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# Structural Pounding Models with Hertz Spring and Nonlinear Damper

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**Abstract:** The aim of the study is to show a comparison between the nonlinear viscoelastic model and the Hertzdamp model, both of them considered as Hertz contact law force-based models in conjunction with nonlinear damper. The results for two different impact experiments as well as for shaking table experiments on pounding between two steel towers excited by harmonic waves are used in this study. In addition, a suit of thirty ground motion records from thirteen different earthquakes is applied to simulate pounding between two single degree of freedom systems of different period ratios. The results of the study show that the nonlinear viscoelastic model gives smaller simulation errors in the impact force time histories comparing to the Hertzdamp model. It also provides smaller displacement and acceleration amplifications of the pounding-involved structural response under earthquake excitation. On the other hand, the Hertzdamp model has been found to be more accurate than the nonlinear viscoelastic model in simulation of impact velocity for pounding of structures under harmonic excitation.

**Key words:** Hertz model, hertzdamp model, nonlinear viscoelastic model, pounding force, earthquakes

## INTRODUCTION

The nonlinear model of impacts based on the Hertz contact law has been used in numerous studies of earthquake-induced structural pounding (e.g., Jing and Young, 1991; Pantelides and Ma, 1998; Chau and Wei, 2001). It simulates the relation between the pounding force and deformation during impact more realistically than the linear models (Goldsmith, 1960). However, the application of the Hertz elastic spring alone prevents us from simulation of energy dissipation during impact.

Lankarani and Nikravesh (1990) presented an improved version of the Hertz contact model based on the Hertz contact law, where a nonlinear damper is used along with the nonlinear spring, to represent the dissipated energy in impact.

Marhefka and Orin (1999) developed a compliant contact model with a nonlinear spring in parallel with a nonlinear damping, in order to overcome problems in the use of rigid body models with Coulomb friction and to eliminate the tension forces arising in the Kelvin-Voigt model.

Muthukumar and DesRoches (2006) and Muthukumar (2003) used the Hertzdamp model, which is equivalent to the improved version of the Hertz contact law model (Lankarani and Nikravesh, 1990; Marhefka and Orin, 1999), to study pounding simulation in structural engineering. Two Single Degree Of Freedom (SDOF) systems with different period ratios and a suit of 27 ground motion records of different Peak Ground Acceleration (PGA) levels were used to compare the Hertzdamp, linear spring, Kelvin, Hertz and stereomechanical models. Numerical results indicate that the Hertz model provides adequate results at low PGA levels, while the Hertzdamp model is recommended at moderate and high PGA levels (Muthukumar and DesRoches, 2006).

Jankowski (2005b) proposed a nonlinear viscoelastic model for more precise simulation of structural pounding based on the Hertz law of contact. This model gave the smallest simulation errors in the response time histories of the structural pounding experiments considered in comparison with the Hertz model and the linear viscoelastic model introduced by Anagnostopoulos (1988, 2004). Additionally, this model and the linear viscoelastic model gave the smallest simulation errors in the pounding force time histories during contact.

The nonlinear viscoelastic model was used to investigate earthquake induced pounding between two buildings (Jankowski, 2008) as well as to determine the pounding force response spectra under ground motion

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excitation (Jankowski, 2005a, 2006b). Numerical results confirmed the ability of the nonlinear viscoelastic model to accurately simulate seismic pounding.

Two different models have been considered as improved versions of the Hertz model by adding a nonlinear damper. However, no comprehensive comparison has been given so far. The aim of this study is to compare the accuracy of the nonlinear viscoelastic model and the Hertzdamp model in seismic pounding analysis. The first comparison uses the results of two impact experiments as well as the results of shaking table experiments on pounding between two steel towers excited by harmonic waves. The second comparison applies a suite of thirty ground motion records from thirteen different earthquakes applied to pounding between two SDOF systems with varying period ratios.

Fung (2001, 2002) used the differential quadrature method to solve both dynamic problems governed by second-order ordinary differential equations in time and first order initial value problems. Civalek (2007) used the method of Harmonic Differential Quadrature (HDQ) for the nonlinear dynamic response of Multi Degree Of Freedom (MDOF) systems. In this study, we use Implicit Runge-Kutta (IRK) methods to solve the system of first order ordinary differential equations.

## HERTZ CONTACT LAW FORCE-BASED MODELS

**Hertz model:** A nonlinear spring stiffness, depending on elastic properties of the colliding structures, is used in the Hertz model to simulate structural pounding. The impact force between the two structures of masses $m_i$ (i = 1,2) follows the relation (Goldsmith, 1960; Lankarani and Nikravesh, 1990, 1994).

$$\begin{aligned} &F(t) = k_h \delta^{\frac{3}{2}}(t); && \delta(t) > 0 \\ &F(t) = 0; && \delta(t) \le 0 \end{aligned} \tag{1}$$

where, $\delta$ (t) is the relative displacement. Assuming that the colliding structures are spherical of density $\rho$ and the radius $R_i$ estimation can be calculated through the following relation (Goldsmith, 1960):

$$R_i = \sqrt[3]{\frac{3m_i}{4\pi\rho}}, \qquad i = 1,2 \tag{2}$$

The nonlinear spring stiffness $k_h$ is linked to the material properties and the radii of the colliding structures as stated through the following formula (Goldsmith, 1960):

$$k_h = \frac{4}{3\pi(h_1 + h_2)}\left[\frac{R_1 R_2}{R_1 + R_2}\right]^{1/2} \tag{3}$$

where, $h_1$ and $h_2$ are the material parameters defined by the formula (Goldsmith, 1960):

$$h_i = \frac{1-\gamma_i}{\pi E_i} \qquad i = 1,2 \tag{4}$$

Here, $\gamma_i$ and $E_i$ are Poisson's ratio and Young's modulus, respectively. When $R_2 \to \infty$, i.e., the second structure becomes a massive plane surface, the nonlinear spring stiffness $k_h$ is defined as (Goldsmith, 1960):

$$k_h = \frac{4}{3\pi(h_1 + h_2)} R_1^{1/2} \tag{5}$$

Since the Hertz model is fully elastic (Eq. 1), it does not allow us to consider the energy dissipation during the collision.

**Hertzdamp model:** Hertzdamp model has been considered by Muthukumar and DesRoches (2006) and Muthukumar (2003) to study the pounding phenomenon in the field of structural engineering. The energy loss during impact has been taken into account by adding nonlinear damping to the Hertz model. The pounding force is written as (Lankarani and Nikravesh, 1990):

$$\begin{aligned} &F(t) = k_h \delta^{\frac{3}{2}}(t)\left[1 + \frac{3(1-e^2)}{4(v_1 - v_2)}\dot{\delta}(t)\right]; && \delta(t) > 0 \\ &F(t) = 0; && \delta(t) \le 0 \end{aligned} \tag{6}$$

where, e is the coefficient of restitution (Goldsmith, 1960), $\dot{\delta}(t)$ is the relative velocity during contact and $v_1 - v_2$ is the relative approaching velocity prior to contact which can be expressed in terms of nonlinear spring and maximum deformation (Lankarani and Nikravesh, 1990):

$$(v_1 - v_2)^2 = \frac{4(m_1 + m_2)k_h}{5m_1 m_2}\delta_m^{5/2} \tag{7}$$

**Nonlinear viscoelastic model:** Another improved version of the Hertz model has been introduced by Jankowski (2005b) by connecting a nonlinear damper in unison with the nonlinear spring. The contact force for this model is expressed as:

$$\begin{aligned} &F(t) = \overline{\beta}\delta^{\frac{3}{2}}(t) + \overline{c}(t)\dot{\delta}(t); && \delta(t) > 0, && \dot{\delta} > 0 && \text{(approach period)} \\ &F(t) = \overline{\beta}\delta^{\frac{3}{2}}(t); && \delta(t) > 0, && \dot{\delta} \le 0 && \text{(restitution period)} \\ &F(t) = 0; && \delta(t) \le 0 \end{aligned} \tag{8}$$

where, $\overline{\beta}$ is the impact stiffness parameter and

$$\bar{c}(t) = 2\bar{\xi}\sqrt{\bar{\beta}\sqrt{\delta(t)}\frac{m_1 m_2}{m_1 + m_2}} \quad (9)$$

is the impact element's damping. Here $\bar{\xi}$ is an impact damping ratio corresponding to a coefficient of restitution e which can be defined as Jankowski (2006a):

$$\bar{\xi} = \frac{9\sqrt{5}}{2}\frac{1-e^2}{e(e(9\pi-16)+16)} \quad (10)$$

## MATERIALS AND METHODS

To investigate the performance of the Hertzdamp and the nonlinear viscoelastic models in capturing pounding, three different procedures of comparison are used. The first one is based on two impact experiments conducted for different types of structural members with various materials and contact surface geometries. The second procedure uses the shaking table experiments on pounding between two steel towers. The accuracy of each model in the first and second procedure is assessed by calculating the Normalized Error (NE) to indicate the difference between the experimental and numerical results (Jankowski, 2005b).

$$NE = \frac{\|\mathbf{F} - \bar{\mathbf{F}}\|}{\|\mathbf{F}\|}\cdot 100\% \quad (11)$$

where, $\mathbf{F}$ is the response time history obtained experimentally, $\bar{\mathbf{F}}$ is the response time history obtained numerically and $\|\cdot\|$ is the Euclidean norm. $\|\mathbf{F} - \bar{\mathbf{F}}\|$ and $\|\mathbf{F}\|$ in case of the time histories given in a discrete form can be calculated as:

$$\|\mathbf{F} - \bar{\mathbf{F}}\| = \sqrt{\sum_{i=1}^{n}(F_i - \bar{F}_i)^2}, \qquad \|\mathbf{F}\| = \sqrt{\sum_{i=1}^{n}F_i^2} \quad (12)$$

where, n is a number of values in the time history record.

The third procedure is based on simulation using thirty ground motion records from thirteen different earthquakes of Peak Ground Acceleration (PGA) levels ranging from 0.1 to 1 (Muthukumar and DesRoches, 2006; Muthukumar, 2003). The ground motion records were selected from PEER Strong Motion Database (http://peer.berkely.edu/smcat/). Two Single Degree Of Freedom (SDOF) systems of structures with equal masses and three different period ratios are subjected to the ground motion records. Two cases are considered. In the first case, the initial separation distance is chosen such that no pounding takes place. In the second case, the initial separation distance is chosen such that pounding occurs. The performance of the models is evaluated by comparing the numerically obtained displacement and acceleration amplifications to each other.

## COMPARISON BASED ON CONDUCTED IMPACT EXPERIMENTS

The results of two impact experiments are considered in this study.

**Steel-to-steel impact:** Goland *et al.* (1955) carried out an experiment to measure load time histories and strain propagation in a square beam of different dimensions subjected to sharp lateral impacts letting a steel ball with diameter ranging from $\frac{1}{8}$ to $\frac{9}{32}$ inch, drops onto the top of the beam from a specified height. A force gauge was used to measure the force-time history exerted by the ball on the beam. The strain gauges were placed at different locations on the beam to record the strain time histories at those locations. The dynamic equation of motion for pounding between a ball of mass $m_1$ dropping onto a beam can be written by drawing the free body diagram (Jankowski, 2005b) as shown in Fig. 1:

$$m_1\ddot{u}_1(t) + F(t) = m_1 g \quad (13)$$

where, $\ddot{u}_1(t)$, g and F(t) denote the acceleration, gravitational acceleration and the pounding force, respectively. The pounding force follows the relations (6) or (8). In the experiment the maximum pounding force was 80.7 N when a ball of diameter $\frac{5}{32}$ inch fell from a height of 2 inches (Jankowski, 2005b). As for the stiffness parameter in the nonlinear viscoelastic model, we set $\bar{\beta}$ = 1.03 $10^{10}$ N/m$^{3/2}$ which was determined through an iterative procedure in order to keep the maximum pounding force in the numerical analysis and experiment

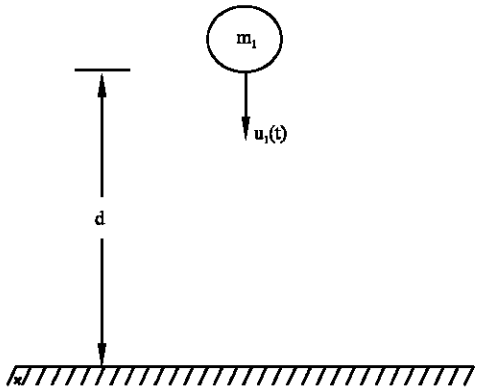


Fig. 1: Free body diagram of a ball dropping onto a beam

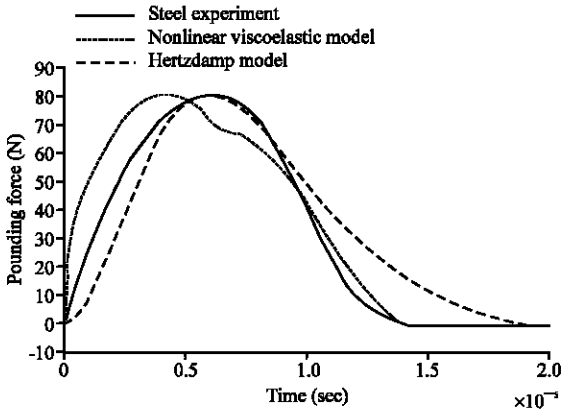


Fig. 2: Pounding force time histories during impact between falling steel ball and a steel hemisphere mounted on a beam

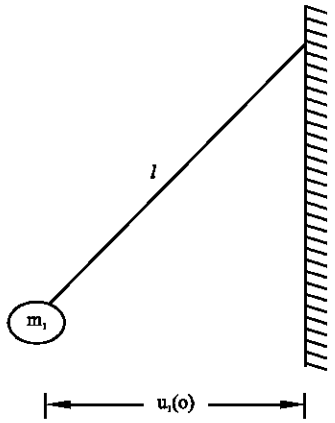


Fig. 3: Free body diagram for pounding between a concrete pendulum striker and prestressed concrete pile

to be the same (Jankowski, 2005b). The same procedure is used to obtain the stiffness parameter of the Hertzdamp model, which is found to be $k_h = 0.60\ 10^{10}$ N/m$^{3/2}$. For the two models, the coefficient of restitution e = 0.6 is used. The results from the numerical analysis and the experiment are shown in Fig. 2. Using Eq. 11 the normalized errors for pounding force histories are found to be equal to 21.45% for the nonlinear viscoelastic model and 24.18% for the Hertzdamp model.

**Concrete-to-concrete impact:** Van Mier *et al.* (1991) carried out an experiment on collisions between a prestressed concrete pile and a concrete striker. The dynamic equation of motion for pounding between a striker of mass $m_1$ and a prestressed fixed pile can be written by drawing the free body diagram as shown in Fig. 3 (Jankowski, 2005b).

$$m_1\ddot{u}_1(t) + \frac{m_1 g}{l}u_1(t) + F(t) = 0 \quad (14)$$

The pounding force F (t) follows the relation (6) for the Hertzdamp model and relation (8) for the nonlinear viscoelastic model. The stiffness parameter used for the nonlinear viscoelastic model is $\bar{\beta} = 2.75\times10^9$ N/m$^{3/2}$ (Jankowski, 2005b). For the Hertzdmap model, $k_h = 1.54\times10^9$ N/m$^{3/2}$ was found through an iterative procedure in order to keep the maximum pounding force in the numerical analysis equals the maximum pounding force of the experiment as 102.5 N. For the two models, the coefficient of restitution is e = 0.6. The results from the numerical computations and the experiment are shown in Fig. 4. It is found that the normalized error equals 25.04% for the nonlinear viscoelastic model and 54.01% for the Hertzdamp model.

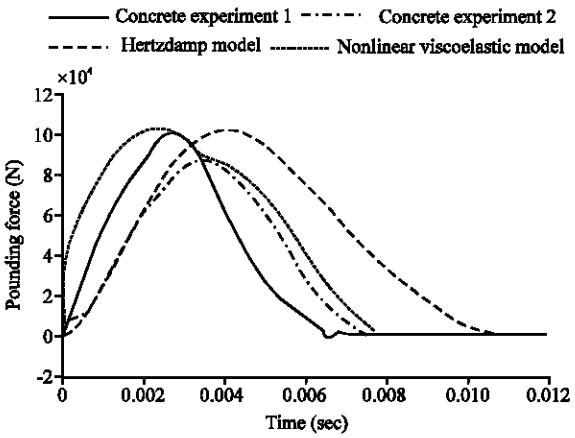


Fig. 4: Pounding force time histories during impact between a concrete pendulum striker and prestressed concrete pile

## COMPARISON BASED ON SHAKING TABLE EXPERIMENTS

Chau *et al.* (2003) carried out shaking table tests to investigate the pounding phenomenon between two steel towers of different natural frequencies and damping ratios subject to different combinations of stand-of distance and seismic excitations. The two Single Degree Of Freedom (SDOF) building systems shown in Fig. 5 were used in the numerical simulation as a representative for the two towers. For i =1, 2, let $m_i$ be the masses, $c_i$ be the viscous damping coefficients and $k_i$ be the stiffness for SDOF 1 and SDOF 2, accordingly. The coupling equation of motion for two adjacent buildings subjected to horizontal ground motion $\ddot{u}_g(t)$ has the following form:

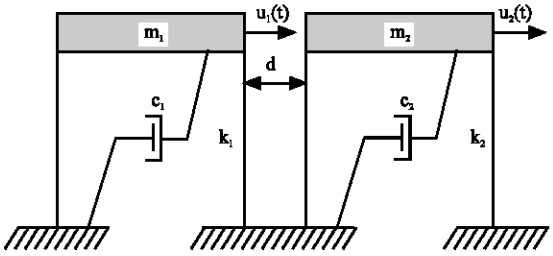


Fig. 5: Model idealization of adjacent structures

$$\begin{aligned} m_1\ddot{u}_1(t) + c_1\dot{u}_1(t) + k_1u_1(t) + F(t) &= -m_1\ddot{u}_g(t) \\ m_2\ddot{u}_2(t) + c_2\dot{u}_2(t) + k_2u_2(t) - F(t) &= -m_2\ddot{u}_g(t) \end{aligned} \quad (15)$$

where, $u_i(t), \dot{u}_i(t)$ and $\ddot{u}_i(t)$ represent the displacement, velocity and acceleration of the system, respectively. The pounding force F (t) follows the relation (6) for Hertzdamp model and relation (8) for nonlinear viscoelastic model. The values of structural stiffness and damping coefficients: $k_i$, $c_i$ can be calculated from the formulas (Harris and Piersol, 2002).

$$k_i = \frac{4\pi^2 m_i}{T_i^2}; \qquad c_i = 2\xi_i\sqrt{k_i m_i} \quad (16)$$

where, $T_i$, $\xi_i$ (i = 1, 2) denote the natural structural vibration period and structural damping, respectively. We solve the equation of motion (15) for the harmonic waves as excitations and the two different models of pounding force considered herein. Chau *et al.* (2003) performed a series of shaking table experiments for n = 1 (1 pounding per cycle) and n = 2 (1 pounding per 2 cycles) using various harmonic waves as excitations for the two SDOF systems shown in Fig. 5. Experimental, numerical and analytical results of the relative impact velocity versus the excitation frequency had been observed and computed for the comparison purposes between experiments and theories. In this study, the results of two different sinusoidal excitations as input are presented. We consider first the input shaking table $\ddot{u}_g(t) = 2.6\ (2\pi f_g t)$ as an excitation acting on SDOF 1 and SDOF 2 with the following dynamic characteristics $m_1$ = 98.0 kg, $f_1$ = 5.04 Hz, $\xi_1$ = 7.2%, $m_2$ = 146.4 kg, $f_2$ = 2.76 Hz, $\xi_2$ = 1.5% and a separation distance of 19.8 mm (Chau *et al.*, 2003). The obtained numerical solutions for the steady state relative impact velocity using both the nonlinear viscoelastic model and the Hertzdamp model are compared with the experimental results which are the average of 10 cycles in the steady state (Chau *et al.*, 2003). The results from the numerical analysis and the experiment are shown in Fig. 6. The simulation errors for the obtained

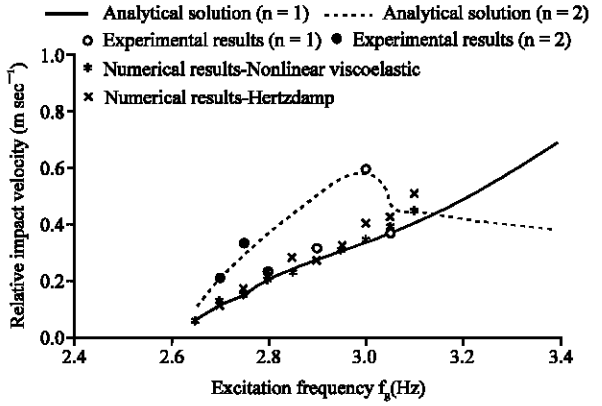


Fig. 6: The steady state relative impact velocity versus the excitation frequency. Numerical results obtained by nonlinear viscoelastic and Hertzdamp models. The input shaking table is $\ddot{u}_g = 2.6\sin(2\pi f_g t)$

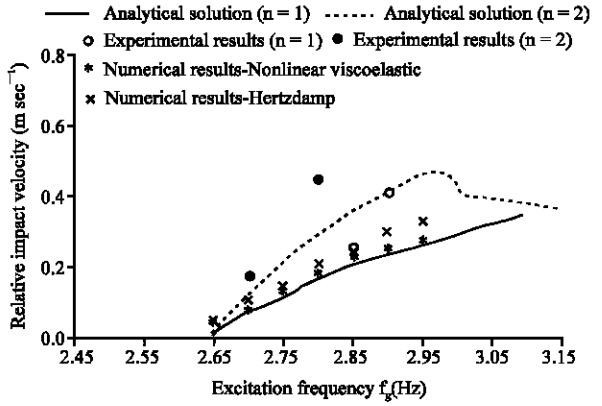


Fig. 7: The steady state relative impact velocity versus the excitation frequency. Numerical results obtained by nonlinear viscoelastic and Hertzdamp models. The input shaking table is $\ddot{u}_g = 1.9\sin(2\pi f_g t)$

numerical impact velocity compared with the experimental results for n = 1 and n = 2 are 25.46 and 47.25% for the Hertzdamp model and 31.66 and 51.26% for the nonlinear viscoelastic model. As a second input shaking table, we consider the harmonic excitation $\ddot{u}_g(t) = 1.9\sin(2\pi f_g t)$ applied to the SDOF 1 and SDOF 2 with the same dynamic characteristics and separation distance as used before. The results from the numerical analysis and the experiment are shown in Fig. 7. The simulation errors for the obtained numerical impact velocity compared with the experimental results for n = 1 and n = 2 are 24.97 and 54.25% for the Hertzdamp model and 33.75 and 59.80% for the nonlinear viscoelastic model.

## COMPARISON BASED ON GROUND MOTION RECORDS

Muthukumar and DesRoches (2006) and Muthukumar (2003) assessed the performance of the Hertzdamp model by comparing it to the linear spring, Kelvin, Hertz and stereomechanical models using two Single Degree Of Freedom (SDOF) building systems shown in Fig. 5 with the parameters shown in Table 1. The systems were subjected to a suite of thirty ground motion records (Muthukumar, 2003) from thirteen different earthquakes with Peak Ground Accelerations (PGA) levels varying from 0.1 to 1 which were carefully chosen to fall within zone I. Three ground motion records were used at each PGA level. In the present study, the performance of the two models is investigated in the process of studying buildings pounding. As suggested by Muthukumar and DesRoches (2006), two cases are considered. First is a no pounding case where the gap distance d is set to be large enough to avoid pounding. In the second case, the gap distance d is chosen to be small enough to induce pounding. The ratio of the maximum responses obtained in second case and the first case, called the amplification response, is used in the analysis. We solve the equation of motion (15) for the thirty ground motion records and the three different period ratios. The obtained amplification factor for displacement and acceleration of both SDOF 1 and SDOF 2 indicates that the nonlinear viscoelastic model provides the smallest displacement and acceleration amplifications for the whole PGA levels and the different period ratios considered herein as shown in Fig. 8-13.

Table 1: Properties of SDOF systems used for the impact model

| Parameters | Values |
|---|---|
| $m_1$, $m_2$ | 7.8 kip-s$^2$/in |
| $\xi_1$,$\xi_2$ | 5% |
| $T_1/T_2$ | 0.30, 0.50, 0.70 |
| $k_h$, $\bar{\beta}$ | 25000 kip-in$^{-3/2}$ |
| e | 0.6 |
| d | 0.5 in |

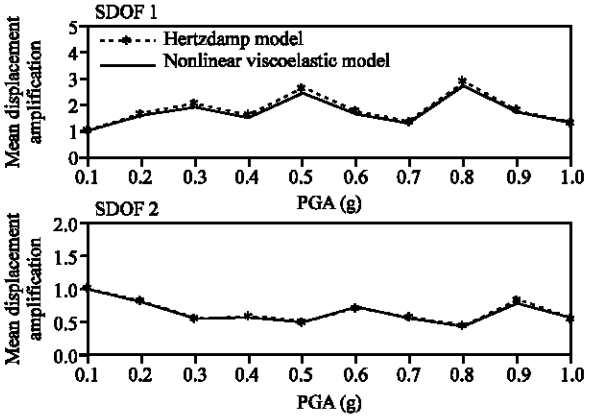


Fig. 8: Mean displacement amplification for $T_1/T_2$ = 0.3

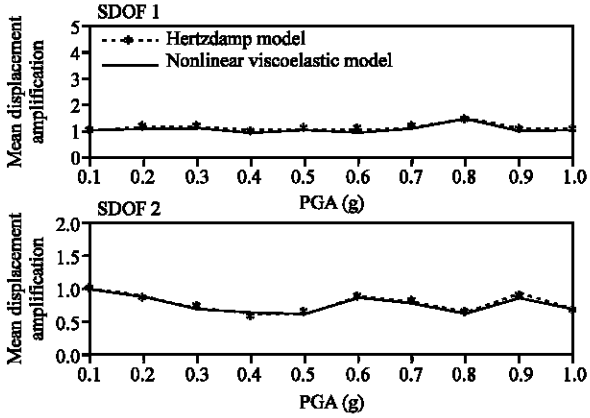


Fig. 9: Mean displacement amplification for $T_1/T_2$ = 0.5

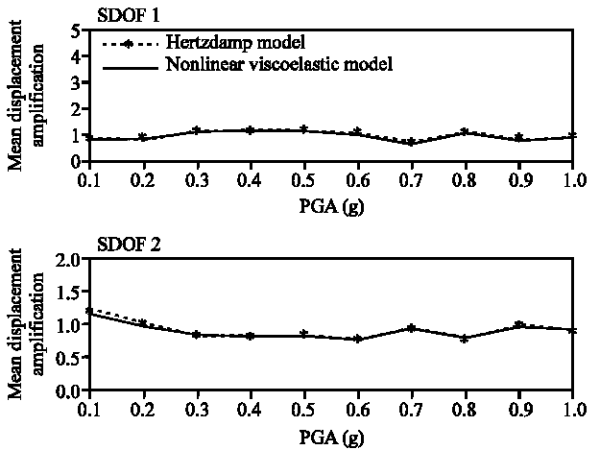


Fig. 10: Mean displacement amplification for $T_1/T_2$ = 0.7

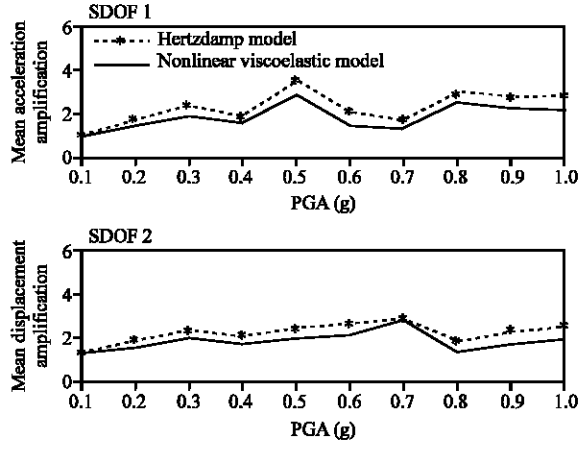


Fig. 11: Mean acceleration amplification for $T_1/T_2$ = 0.3

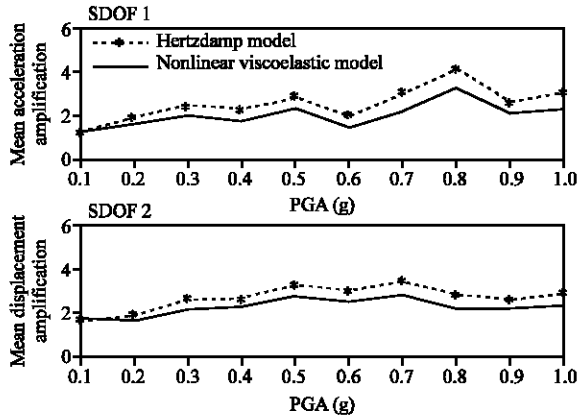


Fig. 12: Mean acceleration amplification for $T_1/T_2 = 0.5$

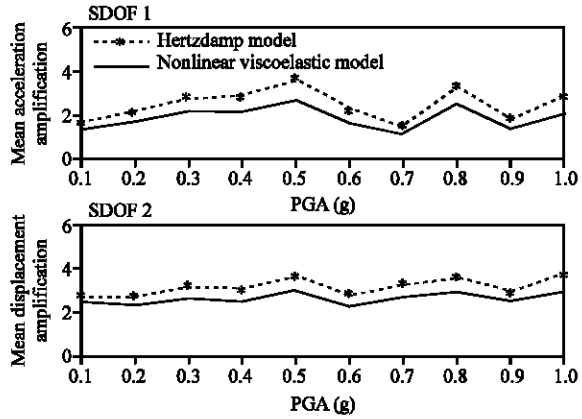


Fig. 13: Mean acceleration amplification for $T_1/T_2 = 0.7$

## CONCLUSIONS

The comparison between two nonlinear models for earthquake-induced structural pounding, i.e., the Hertzdamp model and the nonlinear viscoelastic model, has been conducted in this research. Both models are based on the Hertz contact law and incorporate additional damping in order to simulate the energy dissipation during impact. First, the numerical results obtained for each model have been compared with the results of two impact experiments. Then, the comparison has been carried out based on the results from the shaking table experiments. Finally, the investigation has been conduced for pounding between two single degree of freedom systems with different period ratios subjected to thirty ground motion records of different PGA levels. The effectiveness of two models of pounding has been assessed by comparing the displacement and acceleration response amplifications.

The results of the study demonstrate that the use of the nonlinear viscoelastic model leads to smaller normalized errors in the impact force time histories comparing to the Hertzdamp model. This is mainly due to the fact, that the application of the Hertzdamp model results in longer time of contact as can be shown in Fig. 2 and 4. On the other hand, the impact force time histories obtained for the nonlinear viscoelastic model show the change of the curvature after passing the peak force value and this disadvantage is related to the disengagement of the damping term in the restitution period of impact (Eq. 8).

Studies with the use of the results from the shaking table experiments showed that the Hertzdamp model gives smaller normalized error than the nonlinear viscoelastic model in simulation of impact velocity for pounding between two steel towers of different dynamics characteristics subjected to sinusoidal excitations.

The results of further analysis indicate that the nonlinear viscoelastic model provides smaller displacement and acceleration amplifications of the response of colliding single degree of freedom systems for three different period ratios and PGA levels of ground motions. This makes the nonlinear viscoelastic model more convenient in the simulation of earthquake-induced structural pounding.

Both impact force models considered in this paper have been found to have some advantages and disadvantages when used for modeling of structural pounding. The results of the study indicate that the accuracy of each of the models depends on the type of analysis conducted.

## REMARK 1: IMPLICIT RUNGE-KUTTA (IRK) METHODS

The differential Eq. 13-15 can be written in a uniform version

$$\frac{dp(u)}{dt} = f(u), \qquad t \geq 0 \qquad (17)$$
$$u(0) = u_0$$

where, $u \in R^n$, $p : R^n \to R^n$ is a continuously differentiable function and $f : R^n \to R^n$ is a continuous function but not necessarily differentiable. Problem (17) is called a system of nonsmooth ordinary differential equations of the first order. In order to solve the system (17) efficiently we use the Implicit Runge-Kutta (IRK) method (Jay, 2000; Chen and Mahmoud, 2008; Mahmoud and Chen, 2008). One step of an s-stage Implicit Runge-Kutta (IRK) method for solving (17) has the following version (Jay, 2000):

Given a step size h, a coefficient matrix $A \in R^{s \times s}$ and a weight vector $b \in R^s$. Let $U_0 = u_0$. For $k \geq 0$:

**Step 1:** Solve the s x n-dimensional system of nonlinear equations

$$H(x) = \begin{pmatrix} p(x_1) - p(U_k) - h\sum_{j=1}^{s} a_{1j} f(x_j) \\ \vdots \\ p(x_s) - p(U_k) - h\sum_{j=1}^{s} a_{sj} f(x_j) \end{pmatrix} = 0 \quad (18)$$

to get a solution $x^k = (x_1^k, x_2^k, ..., x_s^k)^T \in R^{sxn}$.

**Step 2:** Solve the n-dimensional system of nonlinear equations

$$\tilde{H}(U) := P(U) - P(U_k) - h\sum_{j=1}^{s} b_j f(x_j^k) = 0 \quad (19)$$

and let the solution be $U_{k+1}$.

A practical IRK method can be defined, by choosing appropriate matrix *A* and vector *b*, such as coefficients of Gauss, Radau IA and IIA, Lobatto IIIA, Burrage, etc. (Jay, 2000; Burrage, 1982). Moreover, we use the slanting Newton method (Chen *et al.*, 2000) to solve the system of nonsmooth equations in each iteration of the IRK method. Numerical results show that the IRK method with the slanting Newton method is efficient. Numerical results reported in this study were obtained by using the two-stages Burrage IRK method (Burrage, 1982) which has stable property (Ferracina and Spliker, 2005). No numerical problems were encountered.

## REMARK 2: CONVERGENCE ORDER OF IRK METHODS

Chen and Mahmoud (2008) have studied convergence order of IRK methods and tested various IRK methods with the slanting Newton method on numerous problems in structural oscillation and pounding. In theory, it can be shown that the order of convergence is at least one for the Lipschitz continuous ODEs under mild conditions. Consider the solution u(t) to (17) in a fixed interval [0, T] with the number n of step chosen such that $t_n = nh = T$. Let

$$e_k(h) = U_k - u(t_k), \qquad k = 0,1,...,n \quad (20)$$

and

$$E(h) = \max_{k=0,..,n} \|e_k(h)\|. \quad (21)$$

For simplicity, we consider p (u) = u. Assume that for any $U_k$ there are $x^k$, $U_{k+1}$ such that $H(x^k) = 0$ and $\tilde{H}(U_{k+1}) = 0$. Let

$$G(x,U) = \begin{pmatrix} x_1 - U - h\sum_{j=1}^{s} a_{1j} f(x_j) \\ \vdots \\ x_s - U - h\sum_{j=1}^{s} a_{sj} f(x_j) \end{pmatrix} \quad (22)$$

Since f is Lipschitz continuous, the function G is Lipschitz continuous. By the Rademacher theorem, G is differentiable almost everywhere. Hence, we can define the Clarke generalized Jacobian as ∂G (x,U) (Clarke, 1983). By the implicit function theorem for Lipschitz continuous function (Clarke, 1983), for small h, there exist a neighborhood $u_k$ of $U_k$ and a Lispchitz function $\psi_k(.;h): u_k \to R^{sxn}$ such that $x^k = \psi_k(U_k;h)$ and for every $U \in u_k$, G ($\psi_k$(U, h), U) = 0. Therefore, we can write

$$U_{k+1} = U_k + h\sum_{j=1}^{s} b_j f((\psi_k(U_k;h))_j) =: U_k + h\phi_k(U_k;h) \quad (23)$$

It is easy to verify that $\phi_k$ (.;h): $R^n \to R^n$ is a locally Lipschitz continuous function.

**Theorem 1:** Chen and Mahmoud (2008) suppose that there are positive constants $K_1$ and $K_2$ such that:

$$\|\phi_k(u(t_k);h) - \phi_k(U_k;h)\| \le K_1 \|u(t_k) - U_k\| \quad (24)$$

and

$$\|\phi_k(u(t_k);h) - u'(t_k)\| \le K_2 h \quad (25)$$

Then, there is a constant $\alpha > 0$ such that

$$E(h) \le \alpha h. \quad (26)$$

Figure 14 shows the convergence of IRK methods with different coefficient versus the Explicit Runge-Kutta

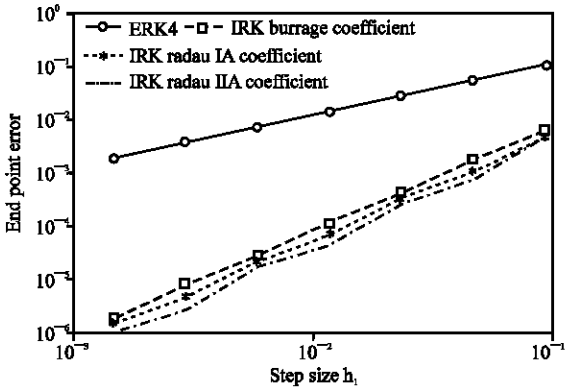


Fig. 14: Convergence of ERK method of fourth order and Burrage and Radau coefficients of the IRK method with various constant step sizes *hl*

(ERK) methods of order four in computation of the response for the collapse of the Tacoma Narrows suspension bridge. The numerical experiments were performed using MATLAB 7.0 on a Dell PC with 2MB memory and 800 MHZ.

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