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Smoothed transition curve for railways

Abstract: The work draws attention to the existing situation in the area of transition curves used in the geometric layouts of the railway track. Difficulties in the practical implementation and maintenance of very small horizontal ordinates of the transition curve and the ordinates of the gradient due to cant in the initial section, appearing on smooth transition curves, were indicated. The main reason for this situation was the excessive smoothing of the curvature in their initial section. Employing the method of curvature identification by differential equations, a new form of the curve was obtained, which was referred to as the "smoothed transition curve". A definite advantage of this curve was shown, from the implementation point of view, over representing the smooth transition curves of the Bloss curve. . It seems that it could successfully compete with the commonly used clothoid, to which it is similar in the initial section, while it differs significantly along its further length, especially in the final section, where it provides a gentle entry from the transition curve into a circular arc.

Keywords: Railway layout; Transition curve; Ordinates analysis

Introduction

As it is commonly known, the use of transition curves is intended to ensure a continuous change of unbalanced lateral acceleration between the sections of the route with different curvature, in a manner favorable for the dynamics of interactions in the road - vehicle system. The issue of transition curves in road and railways is still valid. New curves are being searched for [2-4, 6, 7-9, 15, 20-25]. It should be noted that the transition curves are also defined in various ways. For circular roads, they are often determined by specifying the angle function $\Theta(l)$ by which the direction of the longitudinal axis of the vehicle changes after passing a certain arc. In railways, traditional simplification dominates, involving the use of the $k(x)$ curvature in the rectangular coordinate system.The acceleration values along the transition curve result from the curvature distribution and it is the curvature that should be the basis for identifying the transition curves. In general, it can be linear or nonlinear. For non-linear changes in curvature, it appears to be adequate, e.g. by R. J. Grabowski [5] the term "smooth transition curves", corresponding to the key meaning of the function class describing curvature. Of course, the curvature distribution should be formed for the variable l, determining the location of a given point along the length of the curve. Most transition curves are connected by a common algorithm for determining curvature using differential equations [10-12, 19].

The main disadvantage of the transition curve in the form of clothoid (with a linear curvature) are bends on the graph of its curvature, occurring in the initial and final regions. They are the cause of adverse dynamic interactions in the track - rail vehicle system. Therefore, the use of transition curves with non-linear curvature distribution over the length has been promoted on the railway for many years.

Practical conditions for using smooth transition curves

The theoretical analyses and experimental tests carried out (among others [11, 13, 16-17]) clearly indicate smaller (and therefore more favorable) dynamic interactions during train travel along smooth transition curves. However, despite its indisputable advantages, the scope of application of these curves in operated railway tracks is significantly limited. It seems that the basic reason for the skepticism prevailing in this matter is very low values of horizontal ordinates (and ordinates of the cant ramp) in the area of the initial curves. This often prevents their correct demarcation in the field and in practice leads to a shortening of the transition curve (i.e. elongation of the adjacent straight line) in relation to the design assumptions. The presented conditions cause that still, the most common form of the transition curve is clothoid, with a linear course of curvature (and possibly a rectilinear cant ramp). The use of the third-degree parabola, which is a simplification of the clothoid, with current computational possibilities is no longer justified.

The paper [12] presents an analysis of known forms of transition curves: clothoid, fourth-degree parabola, Bloss curve, cosine, and sinusoid. The curvature $k(l)$ was identified for individual curves by means of differential equations and the parametric equations $x(l)$ and $y(l)$ were determined. Approximate solutions [1], appearing in numerous regulations to this day, have also been presented. In [12] it was also shown that exceeding the value of kinematic parameters - acceleration increase ψ and rolling stock wheel lifting speed on the cant ramp f - occurring on the cloister includes at least half of the length of each smooth transition curve, and the value of this exceeding is significant. Therefore, it seems appropriate to maintain the same ψ_{dop} and f_{dop} limit values for all types of curves, which leads to the need to lengthen individual smooth transition curves relative to clothoid (by introducing the appropriate A factor). It then becomes possible to compare the horizontal ordinates and the cant ramps with each other. This study presents an attempt to find a new form of the transition curve adapted to the conditions presented above. His extended and significantly modified version is work [14], accepted for publication in the Journal of Surveying Engineering ASCE

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Determination of the new form of the transition curve

From the general method of curvature identification $k(l)$ on transition curves [10], it follows that for the radius R of the circular arc and the length l_k of the transition curve, the adopted assumptions determine the following boundary conditions:

$$\begin{cases} k(0) = 0 & k\left(l_k\right) = \dfrac{1}{R} \\ k'(0) = \dfrac{C}{Rl_k} & k'\left(l_k\right) = 0 \end{cases} \tag{1}$$

and differential equation

$$k^{(4)}(l) = 0 \tag{2}$$

with a numerical coefficient $C \geq 0$.

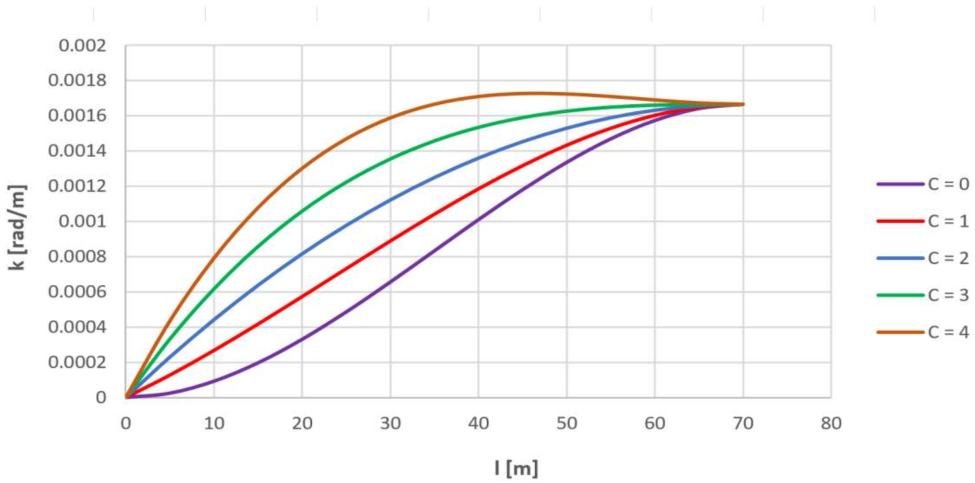
As a result of solving the differential problem (1), (2) we obtain

$$k(l) = \frac{C}{Rl_k} l - \frac{2C-3}{Rl_k^2} l^2 + \frac{C-2}{Rl_k^3} l^3 \tag{3}$$

and the function of the tangent angle *Θ*(*l*) is described by the relation

$$\Theta(l) = \frac{C}{2Rl_k} l^2 - \frac{2C-3}{3Rl_k^2} l^3 + \frac{C-2}{4Rl_k^3} l^4 \tag{4}$$

At the end of the transition curve $\Theta(l_k) = \frac{6+C}{12R} l_k$.

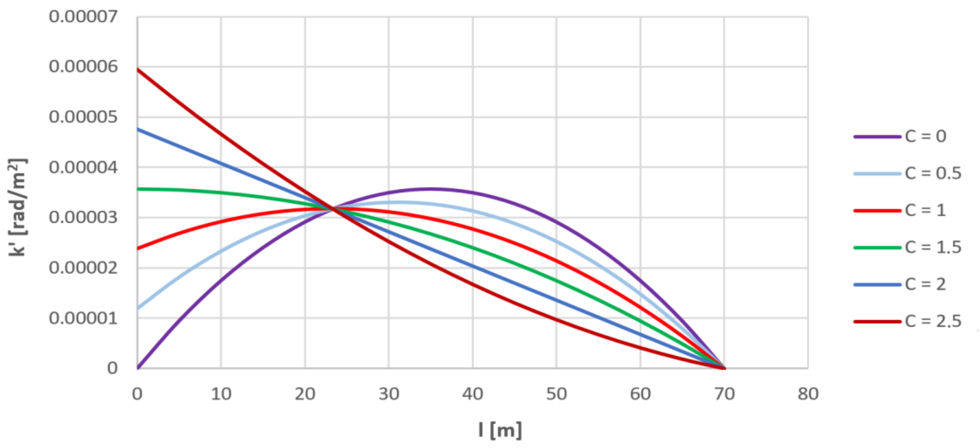


1. Sample curves plots along the length of the new transition curve for selected values of the C coefficient (R = 600 m, l_k = 70 m)

Figure **1** shows examples of length curves for selected values of the C coefficient. As you can see, the curves for $C \in \langle 0;3 \rangle$. The curve for C = 0 has the most gentle course, however - as in the case of other curves - meeting the condition of maintaining the permissible acceleration value requires its elongation in relation to the corresponding curve with a linear curvature. When choosing the most favorable of the curves considered, however, the criterion of the smallest required length should be considered first. This length is determined - in addition to the speed of trains - the permissible value of acceleration increase, which is directly related to the derivative of curvature:

$$k'(l) = \frac{C}{Rl_k} - \frac{2(2C-3)}{Rl_k^2} l + \frac{3(C-2)}{Rl_k^3} l^2 \tag{5}$$

Figure **2** presents graphs of the derivative of curvature over the length of the transition curve for which $C \in \langle 0;2,5 \rangle$.



2. Sample graphs of the derivative of curvature over the length of the new transition curve for selected values of the C coefficient (R = 600 m, l_k = 70 m)

The derivative $k'(l)$ described by equation (5) changes in length, so its maximum value becomes reliable here. For the coefficient $C \in \langle 0;1,5 \rangle$, the value of max $k'(l) = k'(l_0)$, where the position l_0 of the point where the maximum of function $k'(l)$occurs, is determined from

$$k''(l) = -\frac{2(2C-3)}{Rl_k^2} + \frac{6(C-2)}{Rl_k^3} l_0 = 0$$

from which it follows that

$$l_0 = \frac{2C-3}{3(C-2)} l_k \tag{6}$$

The l_0 value defined by the equation (6) substituted for equation (5) determines the maximum of the function $k'(l)$.

$$\max k'(l) = \left[C - \frac{(2C-3)^2}{3(C-2)} \right] \frac{1}{Rl_k} \tag{7}$$

For $C > 1,5$ the l_0 value determined by the formula (7) does not meet the conditions of the task ($l_0 > l_k$ is obtained or the determined l0 refers to the minimum of the function). However, since $k'(l)$ takes the highest value here at the starting point, so max $k'(l) = k'(0)$. The degree of necessary elongation of the sought transition curve relative to the corresponding baseline clothoid, related to the need to maintain the permissible acceleration increase value, is determined by the ratio of the max $k'(l)$ value to the derivative of $k'(l)_{\text{lin}}$ present on the linear curvature, which is a constant value described by the formula

$$k'(l)_{lin} = \frac{1}{Rl_k} \tag{8}$$

Table **1** summarizes the ratio values $\max k'(l) / k'(l)_{lin}$ for the selected $C \in \langle 0;4 \rangle$.

Tab. 1. Values of the ratio max k'(l) / k'(l)$_{lin}$ for selected coefficients C

C	0	0.5	1	1.5	2	2.5	3	3.5	4
max $k'(l)$ / $k'(l)_{lin}$	$\frac{3}{2}$	$\frac{25}{18}$	$\frac{4}{3}$	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4

The analysis shows that for C = 0 $\max k'(l) = {}^{3}\!/\!_{2Rl_k}$,therefore, due to the permissible value of the acceleration increase, the length of the transition curve must be 50% larger than for the linear curvature. As it turns out, from the point of view discussed above, the most favorable solution is to adopt the coefficient C = 1. For this case, it applies $\max k'(l) = {}^{4}\!/\!_{3Rl_k}$ which means that the length of the corresponding curve must be greater than the length of the clothoid only by ⅓. Adoption C = 1 leads to the following equations of the functions $k(l)$ and $\Theta(l)$:

$$k(l) = \frac{1}{Rl_k} l + \frac{1}{Rl_k^2} l^2 - \frac{1}{Rl_k^3} l^3 \tag{9}$$

$$\Theta(l) = \frac{1}{2Rl_k} l^2 + \frac{1}{3Rl_k^2} l^3 - \frac{1}{4Rl_k^3} l^4 \tag{10}$$

At the end of the transition curve, the tangent slope angle $\Theta(l_k) = \frac{7}{12R} l_k$.

The coordinate equations of the sought transition curve can be saved in a parametric form [10]:

$$x(l) = \int \cos\Theta(l) dl \tag{11}$$

$$y(l) = \int \sin\Theta(l) dl \tag{12}$$

The Maxima program was used to expand the functions $\cos\Theta(l)$ i $\sin\Theta(l)$ into the Maclaurin series, and then the individual words were integrated to obtain parametric equations:

$$x(l) = l - \frac{1}{40R^2l_k^2} l^5 - \frac{1}{36R^2l_k^3} l^6 + \frac{5}{504R^2l_k^4} l^7 + \frac{1}{96R^2l_k^5} l^8 + \left(\frac{1}{3456R^4l_k^4} - \frac{3}{864R^2l_k^6} \right) l^9 \tag{13}$$

$$y(l) = \frac{1}{6Rl_k} l^3 + \frac{1}{12Rl_k^2} l^4 - \frac{1}{20Rl_k^3} l^5 - \frac{1}{336R^3l_k^3} l^7 - \frac{1}{192R^3l_k^4} l^8 + \frac{1}{2592R^3l_k^5} l^9 \tag{14}$$

The new form of the transition curve we will refer to as "smoothed transition curve".

Comparative analysis of selected transition curves

In order to determine the location of the new form of the transition curve among other solutions, an appropriate comparative analysis was carried out. The transition curves in the form of clothoid (with linear curvature) and in the form of the Bloss curve (with non-linear curvature) were adopted as reference. The curvature of the transition curve in the form of clothoid is described by the equation [12]

$$k(l) = \frac{1}{Rl_k} l \tag{15}$$

Its Cartesian coordinates are in the form of parametric equations:

$$x(l) = l - \frac{1}{40R^2 l_k^2} l^5 + \frac{1}{3456 R^4 l_k^4} l^9 - \frac{1}{599040 R^6 l_k^6} l^{13} \tag{16}$$

$$y(l) = \frac{1}{6Rl_k} l^3 - \frac{1}{336 R^3 l_k^3} l^7 + \frac{1}{42240 R^5 l_k^5} l^{11} \tag{17}$$

The Bloss curve has a curvature described by the equation [12]

$$k(l) = \frac{3}{Rl_k^2} l^2 - \frac{2}{Rl_k^3} l^3 \tag{18}$$

and parametric equations:

$$x(l) = l - \frac{1}{14R^2 l_k^4} l^7 + \frac{1}{16R^2 l_k^5} l^8 - \frac{1}{72R^2 l_k^6} l^9 \tag{19}$$

$$y(l) = \frac{1}{4Rl_k^2} l^4 - \frac{1}{10Rl_k^3} l^5 - \frac{1}{60R^3 l_k^6} l^{10} + \frac{1}{44R^3 l_k^7} l^{11} \tag{20}$$

To perform the comparative analysis, it becomes necessary to adopt the values of geometrical parameters, i.e. the radius of the circular arc and the length of the transition curve, which is determined for the assumed speed of trains. They are limited by permissible values of appropriate kinematic parameters. The minimum radius of the circular arc R[m] is calculated by the formula

$$R_{\min} = \frac{V^2}{3.6^2 \left(g \frac{h_0}{s} + a_{dop} \right)} \tag{21}$$

where: V – train speed in *km/h*,
a_{dop} – allowable value of unbalanced acceleration in m/s^2,
h_0 – cant value on the curve in *mm*,
g – gravitational acceleration (g = 9,81 m/s^2),
s – track spacing (s = 1500 mm).

The length of the transition curve is determined by the conditions for maintaining the permissible values of the acceleration change speed ψ and the rolling wheel lifting speed on the cant ramp f. In the general case (i.e. except for clothoid) these parameters are variable in length, therefore their maximum values should be taken as the definitive factor. If the relationship between the elevation ramp $h(l)$ and the curvature is taken into account $k(l)$

$$h(l) = h_0 R k(l) \tag{22}$$

the following two conditions are obtained:

$$\psi_{\max} = \frac{V}{3.6}\left[\left(\frac{V}{3.6}\right)^2 - \frac{g}{s}h_0 R\right]\max k'(l) \le \psi_{dop}$$

$$f_{\max} = \frac{V}{3.6}h_0 R \max k'l) \le f_{dop}$$

As it turns out, the max $k'(l)$ value for the transition curves under consideration can be determined from the universal formula

$$\max k'(l) = \frac{A}{Rl_k}$$

with the values of factor A equal to:

- for clothoids $A = 1$,
- for the Bloss curve $A = 3/2$,
- for a new transition curve $A = 4/3$ (Table **1**).

The assumptions of the innovative rail track location system based on mobile measurements based on satellite location discussed in the article indicate the complexity of the problem being solved ψ and f:

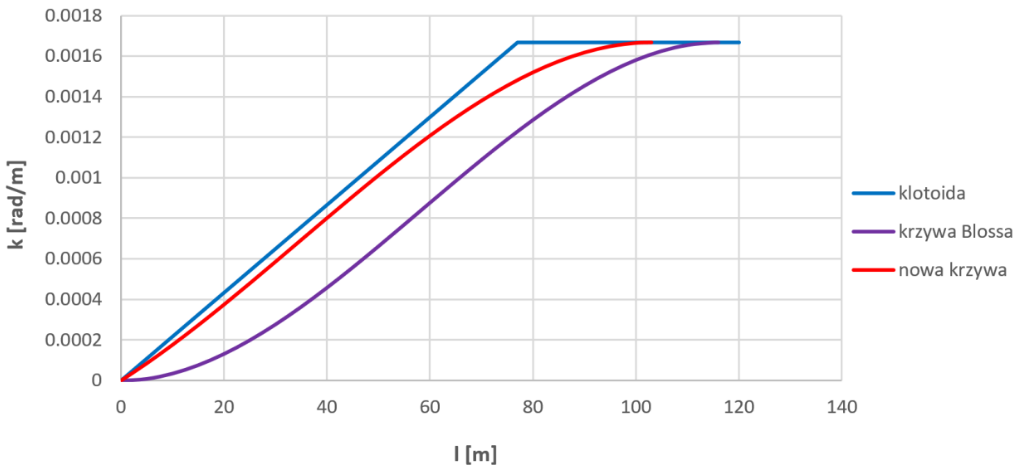
$$\min l_k^{\psi} = \frac{V}{3.6}\left[\left(\frac{V}{3.6}\right)^2 \frac{1}{R} - \frac{g}{s}h_0\right]\frac{A}{\psi_{dop}} \tag{23}$$

$$\min l_k^{f} = \frac{V}{3.6}h_0 \frac{A}{f_{dop}} \tag{24}$$

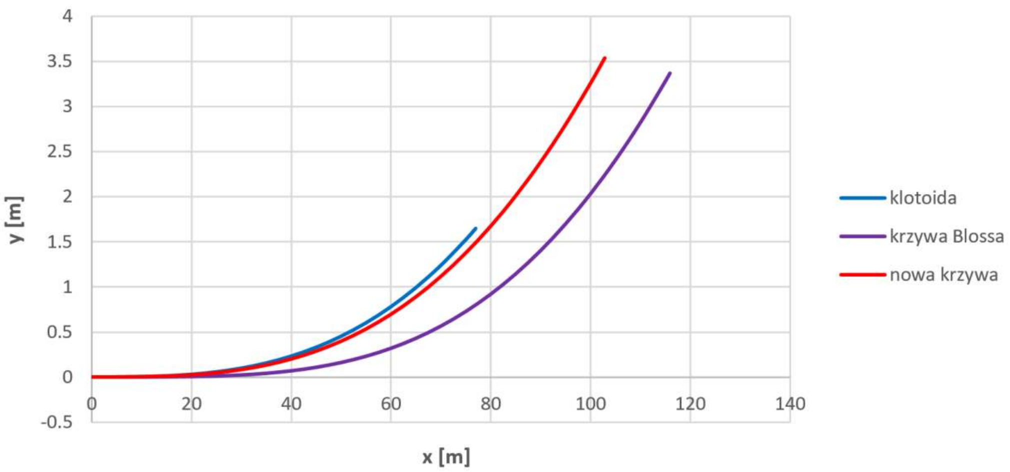
We assume conducting analysis for train speed V = 100 km/h. Assuming the cant value on a circular arc h_0 = 70 mm and the allowable value of unbalanced acceleration a_{dop} = 0,85 m/s^2, based on condition (21) we get the value R_{min} = 590,002 m. In the calculations that follow, we take the radius of the circular arc R = 600 m. Assuming the permissible value of the acceleration change speed ψ_{dop} = 0,3 m/s^3 (as for single transition curves with linear curvature), based on conditions (23) and (24), we determine the minimum lengths of the respective transition curve and finally we assume:

- for clothoids l_k = 77 m,
- for the Bloss curve l_k = 116 m,
- for a new transition curve l_k = 103 m.

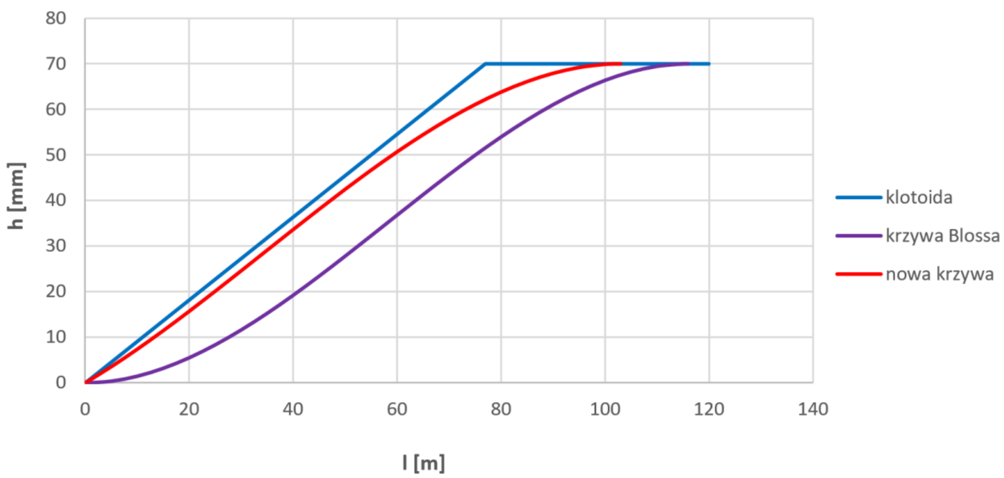
Figure **3** presents curves plots over the length of the compared transition curves, prepared on the basis of formulas (9), (15) and (18). Figure **4** shows the appropriate horizontal ordinates graphs based on formulas (13), (14), (16), (17), (19) and (20). In turn, Figure **5** contains the charts of the ordinates of the cant ramp determined by the formula (22).



3. Curvature charts along the length of the compared transition curves (R = 600 m)



4. Charts of horizontal ordinates along the length of the compared transition curves (on a scale)



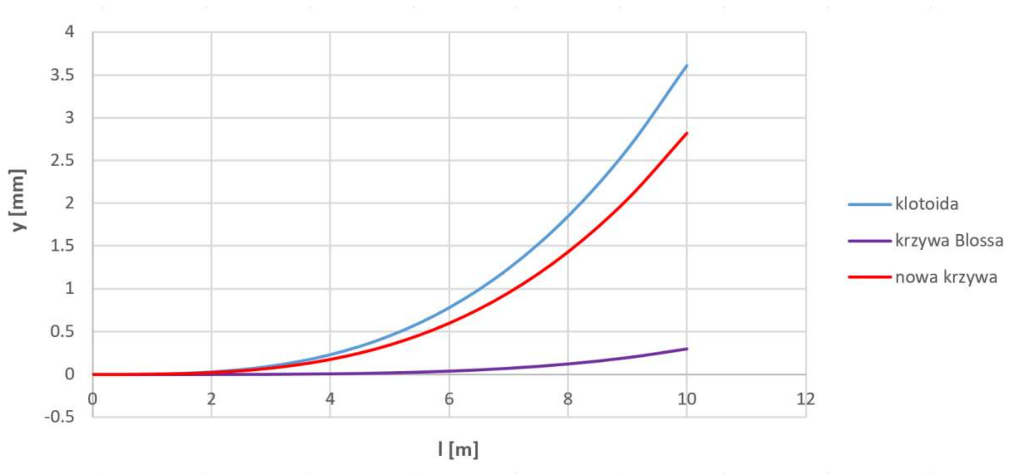
5. Charts of ordinates of the cant ramp over the length of the compared transition curves (h_0 = 70 mm)

As it results from Figure **3**, the curvature graph for the new transition curve determined significantly differs from the curvature graph for the Bloss curve. This applies especially to the initial zone, in which this difference is shaped and makes the discussed curves so much different. On the other hand, the course of the curvature graph for the new curve is similar (though slightly more favorable) over a considerable length to the curvature graph for clothoid. Only in the end region, when switching from a transition curve to a circular arc, does a significant difference become apparent: the curvature chart for the new curve is much softer there.

The comments also fully apply to the elevation ramp elevation charts in Figure **5**. It should be noted that the calculated transition curve does not provide a smooth transition from a straight to a transition curve, so it cannot be classified as a classic smooth transition curve. However, because it is shorter than these curves and was created to meet the implementation requirements, there is no doubt its greater practical usefulness. This can be confirmed by a detailed analysis of the horizontal elevations and the elevations of the cant ramp in the initial region.

Analysis of horizontal ordinates and ordinates of the cant ramp in the initial region

Let's look at how the horizontal ordinates of the compared transition curves and the elevations of the cant ramps are shaped over the first 10 meters. Figure **6** shows the graphs of horizontal ordinates in the initial region. The appropriate numerical values are given in Table **2**.

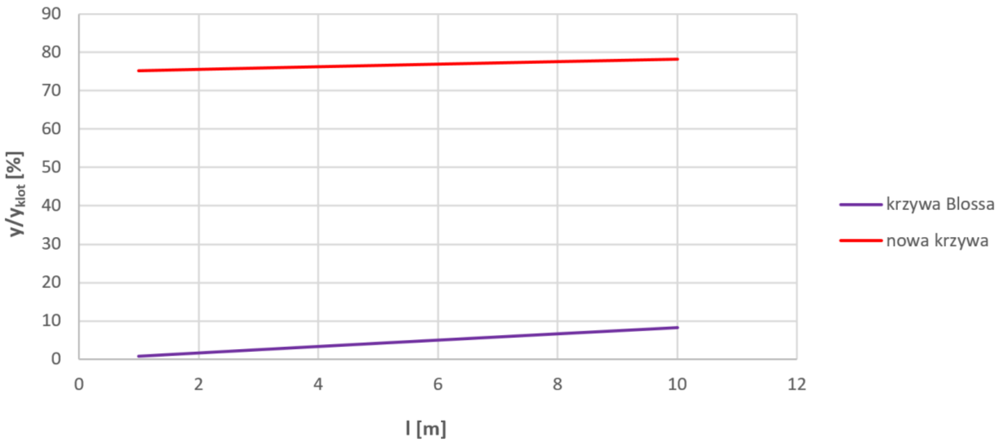


6. Formation of horizontal ordinates of considered transition curves in the initial region (on a contaminated scale)

Tab. 2. Selected values of horizontal ordinates y [mm] in the initial region

Transition curve	*l* = 1 m	*l* = 2 m	*l* = 3 m	*l* = 4 m	*l* = 5 m	*l* = 10 m
Clothoid	0,00361	0,0289	0,0974	0,231	0,451	3,608
Bloss curve	0,00003	0,0005	0,0025	0,008	0,019	0,299
Smoothed transition curve	0,00271	0,0218	0,0739	0,176	0,345	2,820

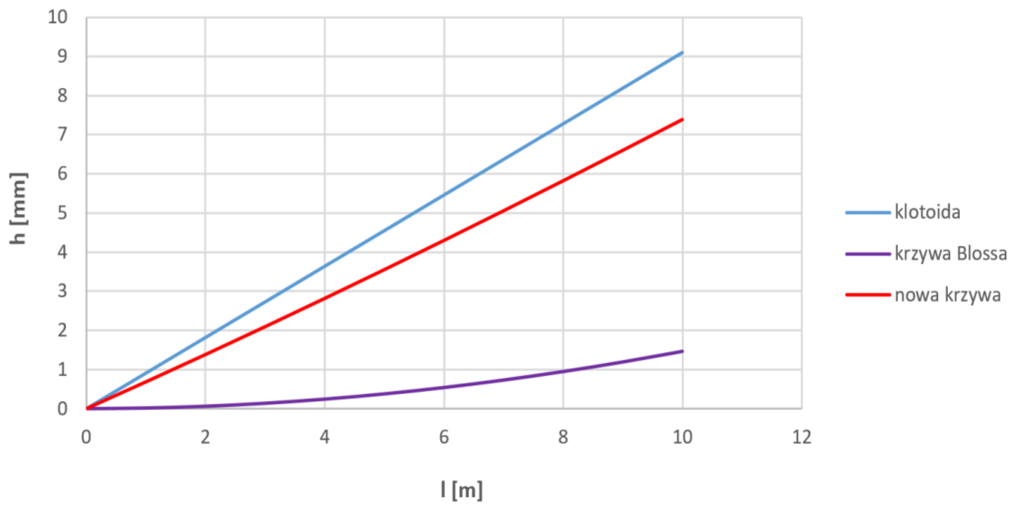
In order to better illustrate the difference between the smoothed transition curve and the Bloss curve, the percentage values of the ratio of the ordinates of the said curves to the ordinates of the clothoid were determined. They are shown in Figure **7**.



7. Percentage values of the ratio of the horizontal ordinates of the Bloss curve and the smoothed transition curve to the ordinates of clothoid in the initial region

As can be seen, the ordinates of the Bloss curve in the initial region are from 1 to 8% of the ordinates of clothoid, while for the new transition curve the appropriate ratio is from 75 to 78%. There is, therefore, no doubt that in the case of the initial region of the Bloss curve, the practical implementation of such small ordinates is absolutely impossible. On the other hand, the implementation possibilities in this zone of the new transition curve do not differ much from the analogous possibilities in relation to clothoid.

A similar analysis should still be carried out for the ordinates of the cant ramp. Figure **8** shows the ordinates of the ramp in the starting area. The appropriate numerical values are given in Table **3**.



8. The shaping of the ordinates of the cant ramp in the initial region on the considered transition curves

Tab. 3. Selected values of the ordinates of the cant ramp *h* [mm] in the initial region

Transition curve	*l* = 1 m	*l* = 2 m	*l* = 3 m	*l* = 4 m	*l* = 5 m	*l* = 10 m
Clothoid	0,909	1,818	2,727	3,636	4,545	9,091
Bloss curve	0,016	0,062	0,138	0,244	0,379	1,471
Smoothed transition curve	0,686	1,385	2,096	2,820	3,555	7,392

The graphs in Figure **8** and the cant ramp ordinate values in Table **3** fully confirm the conclusions of the horizontal ordinate analysis. Practical implementation and subsequent maintenance of very low ordinate values of the cant ramp in the region of the initial Bloss curve do not seem possible. However, as in the case of horizontal ordinates, the implementation possibilities in the initial zone of the cant ramp on the new transition curve are analogous to those for the straight ramp occurring on the clothoid.

Comments formulated for the Bloss curve regarding the feasibility of practical implementation and maintenance of the ordinates of the transition curve and the ordinates of the cant ramp in the initial region refer to the whole family of smooth transition curves. They greatly undermine the desirability of using these curves on railway lines and at the same time explain the skepticism that surrounds these curves in some environments of executive practice.

Conclusions

Smooth transition curves, i.e. curves with a non-linear distribution of curvature over length, have been known for a long time and have a number of indisputable advantages - above all, they are characterized by lower values of dynamic interactions than is the case with clotoids (with a linear distribution of curvature). The scope of their use on railways is, however, so far very limited. These curves, unfortunately, have one major disadvantage - very low values of horizontal ordinates (and ordinates of the cant ramp) in the initial region, in practice often impossible to implement and then maintain. The main reason for the difficulties in the known forms of smooth transition curves is excessive curvature relief in their initial region. It is therefore necessary to find a new form of the transition curve, giving up the condition of zeroing the derivative of curvature at the starting point. Using the method of curvature identification by means of differential equations [12], this form of the curve was obtained, which was called the "smoothed transition curve". The new transition curve determined is characterized by a smooth curvature in the area of entry into the circular arc and its certain disturbance in the initial region (smaller, however, than in the case of clothoids). It showed a clear advantage over the Bloss curve representing smooth transition curves. Practical implementation (and then maintenance) of the initial region of this curve - due to the very low values of horizontal elevations and levels of the cant ramp - turns out to be completely ineffective. On the other hand, the implementation possibilities in this zone of the new transition curve do not differ much from the analogous possibilities in relation to clothoid.

So, it seems, the presented new curve could successfully compete with the commonly used clothoid, to which it is similar in the initial region, while it differs significantly in the longer length, and especially in the final region, where it provides a smooth entry from the transition curve in a circular arc. This is important because other possibilities to improve the existing situation are significantly limited. The demonstrated lack of practical implementation and maintenance of the ordinates of the transition curve and the ordinates of the cant ramp in the initial region largely undermines the desirability of using classic smooth transition curves on railway lines.

It should be noted that among other known forms of transition curves, there may also be solutions (or variants of solutions) that meet the implementation and maintenance requirements of railways, but their respective properties related to the values of horizontal ordinates in the initial region (and ordinates of the cant ramp) have not yet been exposed. As it results from general insight, one of the variants of the so-called polynomial transition curves [7] and two-parameter clothoids [9].

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