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Strong ellipticity conditions and infinitesimal stability within nonlinear strain gradient elasticity

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ABSTRACT

We discuss connections between the strong ellipticity condition and the infinitesimal instability within the nonlinear strain gradient elasticity. The strong ellipticity (SE) condition describes the property of equations of statics whereas the infinitesimal stability is introduced as the positive definiteness of the second variation of an energy functional. Here we establish few implications which simplify the further analysis of stability using formulated SE conditions. The results could be useful for the analysis of solutions of homogenized models of beam-lattice materials at different scales.

1. Introduction

Recently the strain gradient elasticity found various applications in modelling of composite materials with high contrast in mechanical properties, such as beam-lattice materials. Indeed, a homogenization of such discrete structures leads to enhanced models of continua including the strain gradient elasticity, see e.g. [1–6] and the references therein. Due to high flexibility of these materials one can face certain instabilities observed at both micro- and macroscales [7–9]. Obviously, microscale instabilities may essentially change the material response at the macroscale. For example, buckling of cell walls of elastomeric foams results in a plateau in a loading diagram similar to plastic behaviour [10]. So homogenized models should also capture some material instabilities.

Considering a material instability phenomenon it is worth to note the strong ellipticity (SE) condition as a constitutive inequality. Indeed, in nonlinear elasticity of simple materials it is known that SE relates to an infinitesimal stability and vice versa [11–13]. In case of more complex models of continua the analysis of infinitesimal stability of solids and structures may result in rather complex systems of partial differential equations (PDEs), which can demonstrate unusual behaviour, see e.g. [14,15] for micropolar elasticity. In particular, a homogenization of pantographic beam-lattice materials results in so-called gradient incomplete strain gradient models with a particular potential energy density [3,16–18]. As a result, the corresponding system of PDEs is nor strongly elliptic neither elliptic, the corresponding differential properties relate to hypoelliptic operators [19]. A general theory for ellipticity and stability is also crucial even for linear problems for the following reason: sometimes for complex domains it is possible to reduce the problem under consideration to system of singular integral equations using some transformation techniques, see e.g. [20,21]. Unfortunately, their form allows one usually to define only some properties such as the total index but not the partial indices and definitely does not allow to prove the compactness of operators. In this case the preliminary knowledge on the ellipticity and stability gives a chance to clarify the missing information.

Nevertheless, the analysis of strong ellipticity conditions is still meaningful as it relates to solution of algebraic problems, more precisely, to some inequalities given in a point. This analysis could be simpler than the solution of the corresponding complete boundary-value problem. On the other hand it may bring some information about possible instabilities.

So the aim of this paper is to introduce ellipticity conditions for strain gradient elasticity and connect them with the ones for material instabilities. The paper is organized as follows. In Section 2 we briefly introduce the basic equations of the strain gradient elasticity for solids undergoing finite deformations. Hereinafter we use the direct (index-free) tensor calculus as defined in [11,22,23]. In Section 3 we give definitions of strong ellipticity (SE) conditions. The SE conditions are given for the both cases, i.e. for simple and non-simple materials. Section 4 addresses the main results related to ellipticity and stability. Similar to nonlinear elasticity [11–13], here we discuss two implications which clarify the close relations between SE conditions and infinitesimal stability within the framework of the nonlinear strain gradient elasticity.

2. Strain gradient elasticity

In the framework of nonlinear elasticity deformations of an elastic body $\mathcal{B}$ is described through a smooth enough one-to-one mapping from a reference placement into a current one as

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{1}$$

where $\mathbf{x}$ and $\mathbf{X}$ are position vectors of a material particle in $\mathcal{B}$ in the reference and current placements, respectively. As usual, we assume that $\mathbf{x}$ is a differentiable function, so we introduce its gradients

$$\mathbf{F} = \nabla\mathbf{x}, \quad \mathbf{G} = \nabla\nabla\mathbf{x} \equiv \nabla\mathbf{F}, \quad \text{etc.}$$

Here ∇ is the Lagrangian nabla-operator [11,22,23].

Within the strain gradient elasticity constitutive relations depend on $\mathbf{F}$ and $\mathbf{G}$ [3,24,25], whereas within the nonlinear elasticity of simple materials the latter depend on $\mathbf{F}$ only [11,12,26]. For a hyperelastic solid there exists a potential energy density introduced as a function of $\mathbf{F}$ and $\mathbf{G}$

$$W = W(\mathbf{F}, \mathbf{G}). \tag{2}$$

The principle of material frame indifference [11,26] requires the following invariance of W

$$W(\mathbf{F}, \mathbf{G}) = W(\mathbf{F}\cdot\mathbf{Q}, \mathbf{G}\cdot\mathbf{Q}) \tag{3}$$

for any orthogonal tensor $\mathbf{Q}$, $\mathbf{Q}^{-1} = \mathbf{Q}^T$. Here "$\cdot$" stands for the dot product. Eq. (3) results in various representations of W

$$W = \tilde{W}(\mathbf{C}, \mathbf{K}) = \tilde{\tilde{W}}(\mathbf{C}, \widetilde{\mathbf{K}}), \quad \text{etc.,}$$
$$\mathbf{C} = \mathbf{F}\cdot\mathbf{F}^T, \quad \mathbf{K} = \mathbf{G}\cdot\mathbf{F}^T, \quad \widetilde{\mathbf{K}} = \mathbf{G}\cdot\mathbf{F}^{-1}.$$

As we are interesting in a mathematical properties of corresponding boundary-value problems in the following we use form (2). Let us split W into two parts as follows

$$W = W_0(\mathbf{F}) + W_1(\mathbf{F}, \mathbf{G}), \quad W_1(\mathbf{F}, \mathbf{0}) = 0, \tag{4}$$

where

$$W_0 = W(\mathbf{F}, \mathbf{0}), \quad W_1(\mathbf{F}, \mathbf{0}) = W(\mathbf{F}, \mathbf{G}) - W(\mathbf{F}, \mathbf{0}).$$

Obviously, Eq. (4) includes simple nonlinear materials as a particular case with $W_1 = 0$. So one can treat (4) as a certain regularization of constitutive relations of simple materials.

The Lagrangian equations of equilibrium take the form

$$\nabla\cdot\left(\mathbf{P}_0 - \nabla\cdot\mathbf{P}_1\right) + \rho\mathbf{f} = \mathbf{0}, \tag{5}$$

where

$$\mathbf{P}_0 = \frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{P}_1 = \frac{\partial W}{\partial \mathbf{G}}$$

are first Piola–Kirchhoff stress and double stress tensors, respectively, ρ is a referential mass density and $\mathbf{f}$ is a vector mass forces. For simple materials $\mathbf{P}_1 = \mathbf{0}$ and (5) transforms into the classic equilibrium equation [11]

$$\nabla\cdot\mathbf{P}_0 + \rho\mathbf{f} = \mathbf{0}. \tag{6}$$

In what follows, we consider an undistorted reference placement, i.e. energy density and stresses vanish when deformation is absent: $W(\mathbf{1}, \mathbf{0}) = 0$, $\mathbf{P}_0(\mathbf{1}, \mathbf{0}) = \mathbf{0}$, and $\mathbf{P}_1(\mathbf{1}, \mathbf{0}) = \mathbf{0}$. Here $\mathbf{1}$ is the 3D unit tensor. In addition we assume that the double stresses vanish if $\mathbf{G} = \mathbf{0}$:

$$\mathbf{P}_1(\mathbf{F}, \mathbf{0}) = \mathbf{0}. \tag{7}$$

Eq. (7) seems to be a natural assumption as $\mathbf{P}_1$ is conjugated to $\mathbf{G}$ and should vanish if the second deformation gradient is zero.

3. Strong ellipticity

Eqs. (5) and (6) constitute systems of nonlinear partial differential equations of fourth- and second-order, respectively. Using the general theory of partial differential equations [27,28], we can characterize the properties of PDEs under consideration introducing the following strong ellipticity (SE) conditions

$$(\mathbf{k}\otimes\mathbf{a}) : \frac{\partial^2 W}{\partial \mathbf{F}^2} : (\mathbf{k}\otimes\mathbf{a}) \geq C_0(\mathbf{k}\cdot\mathbf{k})(\mathbf{a}\cdot\mathbf{a}), \tag{8}$$

$$(\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a}) \,\vdots\, \frac{\partial^2 W}{\partial \mathbf{G}^2} \,\vdots\, (\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a}) \geq C_1(\mathbf{k}\cdot\mathbf{k})^2(\mathbf{a}\cdot\mathbf{a}), \tag{9}$$

$$\forall \quad \mathbf{k}, \quad \mathbf{a},$$

where $\mathbf{k}$ and $\mathbf{a}$ are constant vectors, C_0 and C_1 are positive constant independent on $\mathbf{k}$ and $\mathbf{a}$, "$\otimes$", "$:$", and "$\vdots$" are dyadic, double dot, and triple dot products respectively. For example for dyads, triads, and tetrads these products result in

$$(\mathbf{a}\otimes\mathbf{b}) : (\mathbf{c}\otimes\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}),$$
$$(\mathbf{a}\otimes\mathbf{b}) : (\mathbf{c}\otimes\mathbf{d}\otimes\mathbf{e}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})\mathbf{e},$$
$$(\mathbf{c}\otimes\mathbf{d}\otimes\mathbf{e}) : (\mathbf{a}\otimes\mathbf{b}) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{e})\mathbf{c},$$
$$(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) \,\vdots\, (\mathbf{d}\otimes\mathbf{e}\otimes\mathbf{f}) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f}),$$
$$(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) \,\vdots\, (\mathbf{d}\otimes\mathbf{e}\otimes\mathbf{f}\otimes\mathbf{g}) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f})\mathbf{g},$$
$$(\mathbf{d}\otimes\mathbf{e}\otimes\mathbf{f}\otimes\mathbf{g}) \,\vdots\, (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = (\mathbf{a}\cdot\mathbf{e})(\mathbf{b}\cdot\mathbf{f})(\mathbf{c}\cdot\mathbf{g})\mathbf{d},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, ... are arbitrary vectors.

Eq. (8) is the strong ellipticity condition used in nonlinear elasticity [11–13], whereas (9) is the strong ellipticity condition within the strain gradient elasticity, see e.g. [29]. Obviously, since the strong ellipticity is determined by highest-order differential terms, both definitions are independent. In other words, (9) does not imply (8) and vice versa. Within the strain gradient elasticity we call (8) *the first-order strong ellipticity* condition and (9) *the second-order strong ellipticity* one.

Inequalities (8) and (9) can be written as a certain convexity conditions

$$B_0(\mathbf{k}, \mathbf{a}) \equiv \frac{d^2}{dt^2} W_0(\mathbf{F} + t\mathbf{k}\otimes\mathbf{a})\Big|_{t=0} \geq C_0(\mathbf{k}\cdot\mathbf{k})(\mathbf{a}\cdot\mathbf{a}), \tag{10}$$

$$B_1(\mathbf{k}, \mathbf{a}) \equiv \frac{d^2}{dt^2} W_1(\mathbf{F}, \mathbf{G} + t\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a})\Big|_{t=0} \geq C_1(\mathbf{k}\cdot\mathbf{k})^2(\mathbf{a}\cdot\mathbf{a}). \tag{11}$$

So one can see that (8) and (9) or (10) and (11) relate to positivity of the second variation of the potential energy density on particular perturbations, i.e. perturbations in the form of dyads and triads, respectively. Moreover, (10) and (11) relate from the more general condition

$$B(\mathbf{k}, \mathbf{a}) \equiv \frac{d^2}{dt^2} W(\mathbf{F} + t\mathbf{k}\otimes\mathbf{a}, \mathbf{G} + t\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a})\Big|_{t=0} > 0, \quad \forall \quad \mathbf{k}, \mathbf{a} \neq \mathbf{0}. \tag{12}$$

Nevertheless, the inequalities (10) and (11) do not imply (12) as well as (12) does not imply the latter, in general. In fact, from (12) it follows only a weak form of (11) that is the inequality

$$B_1(\mathbf{k}, \mathbf{a}) \geq 0, \quad \forall \quad \mathbf{k}, \mathbf{a}, \tag{13}$$

which has a sense of Hadamard's inequality in nonlinear elasticity [11–13]. If B_1 is degenerated for some $\mathbf{k}$, then B_0 should be positive for the same value of $\mathbf{k}$. In other words, from (12) it follows that B_1 is non-negative. Moreover, if $B_1(\mathbf{k}, \mathbf{a}) = 0$ for some $\mathbf{k} = \mathbf{k}^\star$ then $B_0(\mathbf{k}^\star, \mathbf{a}) > 0$.

4. Ellipticity and stability

In the framework of nonlinear elasticity of simple materials it is known that the strong ellipticity condition relates to infinitesimal stability [11–13]. Indeed, we have implications

- positive definiteness of the second variation of the total energy functional implies the weak form of strong ellipticity (SE), i.e. Hadamard's inequality;
- the strong ellipticity implies the infinitesimal stability of affine deformations of a homogeneous body with clamped boundary.

Let us discuss the strong ellipticity and stability within the strain gradient elasticity.

4.1. Infinitesimal stability

In what follows let us restrict ourselves to conservative external forces, couples, double forces and other admissible type of loads. In other words, we assume that there exists a functional $\mathcal{A}$ such that its first variation $\delta\mathcal{A}$ gives the work of external loadings. Thus equilibrium of the solid can be described by the Lagrange variational principle for admissible displacements

$$\delta\mathcal{L} \equiv \delta\mathcal{E} - \delta\mathcal{A} = 0, \qquad \mathcal{E} = \int_V W\, dV.$$

It states that under certain conditions a solution of a corresponding boundary-value problem is a stationary point of $\mathcal{L}$ and vice versa. Here $V \subset \mathbb{R}^3$ is a volume which $\mathcal{B}$ occupies in the reference placement. Static stability of a solution is based on the analysis of the second variation of $\mathcal{L}$. The second variation takes the form

$$\delta^2\mathcal{L} = \delta^2\mathcal{E}(\mathbf{v}) = \int_V \delta^2 W\, dV, \tag{14}$$

$$\begin{aligned}\delta^2 W =& \frac{d^2}{dt^2} W(\mathbf{F}+t\delta\mathbf{F}, \mathbf{G}+t\delta\mathbf{G})\bigg|_{t=0}\\ =&\delta\mathbf{F} : \frac{\partial^2 W}{\partial \mathbf{F}^2} : \delta\mathbf{F} + \delta\mathbf{G} \,\vdots\, \frac{\partial^2 W}{\partial\mathbf{G}\partial\mathbf{F}} : \delta\mathbf{F}\\ &+ \delta\mathbf{F} : \frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{G}} \,\vdots\, \delta\mathbf{G} + \delta\mathbf{G} \,\vdots\, \frac{\partial^2 W}{\partial \mathbf{G}^2} \,\vdots\, \delta\mathbf{G}\\ =&\nabla\mathbf{v} : \frac{\partial^2 W}{\partial \mathbf{F}^2} : \nabla\mathbf{v} + \nabla\nabla\mathbf{v} \,\vdots\, \frac{\partial^2 W}{\partial\mathbf{G}\partial\mathbf{F}} : \nabla\mathbf{v}\\ &+ \nabla\mathbf{v} : \frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{G}} \,\vdots\, \nabla\nabla\mathbf{v} + \nabla\nabla\mathbf{v} \,\vdots\, \frac{\partial^2 W}{\partial \mathbf{G}^2} \,\vdots\, \nabla\nabla\mathbf{v},\end{aligned} \tag{15}$$

where $\mathbf{v} = \delta\mathbf{x}$. In what follows we consider $\mathbf{v}$ as a twice differentiable functions, $\mathbf{v} \in C_2(V)$. Note that in this definition we have explicitly assumed that $\delta^2\mathcal{A} = 0$. In the following we assume that a part S_0 of the boundary $S = \partial V$ is clamped. We call a solution *stable* if

$$\delta^2\mathcal{E}(\mathbf{v}) > 0, \quad \forall\ \mathbf{v} \neq \mathbf{0} : \quad \mathbf{v}\Big|_{S_0} = \mathbf{0}, \quad \frac{\partial\mathbf{v}}{\partial n}\Big|_{S_0} = \mathbf{0}, \tag{16}$$

where $\partial/\partial n$ denotes the normal derivative.

Let $\mathbf{x} = \mathbf{x}_0$ be a stable solution. So (15) takes the form

$$\begin{aligned}\delta^2 W =& \nabla\mathbf{v} : \mathbf{C}_0(\mathbf{X}) : \nabla\mathbf{v} + \nabla\nabla\mathbf{v} \,\vdots\, \mathbf{C}_1(\mathbf{X}) \,\vdots\, \nabla\nabla\mathbf{v}\\ &+ \nabla\nabla\mathbf{v} \,\vdots\, \mathbf{C}_2(\mathbf{X}) : \nabla\mathbf{v} + \nabla\mathbf{v} : \mathbf{C}_3(\mathbf{X}) \,\vdots\, \nabla\nabla\mathbf{v},\end{aligned} \tag{17}$$

where

$$\mathbf{C}_0 = \frac{\partial^2 W}{\partial \mathbf{F}^2}\bigg|_{\mathbf{F}=\mathbf{F}_0, \mathbf{G}=\mathbf{G}_0}, \quad \mathbf{C}_1 = \frac{\partial^2 W}{\partial \mathbf{G}^2}\bigg|_{\mathbf{F}=\mathbf{F}_0, \mathbf{G}=\mathbf{G}_0},$$
$$\mathbf{C}_2 = \frac{\partial^2 W}{\partial\mathbf{G}\partial\mathbf{F}}\bigg|_{\mathbf{F}=\mathbf{F}_0, \mathbf{G}=\mathbf{G}_0}, \quad \mathbf{C}_3 = \frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{G}}\bigg|_{\mathbf{F}=\mathbf{F}_0, \mathbf{G}=\mathbf{G}_0}$$

are continuous tensor-valued functions, and $\mathbf{F}_0 = \nabla\mathbf{x}_0$, $\mathbf{G}_0 = \nabla\mathbf{F}_0$.

Let us show that for any point $\mathbf{X}_0 \in V$, $\mathbf{X}_0 \notin S \equiv \partial V$, inequality (16) implies (13). Applying the partition of unity technique [27] we can prove that the problem under consideration reduces to the problem with coefficients "frozen" at $\mathbf{X} = \mathbf{X}_0$. So we consider

$$\begin{aligned}\delta^2 W =& \nabla\mathbf{v} : \mathbf{C}_0(\mathbf{X}_0) : \nabla\mathbf{v} + \nabla\nabla\mathbf{v} \,\vdots\, \mathbf{C}_1(\mathbf{X}_0) \,\vdots\, \nabla\nabla\mathbf{v}\\ &+ \nabla\nabla\mathbf{v} \,\vdots\, \mathbf{C}_2(\mathbf{X}_0) : \nabla\mathbf{v} + \nabla\mathbf{v} : \mathbf{C}_3(\mathbf{X}_0) \,\vdots\, \nabla\nabla\mathbf{v}.\end{aligned} \tag{18}$$

Now let us consider a vector-valued function $\mathbf{v}$ with finite support, $\mathbf{v} = \mathbf{v}_\varepsilon$, $\operatorname{supp}\mathbf{v}_\varepsilon = V_\varepsilon \equiv \{\mathbf{X} : |\mathbf{X}-\mathbf{X}_0| \le \varepsilon\}$, where ε is a small positive number. We can take $\mathbf{v}_\varepsilon$ as follows

$$\mathbf{v}_\varepsilon = f\left(\frac{X_1 - X_1^0}{\varepsilon}\right) f\left(\frac{X_2 - X_2^0}{\varepsilon}\right) f\left(\frac{X_3 - X_3^0}{\varepsilon}\right)\mathbf{a},$$

where $\mathbf{a}$ is a constant vector, and $f(X)$ is an even function in $C_0^\infty[-1,1]$, such that

$$\int_{-1}^{1} f(X)\, dX = 1, \quad f(\pm 1) = 0, \quad f'(\pm 1) = 0.$$

For example, the bump function

$$f(X) = \begin{cases} \exp\left(\frac{1}{X^2-1}\right), & |X| \le 1,\\ 0, \quad |X| > 1. \end{cases}$$

could be taken.

As a result, $\delta^2\mathcal{E}$ takes the form

$$\begin{aligned}\delta^2\mathcal{E}(\mathbf{v}) = \int_{V_\varepsilon} &\big[\nabla\mathbf{v}_\varepsilon : \mathbf{C}_0(\mathbf{X}_0) : \nabla\mathbf{v}_\varepsilon\\ &+ \nabla\nabla\mathbf{v}_\varepsilon \,\vdots\, \mathbf{C}_1(\mathbf{X}_0) \,\vdots\, \nabla\nabla\mathbf{v}_\varepsilon\big]\, dV.\end{aligned} \tag{19}$$

Changing variables in (19) as follows $\mathbf{y} = \varepsilon^{-1}(\mathbf{X}-\mathbf{X}_0)$, we get

$$\delta^2\mathcal{E}(\mathbf{v}) = \varepsilon J_0(\mathbf{v}) + \varepsilon^{-1} J_1(\mathbf{v}), \tag{20}$$

where

$$J_0(\mathbf{v}) = \int_{V_1} \nabla\mathbf{v} : \mathbf{C}_0(\mathbf{X}_0) : \nabla\mathbf{v}\, dV,$$
$$J_1(\mathbf{v}) = \int_{V_1} \nabla\nabla\mathbf{v} \,\vdots\, \mathbf{C}_1(\mathbf{X}_0) \,\vdots\, \nabla\nabla\mathbf{v}\, dV,$$

and $\mathbf{v} = \mathbf{v}_\varepsilon\big|_{\varepsilon=1}$, $V_1 = V_\varepsilon\big|_{\varepsilon=1}$, $\nabla_y = \varepsilon^{-1}\nabla$. So we can conclude that (16) it follows that $J_1(\mathbf{v}) \ge 0$ for all $\mathbf{v}$. Moreover, if there exists such $\mathbf{v} = \mathbf{v}^\star \neq \mathbf{0}$ that $J_1(\mathbf{v}^\star) = 0$ then $J_0(\mathbf{v}^\star) > 0$ otherwise (16) will be violated. As we consider $\mathbf{X}_0$ as an arbitrary point in V, we conclude that

$$J_1(\mathbf{v}) \equiv \int_V \nabla\nabla\mathbf{v} \,\vdots\, \mathbf{C}_1(\mathbf{X}) \,\vdots\, \nabla\nabla\mathbf{v}\, dV \ge 0, \tag{21}$$
$$\forall\mathbf{v} : \quad \mathbf{v}\Big|_S = \mathbf{0}, \quad \frac{\partial\mathbf{v}}{\partial n}\Big|_S = \mathbf{0}.$$

Finally, let us show that (21) results in (13). Let us assume the opposite, i.e. that there is a point $\mathbf{X}_0$ such that $B_1(\mathbf{k}^\star, \mathbf{a}) < 0$ for some $\mathbf{k} = \mathbf{k}^\star$. Since $\mathbf{C}_1$ is a continuous function there is a neighbourhood $V_\varepsilon \equiv \{\mathbf{X} : |\mathbf{X}-\mathbf{X}_0| \le \varepsilon\}$ for a small $\varepsilon > 0$, such that for all $\mathbf{X} \in V_\varepsilon$ $B_1(\mathbf{k}^\star, \mathbf{a}) < 0$. Let us consider $\mathbf{v} = \cos(\lambda\mathbf{k}^\star \cdot (\mathbf{X}-\mathbf{X}_0))\varphi(\mathbf{X})\mathbf{a}$, where $\varphi(\mathbf{X})$ ia a smooth function with finite support, $\operatorname{supp}\varphi \subset V_\varepsilon$, $\mathbf{a}$ is a constant vector, and λ is a positive number. Then we can see that

$$J_1(\mathbf{v}) = \lambda^4 B_1(\mathbf{k}^\star, \mathbf{a}) \int_{V_\varepsilon} \varphi^2(\mathbf{X}) \cos^2(\lambda\mathbf{k}^\star \cdot (\mathbf{X}-\mathbf{X}_0))\, dV + o(\lambda^4).$$

Considering a large enough λ we see that $J_1(\mathbf{v}) < 0$ that contradicts (21). Thus we come to (13).

4.2. Stability of affine deformations

In order to analyse the converse statement, that is the ellipticity implies some stability results, let us consider a particular class of deformations and boundary conditions. An affine deformation is given by

$$\mathbf{F} = \mathbf{F}_0 = \text{const}, \quad \mathbf{G} = \mathbf{0} \tag{22}$$

of a homogeneous body with the kinematic boundary conditions on the whole boundary

$$\mathbf{v}\Big|_S = \mathbf{0}, \quad \frac{\partial\mathbf{v}}{\partial n}\Big|_S = \mathbf{0}. \tag{23}$$

From (7) and (22) it follows that $\mathbf{P}_1 = \mathbf{0}$ and

$$\frac{\partial^2 W}{\partial \mathbf{F}\partial \mathbf{G}} = \frac{\partial^2 W}{\partial \mathbf{G}\partial \mathbf{F}} = \mathbf{0}.$$

So $\delta^2 W$ became a quadratic form with constant coefficients

$$\delta^2 W = \nabla \mathbf{v} : \mathbf{C}_0 : \nabla \mathbf{v} + \nabla\nabla \mathbf{v} \vdots \mathbf{C}_1 \vdots \nabla\nabla \mathbf{v}, \tag{24}$$
$$\mathbf{C}_0 = \left.\frac{\partial^2 W}{\partial \mathbf{F}^2}\right|_{\mathbf{F}=\mathbf{F}_0, \mathbf{G}=\mathbf{0}}, \quad \mathbf{C}_1 = \left.\frac{\partial^2 W}{\partial \mathbf{G}^2}\right|_{\mathbf{F}=\mathbf{F}_0, \mathbf{G}=\mathbf{0}}.$$

For $\mathbf{v}$ satisfying (23) we can extend it to the whole space as follows

$$\mathbf{u}(\mathbf{X}) = \begin{cases} \mathbf{v}(\mathbf{X}), & \mathbf{X} \in V; \\ \mathbf{0}, & \mathbf{X} \in \mathbb{R}^3 \backslash V. \end{cases}$$

So we get

$$\delta^2 \mathcal{E} = \int_{\mathbb{R}^3} \left(\nabla \mathbf{u} : \mathbf{C}_0 : \nabla \mathbf{u} + \nabla\nabla \mathbf{u} \vdots \mathbf{C}_1 \vdots \nabla\nabla \mathbf{u} \right) dX_1\, dX_2\, dX_3. \tag{25}$$

Here we extend the proof given in [11] or in general form in [27]. We use the Fourier transform of $\mathbf{u}$ and the Plancherel theorem, see e.g. [30]. The direct and inverse transforms are given by

$$\hat{\mathbf{u}}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-\mathbf{k}\cdot\mathbf{X}} \mathbf{u}(\mathbf{X})\, dX_1\, dX_2\, dX_3,$$
$$\mathbf{u}(\mathbf{X}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{\mathbf{k}\cdot\mathbf{X}} \hat{\mathbf{u}}(\mathbf{k})\, dk_1\, dk_2\, dk_3,$$

where i is the imaginary unit, $i^2 = -1$. The Plancherel theorem states that for two functions $f(x)$, $g(x) \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x) g(x)\, dx = \int_{\mathbb{R}} \hat{f}(k) \bar{\hat{g}}(k)\, dk.$$

Hereinafter the overbar stands for the complex conjugate quantities and L_p denotes the Lebesgue spaces [31]. Using the technique [11] we replace $\delta^2 \mathcal{E}$ by

$$\delta^2 \mathcal{E} = \int_{\mathbb{R}^3} \Big[(\mathbf{k} \otimes \hat{\mathbf{u}}(\mathbf{k})) : \mathbf{C}_0 : (\mathbf{k} \otimes \bar{\hat{\mathbf{u}}}(\mathbf{k}))$$
$$+ (\mathbf{k} \otimes \mathbf{k} \otimes \hat{\mathbf{u}}(\mathbf{k})) \vdots \mathbf{C}_1 \vdots (\mathbf{k} \otimes \mathbf{k} \otimes \bar{\hat{\mathbf{u}}}(\mathbf{k})) \Big]\, dk_1\, dk_2\, dk_3. \tag{26}$$

Using symmetries of $\mathbf{C}_0$ and $\mathbf{C}_1$ and (8) and (9) we came to the inequality

$$\delta^2 \mathcal{E} \ge \int_{\mathbb{R}^3} \Big[C_0 (\mathbf{k} \otimes \hat{\mathbf{u}}(\mathbf{k})) : (\mathbf{k} \otimes \bar{\hat{\mathbf{u}}}(\mathbf{k}))$$
$$+ C_1 (\mathbf{k} \otimes \mathbf{k} \otimes \hat{\mathbf{u}}(\mathbf{k})) \vdots (\mathbf{k} \otimes \mathbf{k} \otimes \bar{\hat{\mathbf{u}}}(\mathbf{k})) \Big]\, dk_1\, dk_2\, dk_3$$
$$= \int_{\mathbb{R}^3} \left[C_0 \nabla \mathbf{u} : \nabla \mathbf{u} + C_1 \nabla\nabla \mathbf{u} \vdots \nabla\nabla \mathbf{u} \right] dk_1\, dk_2\, dk_3$$
$$= \int_V \left[C_0 \nabla \mathbf{v} : \nabla \mathbf{v} + C_1 \nabla\nabla \mathbf{v} \vdots \nabla\nabla \mathbf{v} \right] dV. \tag{27}$$

Thus, using *both* SE conditions we have proven the infinitesimal stability of an affine deformations. In other words (8) and (9) play a role of conditions which are *sufficient* for stability. In other words, within the strain gradient elasticity SE condition (9) does not imply stability, in general.

In fact, SE condition (8) could be violated. Indeed, from (23) it follows that $\nabla \mathbf{u}\big|_S = \mathbf{0}$ and one can apply Friedrichs' inequality [31]

$$\|\nabla \mathbf{u}\|_{L_2(V)} \le C_3 \|\nabla\nabla \mathbf{u}\|_{L_2(V)}$$

with a positive constant C_3 which depends on V. So for the positive definiteness of $\delta^2 \mathcal{E}$ it is enough to require the inequality $C_0 > -C_3$.

Moreover, considering the given proof one can see that even 2nd SE condition could be also violated. Indeed, we replace (8) and (9) by their combination

$$(\mathbf{k} \otimes \mathbf{a}) : \frac{\partial^2 W}{\partial \mathbf{F}^2} : (\mathbf{k} \otimes \mathbf{a})$$
$$+ (\mathbf{k} \otimes \mathbf{k} \otimes \mathbf{a}) \vdots \frac{\partial^2 W}{\partial \mathbf{G}^2 F} \vdots (\mathbf{k} \otimes \mathbf{k} \otimes \mathbf{a}) \ge P(\mathbf{k})(\mathbf{a} \cdot \mathbf{a}), \tag{28}$$

where $P(\mathbf{k})$ is a polynomial of $\mathbf{k} = (k_1, k_2, k_3)$ such that

$$P(\mathbf{k}) = A_{mn} k_m k_n + B_{mnpt} k_m k_n k_p k_t \ge \alpha_j k_j^2 + \beta_j k_j^4 \tag{29}$$

with coefficients A_{mn} and B_{mnpt}. Here α_j and β_j, $j = 1, 2, 3$, are positive constants which cannot be both vanished, $\alpha_j + \beta_j > 0$ for each j. For example, such P is admissible

$$P = \alpha_1 k_1^2 + \beta_2 k_2^4 + \beta_3 k_3^4.$$

Obviously, inequality (28) with (29) do not constitute a condition of strong ellipticity. Nevertheless, under certain conditions the uniqueness of a solution can be proven even in this case using the anisotropic Sobolev spaces, see [19] for details. In other words, within the strain gradient elasticity (9) is not a necessary condition for infinitesimal stability, in general.

Conclusions

We have formulated two inequalities (8) and (9) of strong ellipticity of a simple elastic material and its extension towards a strain gradient material, respectively. For nonlinear strain gradient elasticity we call these conditions first- and second-order strong ellipticity conditions (1st SE and 2nd SE). We have proven three implications:

- infinitesimal stability, i.e. the positive definiteness of $\delta^2 \mathcal{E}$, implies weak 2nd SE condition;
- 1st SE and 2nd SE conditions imply infinitesimal stability of affine deformations of a homogeneous body $\mathcal{B}$ with Dirichlet-type boundary conditions. So 1st SE and 2nd SE are sufficient conditions;
- under certain conditions even without 1st SE condition, 2nd SE condition may result in infinitesimal stability of deformations described above in the previous case.

Let us note that the first case has the same form as in nonlinear elasticity, whereas other implications present more complex picture of interrelations between SE and stability. Nevertheless, unlike nonlinear elasticity the relation between SE condition and infinitesimal stability is not straightforward. Indeed, stability could be proven even for not strongly elliptic systems.

These implications mean that both SE conditions may be useful for the analysis of material instabilities in solids modelled within the strain gradient elasticity as they may guarantee stability of some deformations whereas their violation may indicate some instabilities. In particular, a strain gradient regularization may eliminate some instabilities observed within the model of simple materials. In a certain sense this phenomenon is similar to strain localization with strain gradient plasticity, see e.g. [24,32,33].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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