4} \right\rfloor,$$

where $\mu = ir$, γ, i.

Proof. Let G be a graph for which $\chi' = \Delta$. Now, it remains to repeat the appropriate part of the proof of Theorem 4 from [1]. It is needed for solving some problems of the present paper. From the assumption $\chi' = \Delta$ and Theorem 2.1, by putting $t = (n-\beta)/2-\Delta$, we obtain $\mu\chi' \leq (n-\Delta-\beta)\Delta = ((n-\beta)/2+t)((n-\beta)/2-t) = ((n-\beta)/2)^2 - t^2 \leq ((n-\beta)/2)^2$, and since $(n-\Delta-\beta)\Delta$ is an integer, it follows $\mu\chi' \leq (n-\Delta-\beta)\Delta \leq \lfloor (n-\beta)^2/4 \rfloor$. ■

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Now it is studied the equality $\mu\chi' = \lfloor (n-\beta)^2/4 \rfloor$. It is enough to research this equality for $\mu(G) = i(G)$, i.e.,

$$i(G)\chi'(G) = \left\lfloor \frac{(n-\beta_\Delta(G))^2}{4} \right\rfloor. \tag{8}$$

For class 1 graphs some necessary conditions for (8) are given.

Proposition 2.3. *Let G be a class 1 graph. If (8) holds, then $i(G) = \Delta + j$ and $n - \beta_\Delta(G) = 2\Delta + j$, where $j = -1, 0, 1$.*

Proof. From (8) and $\chi' = \Delta$ it follows that $i = n - \Delta - \beta$ (see the proof of Theorem 2.3). Putting in (8) $n - \beta = i + \Delta$ we obtain the equality $i\Delta = \lfloor (i+\Delta)^2/4 \rfloor$, which is equivalent to $0 \leq (i+\Delta)^2/4 - i\Delta < 1$, i.e., $0 \leq (i-\Delta)^2 < 4$. It implies the equality $i = \Delta + j$, $j = -1, 0, 1$. Hence we have $n - \beta = i + \Delta = 2\Delta + j$. It completes the proof. ■

Now, we present a family of paths for which (8) is satisfied. We use here the above proposition.

Proposition 2.4. *Let $G = P_n$ be the path on n vertices. Then (8) holds if and only if $n = 1, 2, 3, 4, 5, 7$.*

Proof. If (8) holds then from Proposition 2.3 we can deduce $i = \lceil n/3 \rceil = 1, 2, 3$ and hence $n = 1, 2, \ldots, 9$. Among these values only $n = 1, 2, 3, 4, 5, 7$ satisfy the equality (8). Moreover, observe that for any path $ir = \gamma = i$. ■

Theorem 2.3 gives the upper bound for the product $\mu(G)\chi'(G)$, where G is a class 1 graph. Is there a similar bound for any graph G?

Let $t = (n-\beta+1)/2 - (\Delta+1)$. It is clear $\mu\chi' \leq (n-\Delta-\beta)(\Delta+1) = ((n-\beta+1)/2+t)((n-\beta+1)/2-t) = (n-\beta+1)^2/4 - t^2 \leq (n-\beta+1)^2/4$. Thus, we obtain a required bound.

Proposition 2.5. *For any graph G*

$$\mu(G)\chi'(G) \leq \left\lfloor \frac{(n-\beta_\Delta(G)+1)^2}{4} \right\rfloor,$$

where $\mu = ir, \gamma, i$.

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Now, it is studied the equality

$$i(G)\chi'(G) = \left\lfloor \frac{(n - \beta_\Delta(G) + 1)^2}{4} \right\rfloor. \tag{9}$$

For any graph some necessary conditions for (9) are given.

Proposition 2.6. *If* (9) *holds, then* G *is a class* 2 *graph.*

Proof. It is obvious that (9) and the inequalities preceding Proposition 2.5 imply $i(G) = n - \Delta - \beta$ and $\chi' = \Delta + 1$. ■

The next statement we can prove analogously to Proposition 2.3.

Proposition 2.7. *If* (9) *holds, then* $n - \beta_\Delta(G) + 1 = 2\Delta + j$, *where* $j = 1, 2, 3$.

Next, we resolve the equation (9) in the family of odd cycles. If $G = C_{2k+1}$ satisfies (9) then by Proposition 2.7 is $n - \beta + 1 = 2k + 1 - \lfloor (2k-2)/2 \rfloor + 1 = k + 3 = 5, 6$ or 7, therefore $G = C_5, C_7$ or C_9. We can check that only C_5 and C_7 are solutions of (9). It follows from Theorem 1.1 that for any cycle C_n we have $ir(C_n) = \gamma(C_n) = i(C_n)$. Thus, we can formulate the following statement.

Proposition 2.8. *Let* $\mu = ir, \gamma$ *or* i. *Then the equality*

$$\mu(C_n)\chi'(C_n) = \left\lfloor \frac{(n - \beta_\Delta(C_n) + 1)^2}{4} \right\rfloor$$

holds if and only if $n = 5, 7$.

Now assume that G is a regular graph. For this one from (3) we obtain:

Corollary 2.3. *For every regular graph* G,

$$\mu(G) \leq n - \Delta\rho(G), \tag{10}$$

where $\mu = ir, \gamma, i$.

Following [1], we define $t = \rho(\Delta + 1) - (n + \rho)/2$. Combining (10) with the Vizing theorem we can find an upper bound for the product $\mu\chi' : \mu\chi' \leq$

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$(n-\Delta\rho)(\Delta+1) = (((n+\rho)/2)^2 - t^2)/\rho \leq (n+\rho)^2/4\rho$. Therefore we have the following statement.

Proposition 2.9. *For every regular graph G,*

$$\mu(G)\chi'(G) \leq \left\lfloor \frac{(n+\rho(G))^2}{4\rho(G)} \right\rfloor, \tag{11}$$

where $\mu = ir, \gamma, i$.

We show that the bound in (11) is not sharp. To this effect we research the following equality:

$$i(G)\chi'(G) = \left\lfloor \frac{(n+\rho(G))^2}{4\rho(G)} \right\rfloor. \tag{12}$$

We prove that there is no a regular graph G for which (12) holds. For this result we need the following fact.

Proposition 2.10. *Let G be a regular graph. If* (12) *holds, then*

(a) $i(G) = n - \Delta\rho(G)$
and

(b) *G is of class* 2 *and thus* $\Delta \geq 2$.

Proof. The above statement is a consequence of (10) and the remark before Proposition 2.9. ∎

We are now in a position to prove the following result.

Theorem 2.4. *There does not exist any regular graph G such that* (12) *holds.*

Proof. Suppose that there exists a regular graph G for which (12) is satisfied. Hence, by notice before Proposition 2.9 we obtain $(n+\rho)^2/4\rho - t^2/\rho = \lfloor (n+\rho)^2/4\rho \rfloor$. By using Fact 1.2 and putting $t = \rho(\Delta+1) - (n+\rho)/2$ we obtain the following sequence of equivalent inequalities $0 \leq t^2/\rho < 1 \iff -1 < \sqrt{\rho}(\Delta+1) - (n+\rho)/2\sqrt{\rho} < 1 \iff 2\rho\Delta + \rho - n < 2\sqrt{\rho}$ and $n - 2\rho\Delta - \rho < 2\sqrt{\rho}$.

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It follows from Fact 1.1 that $2\rho\Delta + \rho - n \leq \lceil 2\sqrt{\rho} \rceil - 1$ and $n - 2\rho\Delta - \rho \leq \lceil 2\sqrt{\rho} \rceil - 1$, i.e.,

$$2\rho\Delta + \rho - (\lceil 2\sqrt{\rho} \rceil - 1) \leq n \leq 2\rho\Delta + \rho + (\lceil 2\sqrt{\rho} \rceil - 1). \tag{13}$$

Let $A \subseteq V(G)$ be a packing set of the maximum cardinality, i.e., $|A| = \rho(G) = \rho$, and $V(G) = N[A] \cup B$, where $N[A] \cap B = \emptyset$. Notice that for each $v \in B$ there exists an edge $e = vw$ such that $w \in N(A)$. Really, otherwise we would obtain a packing set of cardinality greater than ρ, a contradiction. Let l be the number of all edges $e = vw$ such that $v \in B$ and $w \in N(A)$.

In connection with (13) let us consider the following cases.

Case 1. Let $n = 2\rho\Delta + \rho - j$, where $j = 1, \ldots, \lceil 2\sqrt{\rho} \rceil - 1$. It follows from Proposition 2.11 that $i(G) = n - \rho\Delta = \rho\Delta + \rho - j$. In this case $|B| = n - \rho(\Delta + 1) = \rho\Delta - j$. We show that the set B is nonempty. Suppose that $B = \emptyset$, i.e., for a certain j, $j = 1, \ldots, \lceil 2\sqrt{\rho} \rceil - 1$, the equality $\rho\Delta - j = 0$ is satisfied. Hence $\rho\Delta \leq \lceil 2\sqrt{\rho} \rceil - 1$ and by Fact 1.1 we have $\rho\Delta < 2\sqrt{\rho}$. From here we have $\sqrt{\rho}\Delta < 2$, therefore $\Delta = 0$ or 1, because $\rho \geq 1$. But $\Delta \geq 2$ by Proposition 2.10, thus B is nonempty.

Case 1.1. B is independent. Then

$$l = (\rho\Delta - j)\Delta = \rho\Delta^2 - j\Delta. \tag{14}$$

Denote $G_1 = G - A$ and calculate the sum of degrees in G_1 of vertices $u \in N(A)$. On the one hand $\sum_{u \in N(A)} d_{G_1}(u) = \rho\Delta(\Delta - 1) = \rho\Delta^2 - \rho\Delta$, and on the other hand $\sum_{u \in N(A)} d_{G_1}(u) = l + 2|D|$, where D is the set of edges $e = xy$ such that $x, y \in N(A)$. Therefore, we can deduce $l \leq \rho\Delta^2 - \rho\Delta$. Suppose that $l = \rho\Delta^2 - \rho\Delta$. Then the set $N(A)$ would be independent and G would be a graph of class 1 and exactly a bipartite graph $G = G(V_1, V_2)$ with $V_1 = A \cup B$, $V_2 = N(A)$, a contradiction. Therefore, we have

$$l < \rho\Delta^2 - \rho\Delta. \tag{15}$$

From (14) and (15) we can deduce that there exists an integer p, $p > 0$, such that $\rho\Delta^2 - \rho\Delta - p = \rho\Delta^2 - j\Delta$. Hence we obtain $(j - \rho)\Delta = p$, thus $j - \rho > 0$, and all the more $\lceil 2\sqrt{\rho} \rceil - 1 - \rho > 0 \Longrightarrow \rho + 1 \leq \lceil 2\sqrt{\rho} \rceil - 1$. From here using Fact 1.1 we obtain $\rho + 1 < 2\sqrt{\rho} \Longrightarrow (\sqrt{\rho} - 1)^2 < 0$, a contradiction.

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Case 1.2. B is dependent, i.e. in B there exist vertices, which are adjacent. Let M be an independent dominating set in the induced subgraph $G[B]$. Observe that $|M| \leq |B| - 1 = \rho\Delta - j - 1$ and $A \cup M$ is an independent dominating set in G. Since $i(G)$ is the minimum cardinality of an independent dominating set in G, we have $i(G) = \rho + \rho\Delta - j \leq |A \cup M| = |A| + |M| \leq \rho + \rho\Delta - j - 1$, a contradiction.

Case 2. Let $n = 2\rho\Delta + \rho + j$, where $j = 0, 1, \ldots, \lceil 2\sqrt{\rho} \rceil - 1$. In this case $i(G) = n - \rho\Delta = \rho\Delta + \rho + j$ and $|B| = n - \rho(\Delta + 1) = \rho\Delta + j$. It is clear that $B \neq \emptyset$. Really, it follows from $|B| = \rho\Delta + j = 0$ that $\Delta = 0$, which contradicts $\Delta \geq 2$.

Case 2.1. B is independent. Then $l = (\rho\Delta + j)\Delta = \rho\Delta^2 + j\Delta$ and $l \leq \rho\Delta^2 - \rho\Delta$. If $\Delta > 0$ then on the one hand $l \geq \rho\Delta^2$ and on the other hand $l < \rho\Delta^2$. For $\Delta = 0$ we obtain a contradiction.

Case 2.2. B is dependent. Let M be an independent dominating set in $G[B]$. We have $i(G) = \rho + \rho\Delta + j \leq |A \cup M| = |A| + |M| \leq \rho + \rho\Delta + j - 1$, a contradiction. This completes the proof of the theorem. ∎

By Theorem 2.4 we have improved the upper bound in (11).

Corollary 2.4. *For every regular graph G,*

$$\mu(G)\chi'(G) \leq \left\lfloor \frac{(n + \rho(G))^2}{4\rho(G)} \right\rfloor - 1,$$

where $\mu = ir, \gamma, i$.

It is easy to see that the above bound is sharp. Really, for $G = C_7$ we have $\mu(G)\chi'(G) = \lfloor (n + \rho(G))^2/4\rho(G) \rfloor - 1 = 9$.

3. The upper domination parameters

A set D is called a *vertex covering set* of G if every edge of G has at least one end vertex in D. The *vertex covering number* $\alpha_0(G)$ is the smallest size of a vertex covering set of G. The well-known result of Gallai [4] establishes the relationship between the independence number and the vertex covering number.

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Theorem 3.1. *For any graph G,*

$$\alpha(G) + \alpha_0(G) = n. \tag{16}$$

The edge independence number and the vertex covering number are dual parameters of a graph. The classical König theorem [7] concerns the above parameters for bipartite graphs.

Theorem 3.2. *For any bipartite graph G,*

$$\alpha_0(G) = \beta(G). \tag{17}$$

In this section we give a lower bound for the sum of the upper domination parameters and the chromatic index. To this effect we define a new graph parameter which is expressed in terms: the vertex covering number and the maximum degree. For any vertex $v \in V(G)$ of degree Δ, we define: $\alpha_0^v(G) = \alpha_0(G - N[v])$ and $\alpha_0^{\Delta}(G) = min\{\alpha_0^v(G) : d(v) = \Delta\}$. The new parameter $\alpha_0^{\Delta}(G)$ and $\beta_{\Delta}(G)$ are dual.

In this section we show a lower bound for the sum of $\mu(G)$ and $\chi'(G)$, where $\mu = \alpha, \Gamma, IR$; this bound is a function of n and $\alpha_0^{\Delta}(G)$.

At first we bound $\mu(G)$ by means of an independent set of vertices.

Theorem 3.3. *Let $I \subseteq V(G)$ be an independent set of a graph G and let $R = G - N[I]$. Then for $\mu = \alpha, \Gamma, IR$ we have*

$$\mu(G) \geq n - |N(I)| - \alpha_0(R). \tag{18}$$

Proof. By (16) we have $\mu(G) \geq \alpha(G) \geq |I| + \alpha(R) = |I| + n(R) - \alpha_0(R) = |I| + n(G) - |N[I]| - \alpha_0(R) = n(G) - |N(I)| - \alpha_0(R)$. ∎

Now we put $I = \{v\}$, where v is a vertex of degree Δ such that $\alpha_0^v(G) = \alpha_0^{\Delta}(G)$. Then from (18) we obtain:

Corollary 3.1. *For any graph G,*

$$\mu(G) \geq n - \Delta - \alpha_0^{\Delta}(G), \tag{19}$$

where $\mu = \alpha, \Gamma, IR$.

The following result is an immediate consequence of (1) and (19).

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Corollary 3.2. *For any graph G,*

$$\mu(G) + \chi'(G) \geq n - \alpha_0^{\Delta}(G), \tag{20}$$

where $\mu = \alpha, \Gamma, IR$.

Now it is studied the equality in (20):

$$\mu(G) + \chi'(G) = n - \alpha_0^{\Delta}(G). \tag{21}$$

It is easy to check that for $G = C_n, K_n$ is $\alpha(G) = \Gamma(G) = IR(G)$ and (21) holds for even values of n. Observe that if G satisfies (21) for $\mu = \alpha$ then G is a graph of class 1. In this paper we research the equality (21) for bipartite graphs and in particular for trees. For bipartite graphs we have $\mu = \alpha = \Gamma = IR$ by Theorem 1.2.

Let G be a connected bipartite graph, and pick a vertex $v \in V(G)$. For any $w \in V(G)$ the distance $d(v, w)$ is odd or even. This defines the unique bipartition of $V(G) : V(G) = A \cup B$. Therefore, we can denote $G = G(A, B)$, where $|A| \leq |B|$.

The following result describes a family of bipartite graphs for which (21) holds.

Theorem 3.4. *Let $G = G(A, B)$ be a connected bipartite graph such that $|A| \leq |B|$ and $\alpha(G) = |B|$. If there exists a vertex $v \in B$ for which $d(v) = \Delta$, then* (21) *holds.*

Proof. Denote $|A| = p$ and $|B| = q$. From the assumption and Theorem 1.2 we deduce that $\mu(G) = \alpha(G) = q$ and by (20) we obtain $q + \Delta \geq p + q - \alpha_0^{\Delta}(G)$ and from here

$$\alpha_0^{\Delta}(G) \geq p - \Delta. \tag{22}$$

Let $v \in B$ be a vertex of maximum degree Δ. From (17) we have $\alpha_0^{\Delta}(G) \leq \alpha_0^{v}(G) = \beta_v(G) \leq p - \Delta$ and according to (22) we conclude that $\alpha_0^{\Delta}(G) = p - \Delta$ which leads to the equality (21). ∎

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Received 19 September 2008
Revised 23 October 2008
Accepted 13 November 2008