Stark-induced magnetic anapole moment in the ground state of the relativistic hydrogenlike atom: Application of the Sturmian expansion of the generalized Dirac-Coulomb Green function

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The Sturmian expansion of the generalized Dirac-Coulomb Green function [R. Szmytkowski, J. Phys. B **30**, 825 (1997); **30**, 2747(E) (1997)] is used to derive an analytical formula for the static magnetic anapole (toroidal dipole) moment induced in the ground state of the relativistic hydrogenlike atom by a weak, spatially uniform, static electric field. An expression for the anapole polarizability for the system in question is found. This expression contains a single generalized hypergeometric series ${}_3F_2$ with the unit argument. In the nonrelativistic limit our result agrees with that of Lewis and Blinder [Phys. Rev. A **52**, 4439 (1995)].

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I. INTRODUCTION

Over the past three decades, in atomic and molecular physics an interest arose in effects associated with magnetic *anapole moments*. These effects may be divided into two principal categories. The first category comprises those atomic and molecular phenomena in which a *nuclear* magnetic anapole moment, resulting from parity nonconservation in nuclear forces, manifests itself (cf., e.g., Refs. 1–6). The second category includes those effects which evince the magnetic anapole moment of an *atomic* or a *molecular electronic cloud* [7–22].

In this paper, we shall be concerned with an effect falling into the second of the aforementioned categories. Specifically, we shall be interested in the anapole moment induced in the ground state of the one-electron Dirac atom by a weak, spatially uniform, static electric field. This problem was already considered a decade ago, both nonrelativistically and relativistically, by Lewis and Blinder [16]; the reason for which we have decided to revisit it is that the relativistic calculations carried out in Ref. 16 were approximate in character. Exploiting the Sturmian expansion of the generalized Dirac-Coulomb Green function [23–27], in the present paper we shall calculate the Stark-induced anapole moment exactly.

The structure of the paper is as follows. In Sec. II we define the anapole moment. In Sec. III we consider the one-electron Dirac atom in the weak, spatially uniform, static electric field and apply the first-order Rayleigh-Schrödinger perturbation theory, together with the Green functions technique, to derive an approximate expression for a perturbed electronic wave function. Next, in Sec. IV, this approximate wave function is used to determine an induced electric current density in the atom. Subsequently, the anapole moment of the resulting (i.e., unperturbed plus induced) current distribution is considered. It is found that only the induced current contributes and the form of this contribution is calculated analytically exploiting the Sturmian functions technique. The paper ends with conclusions, constituting Sec. V, and with several appendixes containing supplementary material.

II. DEFINITION OF THE MAGNETIC ANAPOLE MOMENT

Consider a bounded sourceless system of stationary currents characterized by the current density $\mathbf{j}(\mathbf{r})$. The magnetic anapole moment of such a system is defined as [28]

$$\mathbf{t} = \frac{1}{2}\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}. \tag{2.1}$$

The definition (2.1) links the anapole moment with the family of so-called magnetic toroidal (toroid) multipole moments with spherical components [28]

$$T_{lm} = \frac{1}{l+1}\sqrt{\frac{4\pi}{2l+1}}\int_{\mathbb{R}^3} d^3\mathbf{r}\, r^l Y_{lm}(\mathbf{n}_r)\mathbf{r}\cdot\mathbf{j}(\mathbf{r}), \tag{2.2}$$

where $\{Y_{lm}(\mathbf{n}_r)\}$, with $\mathbf{n}_r=\mathbf{r}/r$, are the spherical harmonics (B3). Indeed, since it follows from Eqs. (2.1) and (2.2) that

$$\mathbf{t} = \sum_{m=-1}^{+1} T_{1m}\mathbf{e}_m^*, \tag{2.3}$$

where $\{\mathbf{e}_m\}$ are the spherical versors [29], the magnetic anapole moment appears to be the magnetic toroidal *dipole* moment (in Ref. 17, the alternative name "the displacement current dipole moment" was proposed for $\mathbf{t}$).

Although the definition (2.1) is the most proper one from the point of view of methodology, it is not necessarily the most convenient one from the practical point of view. For this reason, several other *equivalent* expressions for the vector $\mathbf{t}$ appear in the relevant literature (cf. Appendix A). Among them, for the purposes of the present work the most suitable one appears to be [30]

$$\mathbf{t} = -\frac{1}{4}\int_{\mathbb{R}^3} d^3\mathbf{r}\, r^2\mathbf{j}(\mathbf{r}). \tag{2.4}$$

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III. A RELATIVISTIC HYDROGENLIKE ATOM IN A WEAK, SPATIALLY UNIFORM, STATIC ELECTRIC FIELD

Consider a relativistic hydrogenlike atom, with an infinitely heavy, pointlike, and spinless nucleus of charge $+Ze$, placed in a weak, spatially uniform, static electric field $\mathbf{F}$. The energy quasieigenvalue problem for quasibound states of this system is constituted by the Dirac equation

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_ec^2\beta-\frac{Ze^2}{(4\pi\epsilon_0)r}+e\mathbf{F}\cdot\mathbf{r}-E\right]\psi(\mathbf{r})=0, \tag{3.1a}$$

supplemented by the boundary conditions

$$r\psi(\mathbf{r})\overset{r\to0}{\sim}0,\quad r\psi(\mathbf{r})\overset{r\to\infty}{\sim}0. \tag{3.1b}$$

In Eq. (3.1a), and hereafter, $\boldsymbol{\alpha}$ and β are the standard Dirac matrices [31]. Because of our assumption that the electric field is weak, the electron-field interaction potential

$$V^{(1)}(\mathbf{r})=e\mathbf{F}\cdot\mathbf{r} \tag{3.2}$$

may be considered as a small perturbation of the Dirac-Coulomb Hamiltonian describing an isolated atom. In virtue of this, one may attack the problem (3.1a) and (3.1b) with the aid of the Rayleigh-Schrödinger perturbation theory, with the zeroth-order bound state eigenproblem constituted by the Dirac-Coulomb equation

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_ec^2\beta-\frac{Ze^2}{(4\pi\epsilon_0)r}-E^{(0)}\right]\psi^{(0)}(\mathbf{r})=0, \tag{3.3a}$$

supplemented by the boundary conditions

$$r\psi^{(0)}(\mathbf{r})\overset{r\to0}{\sim}0,\quad r\psi^{(0)}(\mathbf{r})\overset{r\to\infty}{\sim}0. \tag{3.3b}$$

Throughout the rest of the work, we shall restrict ourselves to the case when the unperturbed state of the atom is the ground state. Then one has

$$E^{(0)}=m_ec^2\gamma_1, \tag{3.4}$$

where

$$\gamma_\kappa=\sqrt{\kappa^2-(\alpha Z)^2}, \tag{3.5}$$

with $\alpha=e^2/(4\pi\epsilon_0)c\hbar$ (not to be confused with the Dirac matrix $\boldsymbol{\alpha}$) being the Sommerfeld fine-structure constant. The eigenvalue $E^{(0)}$ is doubly degenerate. Two orthonormal eigenfunctions to the problem (3.3a) and (3.3b) associated with $E^{(0)}$ may be chosen to be

$$\psi_\mu^{(0)}(\mathbf{r})=\frac{1}{r}\begin{pmatrix}P^{(0)}(r)\Omega_{-1\mu}(\mathbf{n}_r)\\ iQ^{(0)}(r)\Omega_{+1\mu}(\mathbf{n}_r)\end{pmatrix}, \tag{3.6}$$

with $\mu=\pm1/2$. Here $\{\Omega_{\kappa\mu}(\mathbf{n}_r)\}$ are the spherical spinors (see Appendix B), while the radial functions are

$$P^{(0)}(r)=-\sqrt{\frac{Z}{a_0}\frac{1+\gamma_1}{\Gamma(2\gamma_1+1)}}\left(\frac{2Zr}{a_0}\right)^{\gamma_1}\exp(-Zr/a_0), \tag{3.7a}$$

$$Q^{(0)}(r)=\sqrt{\frac{Z}{a_0}\frac{1-\gamma_1}{\Gamma(2\gamma_1+1)}}\left(\frac{2Zr}{a_0}\right)^{\gamma_1}\exp(-Zr/a_0), \tag{3.7b}$$

with $a_0=(4\pi\epsilon_0)\hbar^2/m_ee^2$ denoting the Bohr radius.

We shall seek approximate solutions to the quasieigenproblem (3.1a) and (3.1b) in the form

$$\psi(\mathbf{r})\simeq\psi^{(0)}(\mathbf{r})+\psi^{(1)}(\mathbf{r}), \tag{3.8a}$$

$$E\simeq E^{(0)}+E^{(1)}, \tag{3.8b}$$

where

$$\psi^{(0)}(\mathbf{r})=a_{1/2}\psi_{1/2}^{(0)}(\mathbf{r})+a_{-1/2}\psi_{-1/2}^{(0)}(\mathbf{r}), \tag{3.9}$$

with the coefficients $a_{\pm1/2}$ constrained to obey

$$|a_{1/2}|^2+|a_{-1/2}|^2=1. \tag{3.10}$$

These coefficients should be determined so that the function (3.9) is perturbation-adapted. The corrections $\psi^{(1)}(\mathbf{r})$ and $E^{(1)}$, which, by assumption, are small quantities of the first order in $F=|\mathbf{F}|$, are solutions to the inhomogeneous problem

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_ec^2\beta-\frac{Ze^2}{(4\pi\epsilon_0)r}-E^{(0)}\right]\psi^{(1)}(r)=-[e\mathbf{F}\cdot\mathbf{r}-E^{(1)}]\psi^{(0)}(\mathbf{r}), \tag{3.11a}$$

$$r\psi^{(1)}(\mathbf{r})\overset{r\to0}{\sim}0,\quad r\psi^{(1)}(\mathbf{r})\overset{r\to\infty}{\sim}0. \tag{3.11b}$$

To make the solution to the boundary value problem constituted by Eqs. (3.11a) and (3.11b) unique, we demand that

$$\int_{\mathbb{R}^3}d^3\mathbf{r}\,\psi_\mu^{(0)\dagger}(\mathbf{r})\psi^{(1)}(\mathbf{r})=0\quad\left(\mu=\pm\frac{1}{2}\right). \tag{3.12}$$

Applying the Green functions technique, this solution is found to be

$$\psi^{(1)}(\mathbf{r})=-\int_{\mathbb{R}^3}d^3\mathbf{r}'\,\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')[e\mathbf{F}\cdot\mathbf{r}'-E^{(1)}]\psi^{(0)}(\mathbf{r}'), \tag{3.13}$$

where $\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')$ is the *generalized* (or *reduced*) Dirac-Coulomb Green function for the unperturbed ground-state energy level (3.4). It solves the inhomogeneous differential equation ($\mathbf{r}'$ fixed)

$$\left[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m_ec^2\beta-\frac{Ze^2}{(4\pi\epsilon_0)r}-E^{(0)}\right]\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')=\mathcal{I}\delta^3(\mathbf{r}-\mathbf{r}')-\sum_{\mu=\pm1/2}\psi_\mu^{(0)}(\mathbf{r})\psi_\mu^{(0)\dagger}(\mathbf{r}') \tag{3.14a}$$

(here $\mathcal{I}$ is the unit 4×4 matrix), with the boundary conditions

$$r\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')\overset{r\to0}{\sim}0,\quad r\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')\overset{r\to\infty}{\sim}0, \tag{3.14b}$$

and obeys the additional orthogonality constraints

$$\int_{\mathbb{R}^3} d^3\mathbf{r}\,\psi_{\mu}^{(0)\dagger}(\mathbf{r})\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}') = 0 \quad \left(\mu = \pm \tfrac{1}{2}\right). \tag{3.14c}$$

On combining Eq. (3.14c) with the easily provable Hermiticity property of $\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')$, one may simplify Eq. (3.13) to the form

$$\psi^{(1)}(\mathbf{r}) = -e\mathbf{F}\cdot\int_{\mathbb{R}^3} d^3\mathbf{r}'\,\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')\mathbf{r}'\,\psi^{(0)}(\mathbf{r}'). \tag{3.15}$$

Proceeding further, in the standard way from Eq. (3.11a) one deduces that the coefficients $a_{\pm 1/2}$ solve the algebraic system

$$\begin{pmatrix} V_{1/2,1/2}^{(1)} - E^{(1)} & V_{1/2,-1/2}^{(1)} \\ V_{-1/2,1/2}^{(1)} & V_{-1/2,-1/2}^{(1)} - E^{(1)} \end{pmatrix}\begin{pmatrix} a_{1/2} \\ a_{-1/2} \end{pmatrix} = 0, \tag{3.16}$$

with

$$V_{\mu\mu'}^{(1)} = e\mathbf{F}\cdot\int_{\mathbb{R}^3} d^3\mathbf{r}\,\psi_{\mu}^{(0)\dagger}(\mathbf{r})\mathbf{r}\psi_{\mu'}^{(0)}(\mathbf{r}) \quad \left(\mu,\mu' = \pm \tfrac{1}{2}\right), \tag{3.17}$$

while the energy correction $E^{(1)}$ is a root of the secular equation resulting from equating the determinant of the system matrix in Eq. (3.16) to zero,

$$\det\begin{pmatrix} V_{1/2,1/2}^{(1)} - E^{(1)} & V_{1/2,-1/2}^{(1)} \\ V_{-1/2,1/2}^{(1)} & V_{-1/2,-1/2}^{(1)} - E^{(1)} \end{pmatrix} = 0. \tag{3.18}$$

Since both the functions (3.6) have the same parity, the integrand in Eq. (3.17) is odd under inversion, which implies

$$V_{\mu\mu'}^{(1)} = 0 \quad \left(\mu,\mu' = \pm \frac{1}{2}\right). \tag{3.19}$$

Hence, it follows immediately that

$$E^{(1)} = 0 \tag{3.20}$$

is a double root of the secular equation (3.18).

The important consequence of Eqs. (3.18)–(3.20) is that whatever one chooses the coefficients $a_{\pm 1/2}$ in Eq. (3.9), this always results in the symmetry-adapted unperturbed eigenfunction $\psi^{(0)}(\mathbf{r})$. Physically, this corresponds to the fact that, in the system under consideration, the external electric field does not change the orientation of the permanent atomic magnetic moment $\mathbf{m}^{(0)}$ (cf. Appendix C and the last paragraph in Sec. IV).

IV. EVALUATION OF THE STARK-INDUCED MAGNETIC ANAPOLE MOMENT OF THE ATOM

According to the Dirac theory, an electric current density associated with the atomic electron in the state $\psi(\mathbf{r})$ is

$$\mathbf{j}(\mathbf{r}) = -ec\psi^{\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi(\mathbf{r}). \tag{4.1}$$

In virtue of Eq. (3.8a), to the first order in the perturbing field $\mathbf{F}$ one has

$$\mathbf{j}(\mathbf{r}) \simeq \mathbf{j}^{(0)}(\mathbf{r}) + \mathbf{j}^{(1)}(\mathbf{r}), \tag{4.2}$$

where

$$\mathbf{j}^{(0)}(\mathbf{r}) = -ec\psi^{(0)\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi^{(0)}(\mathbf{r}) \tag{4.3}$$

is the electronic current density in the unperturbed atomic eigenstate (3.9), while

$$\mathbf{j}^{(1)}(\mathbf{r}) = -2ec\,\mathrm{Re}[\psi^{(0)\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi^{(1)}(\mathbf{r})] \tag{4.4}$$

is an induced electronic current density due to the perturbing field $\mathbf{F}$. On combining Eqs. (2.4) and (4.2)–(4.4), one finds that the magnetic anapole moment of the atom may be written as

$$\mathbf{t} \simeq \mathbf{t}^{(0)} + \mathbf{t}^{(1)}, \tag{4.5}$$

where

$$\mathbf{t}^{(0)} = \frac{1}{4}ec\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\psi^{(0)\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi^{(0)}(\mathbf{r}) \tag{4.6}$$

is the magnetic anapole moment of the atom in the unperturbed eigenstate (3.9), while

$$\mathbf{t}^{(1)} = \frac{1}{2}ec\,\mathrm{Re}\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\psi^{(0)\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi^{(1)}(\mathbf{r}) \tag{4.7}$$

is the first-order Stark-induced magnetic anapole moment.

Exploiting Eqs. (3.9), (3.6), (B8), and (B9), with no difficulty one shows that all spherical components of the vector $\mathbf{t}^{(0)}$, given by

$$\mathbf{e}_m\cdot\mathbf{t}^{(0)} = \frac{1}{4}ec\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\psi^{(0)\dagger}(\mathbf{r})\mathbf{e}_m\cdot\boldsymbol{\alpha}\psi^{(0)}(\mathbf{r}) \quad (m = 0, \pm 1) \tag{4.8}$$

vanish. This implies that the unperturbed one-electron Dirac atom has a vanishing magnetic anapole moment in its ground state,

$$\mathbf{t}^{(0)} = 0. \tag{4.9}$$

(It seems worthwhile to mention at this moment that, under certain conditions, the unperturbed one-electron Dirac atom may still have a nonzero magnetic anapole moment being in some particular of its energetically excited stationary states [32].)

We turn to considering the induced moment $\mathbf{t}^{(1)}$. On inserting Eq. (3.15) into Eq. (4.7), the latter takes the form

$$\mathbf{t}^{(1)} = (4\pi\epsilon_0)\mathsf{T}\cdot\mathbf{F}, \tag{4.10}$$

where

$$\mathsf{T} = -\frac{1}{2}\frac{ce^2}{(4\pi\epsilon_0)}\,\mathrm{Re}\int_{\mathbb{R}^3} d^3\mathbf{r}\int_{\mathbb{R}^3} d^3\mathbf{r}'\,r^2\psi^{(0)\dagger}(\mathbf{r}) \times\boldsymbol{\alpha}\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')\mathbf{r}'\,\psi^{(0)}(\mathbf{r}') \tag{4.11}$$

is a tensor *anapole* (or *toroidal dipole*) polarizability [33] for the system under study. The integrations over the angular variables of $\mathbf{r}$ and $\mathbf{r}'$ in Eq. (4.11) may be done with the aid of Eqs. (3.9) and (3.6), of the well-known partial-wave expansion of the generalized Green function $\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')$,

$$\mathcal{G}^{(0)}(\mathbf{r},\mathbf{r}')=\frac{(4\pi\epsilon_0)}{e^2}\sum_{\substack{\kappa=-\infty\\(\kappa\neq 0)}}^{\infty}\sum_{m=-|\kappa|+1/2}^{|\kappa|-1/2}\frac{1}{rr'}\begin{pmatrix} g^{(0)}_{(++)\kappa}(r,r')\Omega_{\kappa m}(\mathbf{n}_r)\Omega^{\dagger}_{\kappa m}(\mathbf{n}'_r) & -ig^{(0)}_{(+-)\kappa}(r,r')\Omega_{\kappa m}(\mathbf{n}_r)\Omega^{\dagger}_{-\kappa m}(\mathbf{n}'_r) \\ ig^{(0)}_{(-+)\kappa}(r,r')\Omega_{-\kappa m}(\mathbf{n}_r)\Omega^{\dagger}_{\kappa m}(\mathbf{n}'_r) & g^{(0)}_{(--)\kappa}(r,r')\Omega_{-\kappa m}(\mathbf{n}_r)\Omega^{\dagger}_{-\kappa m}(\mathbf{n}'_r)\end{pmatrix}, \tag{4.12}$$

and some of the properties of the spherical versors [29] and the spherical spinors (cf. Appendix B). This yields the following representation of the tensor T in the Cartesian basis $\{\mathbf{n}_x,\mathbf{n}_y,\mathbf{n}_z\}$:

$$\mathsf{T}=(\mathbf{n}_x\ \mathbf{n}_y\ \mathbf{n}_z)\mathsf{T}^{(c)}(\mathbf{n}_x\ \mathbf{n}_y\ \mathbf{n}_z)^{\mathrm{T}}, \tag{4.13}$$

with

$$\mathsf{T}^{(c)}=\tau\begin{pmatrix} 0 & |a_{1/2}|^2-|a_{-1/2}|^2 & -2\,\mathrm{Im}(a^*_{1/2}a_{-1/2}) \\ -|a_{1/2}|^2+|a_{-1/2}|^2 & 0 & 2\,\mathrm{Re}(a^*_{1/2}a_{-1/2}) \\ 2\,\mathrm{Im}(a^*_{1/2}a_{-1/2}) & -2\,\mathrm{Re}(a^*_{1/2}a_{-1/2}) & 0\end{pmatrix}. \tag{4.14}$$

In Eq. (4.14), τ is the anapole polarizability given by

$$\tau=\tau_{+1}+\tau_{-2}, \tag{4.15}$$

where

$$\tau_{+1}=-\frac{1}{18}c\int_0^{\infty}dr\int_0^{\infty}dr'\,r^2r'(Q^{(0)}(r)\quad 3P^{(0)}(r))\mathsf{G}^{(0)}_{+1}(r,r')\times\begin{pmatrix}P^{(0)}(r')\\ Q^{(0)}(r')\end{pmatrix} \tag{4.16}$$

and

$$\tau_{-2}=-\frac{1}{9}c\int_0^{\infty}dr\int_0^{\infty}dr'\,r^2r'(Q^{(0)}(r)\quad 0)\mathsf{G}^{(0)}_{-2}(r,r')\begin{pmatrix}P^{(0)}(r')\\ Q^{(0)}(r')\end{pmatrix}, \tag{4.17}$$

with

$$\mathsf{G}^{(0)}_{\kappa}(r,r')=\begin{pmatrix} g^{(0)}_{(++)\kappa}(r,r') & g^{(0)}_{(+-)\kappa}(r,r') \\ g^{(0)}_{(-+)\kappa}(r,r') & g^{(0)}_{(--)\kappa}(r,r')\end{pmatrix}\quad(\kappa=+1,-2) \tag{4.18}$$

being the symmetry-adapted generalized radial Dirac-Coulomb Green functions for the problem at hand.

To perform the radial integrals in Eqs. (4.16) and (4.17), we shall make use of the following Sturmian expansion of the generalized radial Dirac-Coulomb Green function:

$$\mathsf{G}^{(0)}_{\kappa}(r,r')=\sum_{n=-\infty}^{\infty}\frac{1}{\mu^{(0)}_{n\kappa}-1}\begin{pmatrix}S^{(0)}_{n\kappa}(r)\\ T^{(0)}_{n\kappa}(r)\end{pmatrix}\times(\mu^{(0)}_{n\kappa}S^{(0)}_{n\kappa}(r')\quad T^{(0)}_{n\kappa}(r'))\quad(\kappa=+1,-2), \tag{4.19}$$

found by one of us in Ref. 23. Here

$$\mu^{(0)}_{n\kappa}=\frac{|n|+\gamma_\kappa+N_{n\kappa}}{\gamma_1+1}, \tag{4.20}$$

while

$$S^{(0)}_{n\kappa}(r)=\sqrt{\frac{(1+\gamma_1)(|n|+2\gamma_\kappa)|n|!}{2ZN_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)}}\times\left(\frac{2Zr}{a_0}\right)^{\gamma_\kappa}e^{-Zr/a_0}\left[L^{(2\gamma_\kappa)}_{|n|-1}\left(\frac{2Zr}{a_0}\right)+\frac{\kappa-N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}\left(\frac{2Zr}{a_0}\right)\right] \tag{4.21a}$$

and

$$T^{(0)}_{n\kappa}(r)=\sqrt{\frac{(1-\gamma_1)(|n|+2\gamma_\kappa)|n|!}{2ZN_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)}}\times\left(\frac{2Zr}{a_0}\right)^{\gamma_\kappa}e^{-Zr/a_0}\left[L^{(2\gamma_\kappa)}_{|n|-1}\left(\frac{2Zr}{a_0}\right)-\frac{\kappa-N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}\left(\frac{2Zr}{a_0}\right)\right] \tag{4.21b}$$

are the radial Dirac-Coulomb Sturmian functions, with $L^{(\alpha)}_n(\rho)$ being the generalized Laguerre polynomial [35] (we define $L^{(\alpha)}_{-1}(\rho)\equiv 0$). The "apparent principal quantum number" $N_{n\kappa}$, appearing in Eqs. (4.20), (4.21a), and (4.21b), is defined as

$$N_{n\kappa}=\pm\sqrt{(|n|+\gamma_\kappa)^2+(\alpha Z)^2}=\pm\sqrt{|n|^2+2|n|\gamma_\kappa+\kappa^2} \tag{4.22}$$

(notice that $N_{n\kappa}$ may assume positive as well as negative values). The following sign convention is adopted in the definition (4.22): the plus sign should be chosen for $n>0$ and the minus one for $n<0$; for $n=0$ one chooses the plus sign if $\kappa<0$ and the minus sign if $\kappa>0$. On inserting the expansion (4.19) into Eq. (4.16), making use of the formula

$$\int_0^\infty d\rho\rho^\alpha e^{-\rho}L_n^{(\beta)}(\rho)=\frac{\Gamma(\alpha+1)\Gamma(n+\beta-\alpha)}{n!\Gamma(\beta-\alpha)} \quad [\mathrm{Re}(\alpha)>-1], \tag{4.23}$$

and collecting terms with the same $|n|$, with no difficulty one finds that

$$\tau_{+1}=\alpha c\frac{a_0^4}{Z^4}\frac{(\gamma_1+1)(2\gamma_1+1)(2\gamma_1^2+\gamma_1-2)}{72}. \tag{4.24}$$

Proceeding in the analogous way with Eq. (4.17), after tedious calculations exploiting the definition of the generalized hypergeometric series

$$\begin{aligned}{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)&=\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\\&\quad\times\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\cdots\Gamma(a_p+n)}{\Gamma(b_1+n)\cdots\Gamma(b_q+n)}\frac{z^n}{n!}\end{aligned} \tag{4.25}$$

and the Gauss' formula [35]

$$ {}_2F_1\left(\begin{matrix}a_1,a_2\\ b\end{matrix};1\right)=\frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)} \quad [\mathrm{Re}(b-a_1-a_2)>0], \tag{4.26}$$

one eventually arrives at the following result:

$$\begin{aligned}\tau_{-2}=\alpha c\frac{a_0^4}{Z^4}\Bigg[&\frac{(\gamma_1+2)(2\gamma_1+1)(2\gamma_1+3)(2\gamma_1+5)(2\gamma_1^2+2\gamma_1-3)}{216(4\gamma_1+1)}\\&-\frac{(\gamma_1-2)\Gamma(\gamma_1+\gamma_2+3)\Gamma(\gamma_1+\gamma_2+4)}{288(\gamma_1+1)(4\gamma_1+1)(\gamma_2-\gamma_1)\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)}{}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-3,\gamma_2-\gamma_1-2,\gamma_2-\gamma_1\\ \gamma_2-\gamma_1+1,2\gamma_2+1\end{matrix};1\right)\Bigg].\end{aligned} \tag{4.27}$$

In principle, if Eqs. (4.24) and (4.27) are inserted into Eq. (4.15), the task of evaluating the anapole polarizability τ is accomplished. However, the final result for τ appears to be slightly simpler if the component τ_{-2} is expressed in terms of the generalized hypergeometric series

$$ {}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-1,\gamma_2-\gamma_1-1,\gamma_2-\gamma_1\\ \gamma_2-\gamma_1+1,2\gamma_2+1\end{matrix};1\right),$$

which is the same ${}_3F_2$ series which appears in the known formula for the dipole magnetizability [25,36,37] (and also in one of the recently found representations of the dipole polarizability [38]) of the ground state of the Dirac one-electron atom. This goal is most conveniently achieved in two steps. In the first step, we transform the ${}_3F_2$ series appearing in Eq. (4.27) with the aid of the relation (cf. Appendix D)

$$ {}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ a_3+1,b\end{matrix};1\right)=-\frac{a_3}{a_1-a_3}\frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)}+\frac{a_1}{a_1-a_3}{}_3F_2\left(\begin{matrix}a_1+1,a_2,a_3\\ a_3+1,b\end{matrix};1\right) \quad [\mathrm{Re}(b-a_1-a_2)>0]. \tag{4.28}$$

This yields

$$\begin{aligned}\tau_{-2}=\alpha c\frac{a_0^4}{Z^4}\Bigg[&\frac{(\gamma_1+1)(\gamma_1+2)(2\gamma_1-1)(2\gamma_1+1)(2\gamma_1+3)(2\gamma_1+5)}{216(4\gamma_1+1)}\\&-\frac{(\gamma_1-2)\Gamma^2(\gamma_1+\gamma_2+3)}{144(4\gamma_1+1)(\gamma_2-\gamma_1)\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)}{}_3F_2\left(\begin{matrix}\gamma_2-\gamma_1-2,\gamma_2-\gamma_1-2,\gamma_2-\gamma_1\\ \gamma_2-\gamma_1+1,2\gamma_2+1\end{matrix};1\right)\Bigg].\end{aligned} \tag{4.29}$$

In the second step, the ${}_3F_2$ series in Eq. (4.29) is transformed with the use of the formula (cf. again Appendix D)

$$\begin{aligned}{}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ a_3+1,b\end{matrix};1\right)=&-\frac{a_3[a_1a_2+(a_1+a_2-a_3)(b-a_1-a_2-1)]}{(a_1-a_3)(a_2-a_3)}\frac{\Gamma(b)\Gamma(b-a_1-a_2-1)}{\Gamma(b-a_1)\Gamma(b-a_2)}\\&+\frac{a_1a_2}{(a_1-a_3)(a_2-a_3)}{}_3F_2\left(\begin{matrix}a_1+1,a_2+1,a_3\\ a_3+1,b\end{matrix};1\right) \quad [\mathrm{Re}(b-a_1-a_2)>1].\end{aligned} \tag{4.30}$$

This results in

$$\tau_{-2} = \alpha c \frac{a_0^4}{Z^4}\left[\frac{(\gamma_1+1)(2\gamma_1+1)(2\gamma_1+3)(4\gamma_1^2+9\gamma_1+14)}{864} - \frac{(\gamma_1-2)(4\gamma_1+1)\Gamma^2(\gamma_1+\gamma_2+2)}{576(\gamma_2-\gamma_1)\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)} \times {}_3F_2\left(\begin{matrix} \gamma_2-\gamma_1-1, \gamma_2-\gamma_1-1, \gamma_2-\gamma_1 \\ \gamma_2-\gamma_1+1, 2\gamma_2+1 \end{matrix}; 1 \right) \right]. \quad (4.31)$$

Hence, upon inserting Eqs. (4.24) and (4.31) into Eq. (4.15), one eventually arrives at the following representation of the anapole polarizability:

$$\tau = \alpha c \frac{a_0^4}{Z^4}\left[\frac{(\gamma_1+1)(2\gamma_1+1)(8\gamma_1^3+54\gamma_1^2+67\gamma_1+18)}{864} - \frac{(\gamma_1-2)(4\gamma_1+1)\Gamma^2(\gamma_1+\gamma_2+2)}{576(\gamma_2-\gamma_1)\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)} \times {}_3F_2\left(\begin{matrix} \gamma_2-\gamma_1-1, \gamma_2-\gamma_1-1, \gamma_2-\gamma_1 \\ \gamma_2-\gamma_1+1, 2\gamma_2+1 \end{matrix}; 1 \right) \right]. \quad (4.32)$$

It seems worthwhile to investigate the nonrelativistic limit of Eq. (4.32). Since

$$\gamma_1 \overset{c\to\infty}{\sim} 1, \quad \gamma_2 \overset{c\to\infty}{\sim} 2, \quad (4.33)$$

and

$$ {}_3F_2\left(\begin{matrix} 0,0,1 \\ 2,5 \end{matrix}; 1 \right) = 1, \quad (4.34)$$

we obtain

$$\tau \overset{c\to\infty}{\sim} \frac{9}{8}\frac{\hbar}{m_e}\frac{a_0^3}{Z^4}, \quad (4.35)$$

which agrees with what may be inferred from the finding reported by Lewis and Blinder [16, Eq. (29)].

Before concluding, we shall relate the above found anapole moment $\mathbf{t} \simeq \mathbf{t}^{(1)}$ to the electric ($\mathbf{d}$) and magnetic ($\mathbf{m}$) dipole moments in the perturbed atomic state (3.8a). To this end, we observe that it follows from Eq. (4.14) that the Cartesian components of the anapole polarizability tensor T may be written as

$$T_{ij}^{(c)} = \tau \sum_{k\in\{x,y,z\}} \varepsilon_{ijk}\nu_k \quad (i,j \in \{x,y,z\}), \quad (4.36)$$

where ε_{ijk} is the Levi-Civita completely antisymmetric symbol and $\{\nu_k\}$ are the Cartesian components of the unit vector $\boldsymbol{\nu}$ defined in Eq. (C6). Thus, invoking Eqs. (4.10) and (4.13), we have

$$\mathbf{t}^{(1)} = -(4\pi\epsilon_0)\tau\boldsymbol{\nu}\times\mathbf{F}. \quad (4.37)$$

The perturbing electric field $\mathbf{F}$ induces in the atom the electric dipole moment which, to the first order in $\mathbf{F}$, is given by

$$\mathbf{d}^{(1)} = (4\pi\epsilon_0)\alpha_d\mathbf{F}, \quad (4.38)$$

with α_d being the static dipole polarizability; it has been found [38] that, in terms of this particular ${}_3F_2$ series which appears in Eqs. (4.31) and (4.32), one has

$$\alpha_d = \frac{a_0^3}{Z^4}\left[-\frac{\gamma_1(\gamma_1+1)(2\gamma_1+1)(2\gamma_1^2-9\gamma_1-17)}{36} + \frac{(\gamma_1-2)^2\Gamma^2(\gamma_1+\gamma_2+2)}{24(\gamma_2-\gamma_1)\Gamma(2\gamma_1+1)\Gamma(2\gamma_2+1)} \times {}_3F_2\left(\begin{matrix} \gamma_2-\gamma_1-1, \gamma_2-\gamma_1-1, \gamma_2-\gamma_1 \\ \gamma_2-\gamma_1+1, 2\gamma_2+1 \end{matrix}; 1 \right) \right]. \quad (4.39)$$

Making use of Eq. (4.38), and also of the relationship between the unit vector $\boldsymbol{\nu}$ and the magnetic dipole moment $\mathbf{m}^{(0)}$ of the atom in the unperturbed state (3.9),

$$\mathbf{m}^{(0)} = -\frac{2\gamma_1+1}{3}\mu_{\mathrm{B}}\boldsymbol{\nu} \quad (4.40)$$

(cf. Appendix C), one finds that

$$\mathbf{t}^{(1)} = \frac{3}{2\gamma_1+1}\frac{\tau}{\mu_{\mathrm{B}}\alpha_d}\mathbf{m}^{(0)}\times\mathbf{d}^{(1)}. \quad (4.41)$$

Since it is not difficult to show that, to the first order in the perturbing electric field $\mathbf{F}$, it holds that

$$\mathbf{d} \simeq \mathbf{d}^{(1)}, \quad \mathbf{m} \simeq \mathbf{m}^{(0)}, \quad (4.42)$$

on combining Eqs. (4.5), (4.9), (4.41), and (4.42), one obtains that, to the first order in the perturbing electric field, it holds that

$$\mathbf{t} \simeq \frac{3}{2\gamma_1+1}\frac{\tau}{\mu_{\mathrm{B}}\alpha_d}\mathbf{m}\times\mathbf{d}. \quad (4.43)$$

In the nonrelativistic limit, exploiting Eqs. (4.33) and (4.35) and the fact that

$$\alpha_d \overset{c\to\infty}{\sim} \frac{9}{2}\frac{a_0^3}{Z^4} \quad (4.44)$$

[this follows immediately from Eqs. (4.39), (4.33), and (4.34)], Eq. (4.43) becomes

$$\mathbf{t}_{nr} \simeq \frac{1}{2e}\mathbf{m}_{nr}\times\mathbf{d}_{nr} \quad (4.45)$$

(the suffix denotes the nonrelativistic limit), which again agrees with what may be inferred from the finding of Lewis and Blinder [16, Eq. (9')].

V. CONCLUSIONS

The Dirac-Coulomb Sturmian technique [23,24] allows one to find a series representation of the generalized Dirac-Coulomb Green function. In this paper, we have exploited this fact to derive exact analytical expressions for the first-order Stark-induced magnetic anapole moment $\mathbf{t}^{(1)}$ and for the associated anapole polarizability τ of the Dirac one-electron atom in the ground state. We have managed to write down the expression for τ in the form which involves the same ${}_3F_2$ function with the unit argument which appears in the well-known representation of the dipole magnetizability [25,36,37] (and also in one of the recently found representations of the dipole polarizability [38]) of the relativistic hydrogenlike atom in the ground state. In the nonrelativistic limit, our results agree with the findings of Lewis and Blinder [16].

APPENDIX A: VARIOUS EQUIVALENT EXPRESSIONS FOR THE ANAPOLE MOMENT VECTOR

We start with the observation that if $\mathbf{j}(\mathbf{r})$ is divergenceless,

$$\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r})=0, \tag{A1}$$

and such that

$$\lim_{r\to\infty} r^5\mathbf{j}(\mathbf{r})=0, \tag{A2}$$

then the following relationship holds:

$$\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}=-\frac{1}{2}\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\mathbf{j}(\mathbf{r}). \tag{A3}$$

To prove this statement, consider the integral

$$\mathbf{X}=\int_{\mathbb{R}^3} d^3\mathbf{r}\,\boldsymbol{\nabla}\cdot[r^2\mathbf{j}(\mathbf{r})\mathbf{r}]. \tag{A4}$$

With the aid of the Gauss' divergence theorem, it may be transformed to the form

$$\mathbf{X}=\lim_{r\to\infty} r^5\oint_{4\pi} d^2\mathbf{n}_r[\mathbf{n}_r\cdot\mathbf{j}(\mathbf{r})]\mathbf{n}_r, \tag{A5}$$

hence, in virtue of the assumption (A2), it follows that

$$\mathbf{X}=0. \tag{A6}$$

On the other hand, carrying out the differentiation under the integral sign in Eq. (A4), and exploiting the facts that

$$\boldsymbol{\nabla}r^2=2\mathbf{r}, \qquad \boldsymbol{\nabla}\,\mathbf{r}=\mathsf{I} \tag{A7}$$

(I is the unit dyadic), one has

$$\mathbf{X}=2\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}+\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2[\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}+\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\mathbf{j}(\mathbf{r}). \tag{A8}$$

In virtue of the assumption (A1), the second integral on the right-hand side of the above equation vanishes, which yields

$$\mathbf{X}=2\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}+\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\mathbf{j}(\mathbf{r}). \tag{A9}$$

On combining Eqs. (A6) and (A9), the identity (A3) follows immediately.

Rewriting the definition

$$\mathbf{t}=\frac{1}{2}\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r} \tag{A10}$$

in the form

$$\mathbf{t}=\frac{1}{2}\eta\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}+\frac{1}{2}(1-\eta)\int_{\mathbb{R}^3} d^3\mathbf{r}[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}, \tag{A11}$$

where η is an arbitrary complex number, and applying subsequently the identity (A3) to the second integral on the right-hand side, transforms Eq. (A11) into

$$\mathbf{t}=\frac{1}{4}\int_{\mathbb{R}^3} d^3\mathbf{r}\{2\eta[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}-(1-\eta)r^2\mathbf{j}(\mathbf{r})\}. \tag{A12}$$

Various equivalent expressions for the anapole moment found in the literature may be obtained from Eq. (A12) by making there a suitable choice of η. In particular, the expression (A10), adopted by us in Sec. II as the definition of the anapole moment, corresponds to the choice $\eta=1$ [7,10,28,39]. For $\eta=0$, from Eq. (A12) one has [1–6,10,11,13–16,20,28,39–41]

$$\mathbf{t}=-\frac{1}{4}\int_{\mathbb{R}^3} d^3\mathbf{r}\,r^2\mathbf{j}(\mathbf{r}), \tag{A13}$$

choosing $\eta=\frac{1}{5}$ yields [8,9,17,28,34,42–46]

$$\mathbf{t}=\frac{1}{10}\int_{\mathbb{R}^3} d^3\mathbf{r}\{[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}-2r^2\mathbf{j}(\mathbf{r})\}, \tag{A14}$$

while setting $\eta=\frac{1}{3}$ results in [22,47]

$$\mathbf{t}=\frac{1}{6}\int_{\mathbb{R}^3} d^3\mathbf{r}\{[\mathbf{r}\cdot\mathbf{j}(\mathbf{r})]\mathbf{r}-r^2\mathbf{j}(\mathbf{r})\}. \tag{A15}$$

The latter expression may be rewritten, perhaps more elegantly, as [18,39,48]

$$\mathbf{t}=\frac{1}{6}\int_{\mathbb{R}^3} d^3\mathbf{r}\,\mathbf{r}\times[\mathbf{r}\times\mathbf{j}(\mathbf{r})]. \tag{A16}$$

APPENDIX B: SPHERICAL SPINORS

Unfortunately, in the physical and mathematical literature there are no unique phase conventions for defining the spherical harmonics and the spherical spinors. On the other hand, recurrence relations linking the spherical spinors with various indices appear to be phase-dependent. Therefore, to avoid any misunderstandings, in this appendix we define the spherical spinors (as well as the spherical harmonics) used in this paper, and list these particular recurrence relations obeyed by them which have been exploited by us.

The spherical spinors are defined as

$$\Omega_{\kappa\mu}(\mathbf{n}_r) = \begin{pmatrix} \operatorname{sgn}(-\kappa)\sqrt{\dfrac{\kappa+\frac{1}{2}-\mu}{2\kappa+1}}\,Y_{l,\mu-1/2}(\mathbf{n}_r) \\ \sqrt{\dfrac{\kappa+\frac{1}{2}+\mu}{2\kappa+1}}\,Y_{l,\mu+1/2}(\mathbf{n}_r) \end{pmatrix}, \tag{B1}$$

where $\kappa=\pm 1,\pm 2,\ldots$, $\mu=-|\kappa|+\frac{1}{2},-|\kappa|+\frac{3}{2},\ldots,|\kappa|-\frac{1}{2}$, and

$$l \equiv l(\kappa) = \left|\kappa+\frac{1}{2}\right| - \frac{1}{2} = \begin{cases} -\kappa-1 & \text{for} \quad \kappa<0 \\ \kappa & \text{for} \quad \kappa>0, \end{cases} \tag{B2}$$

while $Y_{lm}(\mathbf{n}_r)$ is the spherical harmonics defined as

$$Y_{lm}(\mathbf{n}_r) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \tag{B3}$$

with $l=0,1,2,\ldots$, $m=-l,-l+1,\ldots,l$, and with $P_l^m(\xi)$ being the associated Legendre function,

$$P_l^m(\xi) = \frac{(-)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{d^{l+m}(\xi^2-1)^l}{d\xi^{l+m}} \quad (-1 \leqslant \xi \leqslant 1). \tag{B4}$$

The definition (B3) and (B4) conforms to the Condon-Shortley phase convention [49].

The following properties of the spherical spinors have proved useful in the course of preparing the present work:

$$\oint_{4\pi} d^2\mathbf{n}_r \Omega^{\dagger}_{\kappa\mu}(\mathbf{n}_r)\Omega_{\kappa'\mu'}(\mathbf{n}_r) = \delta_{\kappa\kappa'}\delta_{\mu\mu'}, \tag{B5}$$

$$\mathbf{e}_0\cdot\mathbf{n}_r\Omega_{\kappa\mu}(\mathbf{n}_r) = -\frac{2\mu}{4\kappa^2-1}\Omega_{-\kappa\mu}(\mathbf{n}_r) + \frac{\sqrt{\left(\kappa+\frac{1}{2}\right)^2-\mu^2}}{|2\kappa+1|}\Omega_{\kappa+1,\mu}(\mathbf{n}_r) + \frac{\sqrt{\left(\kappa-\frac{1}{2}\right)^2-\mu^2}}{|2\kappa-1|}\Omega_{\kappa-1,\mu}(\mathbf{n}_r), \tag{B6}$$

$$\begin{aligned}\mathbf{e}_{\pm 1}\cdot\mathbf{n}_r\Omega_{\kappa\mu}(\mathbf{n}_r) = {}& \pm\sqrt{2}\frac{\sqrt{\kappa^2-\left(\mu\pm\frac{1}{2}\right)^2}}{4\kappa^2-1}\Omega_{-\kappa,\mu\pm 1}(\mathbf{n}_r) + \frac{\sqrt{\left(\kappa\pm\mu+\frac{1}{2}\right)\left(\kappa\pm\mu+\frac{3}{2}\right)}}{\sqrt{2}(2\kappa+1)}\Omega_{\kappa+1,\mu\pm 1}(\mathbf{n}_r) \\ & - \frac{\sqrt{\left(\kappa\mp\mu-\frac{1}{2}\right)\left(\kappa\mp\mu-\frac{3}{2}\right)}}{\sqrt{2}(2\kappa-1)}\Omega_{\kappa-1,\mu\pm 1}(\mathbf{n}_r),\end{aligned} \tag{B7}$$

$$\mathbf{e}_0\cdot\boldsymbol{\sigma}\Omega_{\kappa\mu}(\mathbf{n}_r) = -\frac{2\mu}{2\kappa+1}\Omega_{\kappa\mu}(\mathbf{n}_r) - 2\frac{\sqrt{\left(\kappa+\frac{1}{2}\right)^2-\mu^2}}{|2\kappa+1|}\Omega_{-\kappa-1,\mu}(\mathbf{n}_r), \tag{B8}$$

$$\mathbf{e}_{\pm 1}\cdot\boldsymbol{\sigma}\Omega_{\kappa\mu}(\mathbf{n}_r) = \pm\sqrt{2}\frac{\sqrt{\kappa^2-\left(\mu\pm\frac{1}{2}\right)^2}}{2\kappa+1}\Omega_{\kappa,\mu\pm 1}(\mathbf{n}_r) - \sqrt{2}\frac{\sqrt{\left(\kappa\pm\mu+\frac{1}{2}\right)\left(\kappa\pm\mu+\frac{3}{2}\right)}}{2\kappa+1}\Omega_{-\kappa-1,\mu\pm 1}(\mathbf{n}_r), \tag{B9}$$

$$\mathbf{e}_0\cdot(\mathbf{n}_r\times\boldsymbol{\sigma})\Omega_{\kappa\mu}(\mathbf{n}_r) = i\frac{4\mu\kappa}{4\kappa^2-1}\Omega_{-\kappa\mu}(\mathbf{n}_r) + i\frac{\sqrt{\left(\kappa+\frac{1}{2}\right)^2-\mu^2}}{|2\kappa+1|}\Omega_{\kappa+1,\mu}(\mathbf{n}_r) - i\frac{\sqrt{\left(\kappa-\frac{1}{2}\right)^2-\mu^2}}{|2\kappa-1|}\Omega_{\kappa-1,\mu}(\mathbf{n}_r), \tag{B10}$$

$$\begin{aligned}\mathbf{e}_{\pm 1}\cdot(\mathbf{n}_r\times\boldsymbol{\sigma})\Omega_{\kappa\mu}(\mathbf{n}_r) = {}& \mp i2\sqrt{2}\kappa\frac{\sqrt{\kappa^2-\left(\mu\pm\frac{1}{2}\right)^2}}{4\kappa^2-1}\Omega_{-\kappa,\mu\pm 1}(\mathbf{n}_r) \\ & + i\frac{\sqrt{\left(\kappa\pm\mu+\frac{1}{2}\right)\left(\kappa\pm\mu+\frac{3}{2}\right)}}{\sqrt{2}(2\kappa+1)}\Omega_{\kappa+1,\mu\pm 1}(\mathbf{n}_r) + i\frac{\sqrt{\left(\kappa\mp\mu-\frac{1}{2}\right)\left(\kappa\mp\mu-\frac{3}{2}\right)}}{\sqrt{2}(2\kappa-1)}\Omega_{\kappa-1,\mu\pm 1}(\mathbf{n}_r).\end{aligned} \tag{B11}$$

The Pauli vector $\boldsymbol{\sigma}$ appearing in Eqs. (B8)–(B11) is defined as

$$\boldsymbol{\sigma} = \sigma_x \mathbf{n}_x + \sigma_y \mathbf{n}_y + \sigma_z \mathbf{n}_z, \tag{B12}$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B13}$$

It is to be emphasized that relations (B8)–(B11) have to be modified if the matrix σ_y is defined, as it occasionally happens in the literature, as the negative of that in Eq. (B13).

APPENDIX C: MAGNETIC DIPOLE MOMENT OF THE DIRAC ONE-ELECTRON ATOM IN THE UNPERTURBED STATE (3.9)

The magnetic dipole moment for a stationary bounded electric current distribution characterized by the density $\mathbf{j}(\mathbf{r})$ is defined as

$$\mathbf{m} = \frac{1}{2}\int_{\mathbb{R}^3} d^3\mathbf{r}\, \mathbf{r} \times \mathbf{j}(\mathbf{r}). \tag{C1}$$

For the hydrogenlike atom in the unperturbed energy eigenstate (3.9) it holds that

$$\mathbf{j}(\mathbf{r}) \equiv \mathbf{j}^{(0)}(\mathbf{r}) = -ec\psi^{(0)\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi^{(0)}(\mathbf{r}) \tag{C2}$$

and Eq. (C1) becomes

$$\mathbf{m}^{(0)} = -\frac{1}{2}ec\int_{\mathbb{R}^3} d^3\mathbf{r}\, \psi^{(0)\dagger}(\mathbf{r})\mathbf{r} \times \boldsymbol{\alpha}\psi^{(0)}(\mathbf{r}) \tag{C3}$$

[the superscript added at $\mathbf{m}$ corresponds to the superscript at $\psi^{(0)}(\mathbf{r})$]. Integration over angular variables in Eq. (C3) may be carried out with the aid of the identities (B10) and (B11); this results in

$$\mathbf{m}^{(0)} = m^{(0)}\boldsymbol{\nu}, \tag{C4}$$

where

$$m^{(0)} = \frac{2}{3}ec\int_0^\infty dr\, r P^{(0)}(r) Q^{(0)}(r), \tag{C5}$$

while

$$\boldsymbol{\nu} = 2\,\mathrm{Re}(a_{1/2}^{*}a_{-1/2})\mathbf{n}_x + 2\,\mathrm{Im}(a_{1/2}^{*}a_{-1/2})\mathbf{n}_y + (|a_{1/2}|^2 - |a_{-1/2}|^2)\mathbf{n}_z \tag{C6}$$

is the vector of a unit length [this may be verified after exploiting the normalization constraint (3.10)]. The radial integral in Eq. (C5) is easily performed after making use of the definitions (3.7a) and (3.7b); this yields

$$m^{(0)} = -\frac{2\gamma_1 + 1}{6}\frac{e\hbar}{m_e}, \tag{C7}$$

or equivalently, in terms of the Bohr magneton $\mu_{\mathrm{B}} = e\hbar/2m_e$,

$$m^{(0)} = -\frac{2\gamma_1 + 1}{3}\mu_{\mathrm{B}}. \tag{C8}$$

Because of the normalization constraint (3.10), the coefficients $a_{\pm 1/2}$ may be parametrized by two real variables ϑ and φ, chosen to satisfy $0 \leqslant \vartheta \leqslant \pi$ and $0 \leqslant \varphi < 2\pi$, according to

$$a_{1/2} = e^{i\varphi}\cos(\vartheta/2), \quad a_{-1/2} = e^{i\varphi}\sin(\vartheta/2). \tag{C9}$$

Making use of this parametrization in Eq. (C6) gives

$$\boldsymbol{\nu} = \sin\vartheta\cos\varphi\mathbf{n}_x + \sin\vartheta\sin\varphi\mathbf{n}_y + \cos\vartheta\mathbf{n}_z, \tag{C10}$$

which shows that ϑ and φ may be interpreted as the polar and azimuthal angles, respectively, specifying the orientation of the unit vector $\boldsymbol{\nu}$ in the spherical coordinate system (with the polar and azimuthal axes of the latter directed along the Cartesian versors $\mathbf{n}_z$ and $\mathbf{n}_x$, respectively).

APPENDIX D: DERIVATIONS OF RELATIONS (4.28) AND (4.30)

Consider the generalized hypergeometric series ${}_3F_2(a_1, a_2, a_3; a_3+1, b; z)$. Specializing the definition (4.25) to this particular case, making use of the relationship

$$\Gamma(a_1+n)\Gamma(a_3+n) = -\frac{\Gamma(a_1+n)\Gamma(a_3+1+n)}{a_1 - a_3} + \frac{\Gamma(a_1+1+n)\Gamma(a_3+n)}{a_1 - a_3} \tag{D1}$$

and of the property

$${}_3F_2\left(\begin{matrix} a_1, a_2, a_3+1 \\ a_3+1, b \end{matrix}; z\right) = {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b \end{matrix}; z\right), \tag{D2}$$

one finds that the following recurrence relation holds:

$${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ a_3+1, b \end{matrix}; z\right) = -\frac{a_3}{a_1 - a_3}\,{}_2F_1\left(\begin{matrix} a_1, a_2 \\ b \end{matrix}; z\right) + \frac{a_1}{a_1 - a_3}\,{}_3F_2\left(\begin{matrix} a_1+1, a_2, a_3 \\ a_3+1, b \end{matrix}; z\right). \tag{D3}$$

Shifting in the above equation from a_2 to a_2+1, interchanging in the emerging expression a_1 with a_2, and exploiting the result to transform the right-hand side of Eq. (D3) leads to

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ a_3+1, b \end{matrix}; z\right) = &-\frac{a_3}{a_1 - a_3}\,{}_2F_1\left(\begin{matrix} a_1, a_2 \\ b \end{matrix}; z\right) \\ &- \frac{a_1 a_3}{(a_1 - a_3)(a_2 - a_3)}\,{}_2F_1\left(\begin{matrix} a_1+1, a_2 \\ b \end{matrix}; z\right) \\ &+ \frac{a_1 a_2}{(a_1 - a_3)(a_2 - a_3)} \times {}_3F_2\left(\begin{matrix} a_1+1, a_2+1, a_3 \\ a_3+1, b \end{matrix}; z\right). \end{aligned} \tag{D4}$$

Specializing Eq. (D3) to the case $z=1$ and exploiting the Gauss' formula (4.26) yields

$$ {}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ a_3+1,b\end{matrix};1\right) = -\frac{a_3}{a_1-a_3}\frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)} + \frac{a_1}{a_1-a_3}\,{}_3F_2\left(\begin{matrix}a_1+1,a_2,a_3\\ a_3+1,b\end{matrix};1\right) $$

$$ [\mathrm{Re}(b-a_1-a_2)>0], \tag{D5} $$

i.e., Eq. (4.28). Proceeding in the same way with Eq. (D4) gives

$$ {}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ a_3+1,b\end{matrix};1\right) = -\frac{a_3[a_1a_2+(a_1+a_2-a_3)(b-a_1-a_2-1)]}{(a_1-a_3)(a_2-a_3)} \times\frac{\Gamma(b)\Gamma(b-a_1-a_2-1)}{\Gamma(b-a_1)\Gamma(b-a_2)} + \frac{a_1a_2}{(a_1-a_3)(a_2-a_3)} \times {}_3F_2\left(\begin{matrix}a_1+1,a_2+1,a_3\\ a_3+1,b\end{matrix};1\right) $$

$$ [\mathrm{Re}(b-a_1-a_2)>1], \tag{D6} $$

i.e., Eq. (4.30).

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$$ \mathbf{t} = -\frac{\pi}{c}\int_{\mathbb{R}^3} d^3\mathbf{r}\, r^2 \mathbf{j}(\mathbf{r}), $$

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