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Strong ellipticity within the Toupin–Mindlin first strain gradient elasticity theory

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A B S T R A C T

We discuss the strong ellipticity (SE) condition within the Toupin–Mindlin first strain gradient elasticity theory. SE condition is closely related to certain material instabilities and describes mathematical properties of corresponding boundary-value problems. For isotropic solids, SE condition transforms into two inequalities in terms of five gradient-elastic moduli.

1. Introduction

Nowadays, strain gradient elasticity theory found various applications in modeling of solids and fluids at small scales, in modeling of crystal-defects at small scales as well as in description of composite materials with essential difference of material properties of constituents such as in the case of beam-lattice metamaterials, see e.g., [1–8]. In particular, since the model has internal length-scale parameters, it can capture size-effects observed at the nanoscale. Within the model a potential energy density is introduced as a function of the first and second gradients of a displacement vector. Among the various models describing solids undergoing infinitesimal deformations, it is worth to mention the Toupin–Mindlin gradient approach [9–12], which is the most straightforward version of first strain gradient elasticity theory respecting group theory. For isotropic materials, first strain gradient elasticity theory contains two Lamé constants and five strain gradient parameters leading to two characteristic lengths-scale parameters. Moreover, Toupin and Grazis [13] and Mindlin [14] (see also [15]) showed that first strain gradient elasticity, which is sometimes called gradient elasticity of grade-2, might be considered as the continuum version of a lattice theory with up to second-neighbor interactions (nearest and next-nearest neighbor interactions).

From the mathematical point of view, the Toupin–Mindlin first strain gradient elasticity results in a system of partial differential equations (PDEs) of fourth-order. Its mathematical properties could be described using the general theory of elliptic PDEs [16–18]. Let us note that in the literature, one can find various definitions of ellipticity, see [19–22]. In what follows we use the strong ellipticity condition as used in linear and nonlinear elasticity [23–27]. The SE condition ensures that the governing PDE for elastostatic problems be completely elliptic [27]. It guaranties some “natural properties” of solutions such as existence, uniqueness, and regularity. Violation of SE conditions could be treated as a certain material instabilities, see [23,24] for nonlinear elasticity and [28–30] for nonlinear strain gradient elasticity. Moreover, an important mathematical property of gradient elasticity is that it provides a mathematical regularization based on PDEs of higher-order where the characteristic length-scale parameters play the role of regularization parameters [31].

The aim of this paper is to formulate SE condition within the linear Toupin–Mindlin first strain gradient elasticity theory of isotropic solids.

2. Toupin–Mindlin first strain gradient elasticity and strong ellipticity

Within the Toupin–Mindlin first strain gradient elasticity [9–12] there exists a strain energy density W as a function of strain tensor $\boldsymbol{\varepsilon}$ and its gradient

$$W = W(\boldsymbol{\varepsilon}, \operatorname{grad}\boldsymbol{\varepsilon}),$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}(\operatorname{grad}\boldsymbol{u} + \operatorname{grad}\boldsymbol{u}^T)$ and $\boldsymbol{u}$ is a displacement vector. In Cartesian coordinates, we have

$$\boldsymbol{u} = u_k \mathbf{i}_k, \quad \operatorname{grad}\boldsymbol{u} = \partial_m u_n \mathbf{i}_n \otimes \mathbf{i}_m,$$

$$\boldsymbol{\varepsilon} = \varepsilon_{mn}\mathbf{i}_m \otimes \mathbf{i}_n, \quad \varepsilon_{mn} = \frac{1}{2}(\partial_m u_n + \partial_n u_m),$$

$\partial_k = \partial/\partial x_k$, "$\otimes$" is a dyadic product, x_k and $\mathbf{i}_k$ are Cartesian coordinates and corresponding unit base vectors, respectively. In addition, Einstein's summation rule is utilized.

Following [11], we introduce a strain energy density as a quadratic form of $\boldsymbol{\varepsilon}$ and grad $\boldsymbol{\varepsilon}$

$$W = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{E} \,\vdots\, \mathrm{grad}\,\boldsymbol{\varepsilon} + \frac{1}{2}\mathrm{grad}\,\boldsymbol{\varepsilon} \,\vdots\, \mathbb{D} \,\vdots\, \mathrm{grad}\,\boldsymbol{\varepsilon}, \tag{1}$$

where ":" and "$\vdots$" are the double and triple dot products, $\mathbb{C}$, $\mathbb{E}$, and $\mathbb{D}$ are constitutive tensors of rank four, five, and six, respectively. For the major and minor symmetries of these constitutive tensors, we refer to [7,32–35].

The corresponding equilibrium equation can be written in the following form in terms of the displacement vector [31,35]

$$L(\partial)\,\boldsymbol{u} + \boldsymbol{f} = \mathbf{0}, \tag{2}$$

where $\boldsymbol{f}$ is the body force vector and $L(\partial)$ is the Mindlin differential operator. In Cartesian coordinates, Eq. (2) has the form

$$L_{ik}(\partial)\,u_k + f_i = 0, \qquad i,k = 1,2,3, \tag{3}$$

where the Mindlin operator of first strain gradient elasticity is given by (see, e.g., [35])

$$L_{ik}(\partial) = \mathbb{C}_{ijkl}\partial_j\partial_l + \left(\mathbb{E}_{ijklm} - \mathbb{E}_{klijm}\right)\partial_j\partial_l\partial_m - \mathbb{D}_{ijmkln}\partial_j\partial_l\partial_m\partial_n. \tag{4}$$

The strong ellipticity (SE) condition can be formulated as follows

$$(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{b}) \,\vdots\, \mathbb{D} \,\vdots\, (\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{b}) \geq C_1(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})^2, \tag{5}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary real vectors, C_1 is a positive constant independent of $\boldsymbol{a}$ and $\boldsymbol{b}$, and "$\cdot$" is the dot product. In Cartesian coordinates, Eq. (5) takes the form

$$a_i b_j b_m \mathbb{D}_{ijmkln} a_k b_l b_n \geq C_1 a_i a_i (b_j b_j)^2. \tag{6}$$

Note that the SE condition does not affect other elastic constitutive moduli, because the SE is determined by highest-order differential terms.

If we neglect in Eq. (1) the first strain gradient terms, i.e. assume that $\mathbb{E} = 0$ and $\mathbb{D} = 0$, we recover the strain energy density of classical elasticity

$$W = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}. \tag{7}$$

For classical elasticity, the strong ellipticity condition takes the form

$$(\boldsymbol{a}\otimes\boldsymbol{b}) : \mathbb{C} : (\boldsymbol{a}\otimes\boldsymbol{b}) \geq C_0(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}), \tag{8}$$

or, in components,

$$a_i b_j \mathbb{C}_{ijkl} a_k b_l \geq C_0 a_i a_i b_j b_j \tag{9}$$

with a positive constant C_0.

In order to distinguish SE conditions for strain gradient and classical elasticity, we call (5) and (8) the *second-order* and the *first-order* strong ellipticity conditions, respectively, see [29] for nonlinear case.

3. Strong ellipticity conditions for isotropic solids

For an isotropic solid $\mathbb{E} = 0$, the isotropic representations of the components of the constitutive tensors $\mathbb{C}$ and $\mathbb{D}$ have the form

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{10}$$

and

$$\begin{aligned}\mathbb{D}_{ijmkln} = {} & \frac{a_1}{2}\left(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lmn} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{kl}\delta_{in}\delta_{jm}\right) \\ & + 2a_2\delta_{ij}\delta_{kl}\delta_{mn} \\ & + \frac{a_3}{2}\left(\delta_{jk}\delta_{im}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{jl}\delta_{im}\delta_{kn}\right) \\ & + a_4\left(\delta_{il}\delta_{jk}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn}\right) \\ & + \frac{a_5}{2}\left(\delta_{jk}\delta_{in}\delta_{lm} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{jl}\delta_{km}\delta_{in} + \delta_{il}\delta_{km}\delta_{jn}\right),\end{aligned} \tag{11}$$

Table 1
Lamé moduli, Voigt-type averaged isotropic gradient-elastic moduli and characteristic length-scale parameters for aluminum (Al) and tungsten (W) computed from second nearest-neighbor modified-embedded-atom-method (2NN MEAM) interatomic potential [35].

	Al (fcc)	W (bcc)
λ [eV/Å^3]	0.38649	1.28028
μ [eV/Å^3]	0.19704	1.01812
a_1 [eV/Å]	−0.13862	0.02387
a_2 [eV/Å]	0.22500	0.19215
a_3 [eV/Å]	0.10877	0.43264
a_4 [eV/Å]	0.15309	0.54907
a_5 [eV/Å]	0.21632	0.28799
ℓ_1 [Å]	1.20272	0.94654
ℓ_2 [Å]	1.26566	0.94509

where δ_{ij} is Kronecker's symbol, λ and μ are the two Lamé elastic moduli, a_1, a_2, a_3, a_4, and a_5 are the five gradient-elastic moduli [11].

Using Eq. (11), we get the formula

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{b}) \,\vdots\, \mathbb{D} \,\vdots\, (\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{b}) &= 2a_1(\boldsymbol{a}\cdot\boldsymbol{b})^2\boldsymbol{b}\cdot\boldsymbol{b} + 2a_2(\boldsymbol{a}\cdot\boldsymbol{b})^2\boldsymbol{b}\cdot\boldsymbol{b} \\ & + \frac{a_3}{2}\left[3(\boldsymbol{a}\cdot\boldsymbol{b})^2\boldsymbol{b}\cdot\boldsymbol{b} + (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})^2\right] \\ & + a_4\left[(\boldsymbol{a}\cdot\boldsymbol{b})^2\boldsymbol{b}\cdot\boldsymbol{b} + (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})^2\right] \\ & + \frac{a_5}{2}\left[3(\boldsymbol{a}\cdot\boldsymbol{b})^2\boldsymbol{b}\cdot\boldsymbol{b} + (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})^2\right] \\ &= \left[2a_1 + 2a_2 + \frac{3}{2}a_3 + a_4 + \frac{3}{2}a_5\right](\boldsymbol{a}\cdot\boldsymbol{b})^2(\boldsymbol{b}\cdot\boldsymbol{b}) \\ & + \left[\frac{a_3}{2} + a_4 + \frac{a_5}{2}\right](\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})^2.\end{aligned} \tag{12}$$

Without loss of generality, we can use unit vector $\boldsymbol{b}$, $(\boldsymbol{b}\cdot\boldsymbol{b}) = 1$. Representing $\boldsymbol{a}$ as a sum

$$\boldsymbol{a} = a_{\parallel}\boldsymbol{b} + \boldsymbol{a}_{\perp}, \quad a_{\parallel} = \boldsymbol{a}\cdot\boldsymbol{b}, \quad \boldsymbol{a}_{\perp} = \boldsymbol{a} - a_{\parallel}\boldsymbol{b}, \tag{13}$$

we transform Eq. (12) into the form

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{b}) \,\vdots\, \mathbb{D} \,\vdots\, (\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{b}) &= 2\left[a_1 + a_2 + a_3 + a_4 + a_5\right]a_{\parallel}^2 \\ & + \left[\frac{a_3}{2} + a_4 + \frac{a_5}{2}\right](\boldsymbol{a}_{\perp}\cdot\boldsymbol{a}_{\perp}).\end{aligned} \tag{14}$$

As a result, the strong ellipticity condition (second-order SE condition) reduces to two inequalities

$$a_1 + a_2 + a_3 + a_4 + a_5 > 0, \quad a_3 + 2a_4 + a_5 > 0. \tag{15}$$

For classical elasticity, we have the formula

$$(\boldsymbol{a}\otimes\boldsymbol{b}) : \mathbb{C} : (\boldsymbol{a}\otimes\boldsymbol{b}) = (\lambda+\mu)(\boldsymbol{a}\cdot\boldsymbol{b})^2 + \mu(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}). \tag{16}$$

Using (13), we transform (16) into

$$(\boldsymbol{a}\otimes\boldsymbol{b}) : \mathbb{C} : (\boldsymbol{a}\otimes\boldsymbol{b}) = (\lambda+2\mu)a_{\parallel}^2 + \mu(\boldsymbol{a}_{\perp}\cdot\boldsymbol{a}_{\perp}), \tag{17}$$

that recovers the classic strong ellipticity conditions (first-order SE condition) as two inequalities for the Lamé moduli

$$\lambda + 2\mu > 0, \quad \mu > 0. \tag{18}$$

Note that the five gradient-elastic moduli and the two Lamé moduli for aluminum (Al) and tungsten (W) computed from second nearest-neighbor modified-embedded-atom-method given in Table 1 satisfy the second SE condition (15) and the first SE condition (18), respectively.

4. On Mindlin's isotropic operator

For an isotropic material, the Mindlin operator takes the form [31]

$$L_{ik}(\partial) = (\lambda+2\mu)\left[1 - \ell_1^2\Delta\right]\partial_i\partial_k + \mu\left[1 - \ell_2^2\Delta\right](\delta_{ik}\Delta - \partial_i\partial_k), \tag{19}$$

where Δ is the 3D Laplace operator and two characteristic lengths have been introduced by the formulae

$$\ell_1^2 = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{\lambda + 2\mu}, \tag{20}$$

$$\ell_2^2 = \frac{a_3 + 2a_4 + a_5}{2\mu}. \tag{21}$$

Obviously, if Eqs. (15) and (18) are fulfilled both characteristic lengths are meaningful, namely they are real length scales. If both SE conditions (15) and (18) are satisfied, then the Mindlin operator (19) is an elliptic one.

Note that the two characteristic length-scale parameters for aluminum (Al) and tungsten (W) computed from second nearest-neighbor modified-embedded-atom-method given in Table 1 are real and positive.

5. Strong ellipticity and uniqueness

Let us note that SE conditions guarantee uniqueness of solutions for the first boundary-value problem, that is for Eq. (2) complemented by Dirichlet boundary conditions on the whole boundary

$$\boldsymbol{u}\Big|_S = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{u}}{\partial n}\Big|_S = \boldsymbol{0}, \tag{22}$$

where S is a boundary of a solid and $\partial/\partial n$ denotes the normal derivative. Indeed, if *both* SE conditions, that is (15) and (18), are fulfilled the solution is unique, see, e.g., [29] for details. Thus, the first- and the second-order SE conditions are *sufficient* conditions for uniqueness of solutions. Note that SE conditions are less restrictive than the positive definiteness of W, which involves SE conditions. Nevertheless, Eqs. (15) and (18) are not *necessary* conditions. For example, uniqueness could be proven if we have relaxed inequalities

$$a_1 + a_2 + a_3 + a_4 + a_5 > 0, \quad \lambda + 2\mu = 0,$$
$$a_3 + 2a_4 + a_5 = 0, \quad \mu > 0. \tag{23}$$

or

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0, \quad \lambda + 2\mu > 0,$$
$$a_3 + 2a_4 + a_5 > 0, \quad \mu = 0. \tag{24}$$

These mathematical cases correspond to the following lengths $\ell_1 = \infty$, $\ell_2 = 0$, and $\ell_1 = 0$, $\ell_2 = \infty$, respectively. Note that for (23) or (24) Mindlin's operator is not elliptic, but hypoelliptic [22], see the discussion on reduced gradient models in [36].

6. Conclusions

In this work, we have established SE conditions of the Toupin–Mindlin first strain gradient elasticity theory for isotropic materials. Two SE conditions have been found for the five gradient-elastic moduli. Both the first order SE and the second SE conditions guarantee that the two characteristic lengths of the Toupin–Mindlin first strain gradient elasticity theory are real length-scale parameters. An analysis of inequalities of SE conditions for anisotropic strain gradient materials requires advanced calculations as in the case of linear elasticity, see, e.g., [27,37,38].

Data availability

No data was used for the research described in the article.

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