Sudden death of effective entanglement

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Sudden death of entanglement is a well-known effect resulting from the finite volume of separable states. We study the case when the observer has a limited measurement capability and analyze the effective entanglement (i.e., entanglement minimized over the output data). We show that in the well-defined system of two quantum dots monitored by single-electron transistors, one may observe a sudden death of effective entanglement when real, physical entanglement is still alive. For certain measurement setups, this occurs even for initial states for which sudden death of physical entanglement is not possible at all. The principles of the analysis may be applied to other analogous scenarios, such as estimation of the parameters arising from quantum process tomography.

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I. INTRODUCTION

Sudden death of quantum entanglement [1–3] is one of the phenomena related to the fact that in finite-dimensional systems the set of nonentangled states is of finite volume [4]. The phenomenon was explicitly identified in Ref. [1] (see also its implicit presence in independent analysis [3]). Full recognition of its importance and consequences was established with time [2,5,6] (for the review see Ref. [7]) and demonstrated experimentally [8].

In an earlier analysis of entanglement evolution, it was assumed that the observer has the power to perform arbitrary measurements and can determine the state of the system completely. This is, however, not always true. In particular, there are natural systems, like quantum dots (QD's), which we will consider later, where limited measurement power is a natural and practical constraint (i.e., single-electron transistors (SET's) coupled to QD systems can be used to find a limited amount of information about the QD state [9,10]) In all such cases of limited measurement capability, it is natural to consider the worst case scenario proposed in Ref. [11]: *as real entanglement, one should consider the entanglement (i.e., chosen entanglement measure) minimized over the set of measurement data.* This approach has found wide developments in terms of entanglement witnesses [12–15] with respect to experimental data [16,17]. The minimized entanglement will subsequently be called *effective entanglement*.

Here, we consider the system of a double QD (DQD) interacting with a phonon-bath which leads to an unavoidable partial pure dephasing effect typically on picosecond time scales [18–20]. The time evolution of physical entanglement in this system can be found in Ref. [21]. We take into account the limitations to the knowledge of the system state imposed by a realistic measurement setup (consisting of different configurations of SET's interacting with the double-dot system). This measurement scheme does not allow for state tomography and provides, in fact, a very limited set of observables. In each time step we minimize the value of entanglement with respect to the data which can be measured and find the evolution of the *effective entanglement with respect to the SET-defined observables*. We show that such an evolution, besides leading to a quantitative reduction of entanglement (compared to physical entanglement), demonstrates qualitative changes such as the possibility of the sudden death of effective entanglement in situations when physical entanglement lives for arbitrarily long times. Our approach may be extended to estimate the time evolution of quantities describing quantum processes (e.g., entangling power or fidelity of a quantum process).

II. THE MODEL

The system under study consists of two parts, the measured and the measuring subsystems. The former consists of the DQD ensemble, in which superpositions of excitonic states undergo pure dephasing due to the interaction with the phonon modes of the surrounding crystal. The pure dephasing is only partial, as has been experimentally shown [20] and later explained theoretically [18,19]; experimental and theoretical data yield both qualitative and quantitative agreement [19]. The $|0\rangle$ and $|1\rangle$ states of the qubit correspond to an empty QD and a QD with an exciton in its ground state, respectively (the two qubits are located in separate QD's). A number of SET's which determine the charge distribution in their vicinity constitute the latter. This allows for the measurement of a set of elements of the DQD density matrix (the precise elements measured are determined by the geometry of the measurement device with respect to the DQD), and the subsequent calculation of effective entanglement.

The Hamiltonian which governs the evolution of the excitonic states in the DQD is

$$\begin{aligned}H =& \epsilon_1(|1\rangle\langle 1|\otimes \mathbb{I}) + \epsilon_2(\mathbb{I}\otimes|1\rangle\langle 1|)\\ &+(|1\rangle\langle 1|\otimes \mathbb{I})\sum_{\boldsymbol{k}} f_{\boldsymbol{k}}^{(1)}(b_{\boldsymbol{k}}^\dagger + b_{-\boldsymbol{k}})\\ &+(\mathbb{I}\otimes|1\rangle\langle 1|)\sum_{\boldsymbol{k}} f_{\boldsymbol{k}}^{(2)}(b_{\boldsymbol{k}}^\dagger + b_{-\boldsymbol{k}}) + \sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}} b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}},\end{aligned} \tag{1}$$

where the two states of each QD are denoted by $|0\rangle$ and $|1\rangle$, $\mathbb{I}$ is the unit operator, $\epsilon_{1,2}$ are the transition energies in the two

QD's, $f_{\boldsymbol{k}}^{(1,2)}$ are exciton-phonon coupling constants, $b_{\boldsymbol{k}}^{\dagger}, b_{\boldsymbol{k}}$ are creation and annihilation operators of the phonon modes, and $\omega_{\boldsymbol{k}}$ are the corresponding energies (we put $\hbar = 1$). The explicit tensor notation refers to the DQD but is suppressed for the phonon reservoir components.

Exciton wave functions are modeled by anisotropic Gaussians with the extension $l_{\mathrm{e/h}}$ in the xy plane for the electron or hole, and l_z along z for both particles. Then, the coupling constants for the deformation potential coupling between confined charges and longitudinal phonon modes have the form $f_{\boldsymbol{k}}^{(1,2)} = f_{\boldsymbol{k}} e^{\pm i k_z d/2}$, where

$$f_{\boldsymbol{k}} = \sqrt{\frac{k}{2\varrho V c}} e^{-l_z^2 k_z^2/4} \left[\sigma_{\mathrm{e}} e^{-l_{\mathrm{e}}^2 k_\perp^2/4} - \sigma_{\mathrm{h}} e^{-l_{\mathrm{h}}^2 k_\perp^2/4}\right],$$

V is the normalization volume of the bosonic reservoir, d is the distance between the subsystems, $k_{\perp,z}$ are momentum components in the xy plane and along the z axis, $\sigma_{\mathrm{e,h}}$ are deformation potential constants for electrons and holes, c is the speed of longitudinal sound, and ϱ is the crystal density.

In our calculations we use parameters typical for two self-assembled GaAs/InGaAs QD's stacked on top of each other [21,22]. The material parameters used are $\sigma_{\mathrm{e}} = 8$ eV, $\sigma_{\mathrm{h}} = -1$ eV, $c = 5.1$ nm/ps, $\varrho = 5360$ kg/m^3 (corresponding to GaAs), and $d = 6$ nm, $l_{\mathrm{e}} = 4.4$ nm, $l_{\mathrm{h}} = 3.6$ nm, $l_z = 1$ nm (dot related parameters).

The Hamiltonian can be diagonalized exactly using the Weyl operator method [23] and the evolution is calculated following Ref. [21]. The interaction with the phonon modes leads to partial pure dephasing [19,20,24], leaving the state occupations unchanged; the explicit forms of the time dependence of the off-diagonal density matrix elements may be found in Ref. [21].

The measuring subsystem is taken into account only in principle, in the sense that the information which can be gained about the state of the DQD ensemble is limited by realistic measurement capability. When measuring the state of a DQD by observing the current through a SET, the actual observable depends on its position with respect to the DQD. The number of the DQD density matrix elements which can be obtained is very limited; this restriction is a key point in this article. There are a number of features inherent to this measurement scheme (e.g., finite measurement time and an uncertainty of the outcome) which are not taken into account for the sake of clarity.

Let us consider a SET located near the lower QD (configuration A). The interaction between the SET electron and the exciton in the dot shifts the energy levels in the SET when the QD is occupied, hence affecting the current. An appropriate choice of SET parameters allows for the maximization of the difference in current flow and the measurement of the occupation of the lower dot [9]. The measurement projectors corresponding to this situation are $P_n = |0\rangle\langle 0| \otimes \mathbb{I}$ and $P_e = |1\rangle\langle 1| \otimes \mathbb{I}$, so the measured quantity is

$$x = \langle 00|\rho|00\rangle + \langle 01|\rho|01\rangle. \tag{2}$$

If such a SET is located near the upper QD, the measured quantity is

$$y = \langle 00|\rho|00\rangle + \langle 10|\rho|10\rangle. \tag{3}$$

Configuration B involves a SET located symmetrically between the QD's and in close proximity to them in such a way that the energy level on the SET island is sensitive to the probability of finding an exciton in midpoint [10]. The corresponding projectors are $P_n = |-\rangle\langle -|$ and $P_e = |+\rangle\langle +|$, where

$$|\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}. \tag{4}$$

This allows for the measurement of a linear combination of density matrix elements

$$z = \langle 01|\rho|01\rangle + \langle 10|\rho|10\rangle + 2\mathrm{Re}\langle 01|\rho|10\rangle. \tag{5}$$

If the SET is located further away from the DQD region (symmetrically), it is sensitive to the total number of excitons (configuration C). This allows for the measurement of

$$d = \langle 11|\rho|11\rangle, \tag{6}$$

via projectors $P_e = |11\rangle\langle 11|$ and $P_n = |+\rangle\langle +| + |-\rangle\langle -| + |00\rangle\langle 00|$, or $a = \langle 00|\rho|00\rangle$ via $P_e = |00\rangle\langle 00|$ and $P_n = |+\rangle\langle +| + |-\rangle\langle -| + |11\rangle\langle 11|$ depending on the SET parameters. Switching between the two modes can be accomplished by applying a different voltage to the SET.

III. MINIMIZING ENTANGLEMENT FROM THE SET DATA

Consider the general quantum state ρ. There is a simple lemma:

Lemma. For any convex entanglement measure E (or entanglement parameter) which is invariant under the complex conjugate one has $E(\rho) \geqslant E\big(\mathrm{Re}(\rho)\big)$, where the "Re" symbol means a real part of the quantum state.

Proof. $E\big(\mathrm{Re}(\rho)\big) = E(\frac{1}{2}\rho + \frac{1}{2}\rho^*) \leqslant \frac{1}{2}E(\rho) + \frac{1}{2}E(\rho^*) = E(\rho)$.

Any reasonable entanglement measure should be invariant under the complex conjugate; all known entanglement measures like entanglement of formation, concurrence, log-negativity, relative entropy of entanglement, and all distillable quantities fulfill this condition. Some of them are also convex (i.e., entanglement of formation, concurrence, and relative entropy of entanglement). In what follows we shall consider the concurrence [25,26], which is both convex and invariant under the complex conjugate. The reason for this choice of entanglement measure is its mathematical simplicity and the fact that it is easily converted into entanglement of formation (which has a good physical interpretation). Since the SET measurement provides only real parts of off-diagonal density matrix elements, the first step of minimization is to consider only the real parts of DQD states, $\tilde{\rho} = \mathrm{Re}(\rho)$.

IV. SUDDEN DEATH OF EFFECTIVE ENTANGLEMENT

First, we will consider the situation, when a number of SET's provide all possible information that can be gained about the state of the DQD with this measurement technique. This requires a pair of SET's in configuration A located near the two QD's, giving x and y defined, respectively, by Eqs. (2) and (3), and one in configuration C providing d [Eq. (6)]. Hence, all of the diagonal elements of the density matrix can be

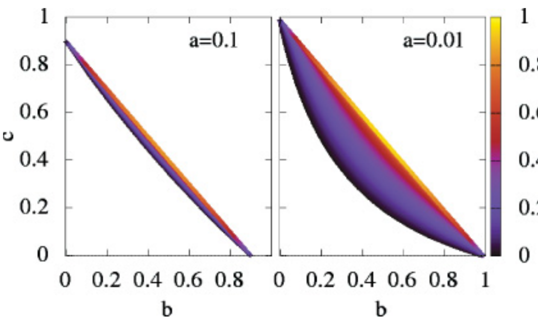

FIG. 1. (Color online) The effective entanglement of coherent states in the first measurement setup as a function of known occupations b and c for two values of a. The plot shows only nonzero effective entanglement.

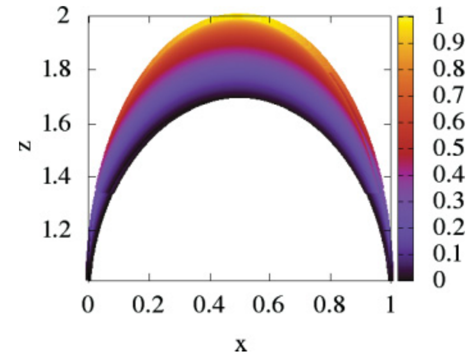

FIG. 2. (Color online) The effective entanglement in the second measurement setup as a function of the observables x and z. The plot shows only nonzero effective entanglement.

found,

$$\begin{aligned}\langle 00|\rho|00\rangle &= x + y + d - 1 \equiv a,\\ \langle 01|\rho|01\rangle &= 1 - y - d \equiv b,\\ \langle 10|\rho|10\rangle &= 1 - x - d \equiv c,\\ \langle 11|\rho|11\rangle &= d.\end{aligned} \tag{7}$$

A SET in configuration B provides z [Eq. (5)] giving the real part of one of the off-diagonal density matrix elements

$$\mathrm{Re}(\langle 01|\rho|10\rangle) = \frac{x+y+z}{2} + d - 1 \equiv \mathrm{Re}(h). \tag{8}$$

Since all of the diagonal density matrix elements are known, the set of initial maximally entangled states which cannot exhibit sudden death of effective entanglement is the same as the set of states with real off-diagonal density matrix elements which do not exhibit sudden death of physical entanglement [21]. The time evolution of effective and physical entanglements in these states under pure dephasing is the same and the concurrence is equal to $C(\tilde{\rho}) = 2\mathrm{Re}(h)$. The situation is different for states where all diagonal density matrix elements are nonzero where the set of effectively entangled coherent states is substantially reduced. In Fig. 1 the effective entanglement of coherent states [$\mathrm{Re}(h) = \sqrt{bc}$] as a function of b and c is plotted for two values of a.

Second, let us consider the situation when only x and z have been measured (configurations A and B), so only linear combinations of some density matrix elements are known. The measurement outcome $x \in [0, 1]$. The outcome $z \in [0, 2]$, but it is easy to show that for nonzero effective entanglement $z > 1$. The minimization of entanglement requires that the quantity ad be maximized, which occurs for $\mathrm{Re}(h) = \sqrt{bc}$ [the unknown density matrix elements a, b, c, $d = 1 - a - b - c$ and $\mathrm{Re}(h)$ are defined in Eqs. (7) and (8)]; then $ad = (x - b)[1 - x - (-\sqrt{b} + \sqrt{z})^2]$. Finding the maximal ad, which leads to set values for a, b, c, $d = 1 - a - b - c$ and $\mathrm{Re}(h)$, and minimizing over the five still unknown off-diagonal density matrix elements produces the effective entanglement for given measurement outcomes x and z. Effective entanglement as a function of x and z is plotted in Fig. 2.

It is interesting to consider here the time evolution of effective entanglement under phonon-induced pure dephasing of an initially maximally entangled state $|+\rangle$ [Eq. (4)]. When measured it will yield $x = 0.5$ and $z = 2$, so the effective concurrence $C_e(|+\rangle\langle +|) = 1$ (note that the state $|-\rangle$ has $x = 0.5$, but $z = 0$ and $C_e(|-\rangle\langle -|) = 0$). Phonon-induced evolution of the state does not change x, but z decreases with decreasing $\mathrm{Re}(h)$ leading to the sudden death of effective entanglement for sufficiently dephased states. This state does not exhibit sudden death of physical entanglement.

Third, we will consider the simplest situation, where the measurement device is limited to configuration B and the measurement data yield only z as defined in Eq. (5). The amount of information gained by the measurement is very limited. Minimizing effective entanglement requires the maximisation of ad the same as in the previous measurement setup, yet now the maximum is easily found for $b = c = \mathrm{Re}(h) = z/4$. The dependence of effective entanglement on z is plotted in the inset of Fig. 3.

The evolution of effective entanglement of the initial state $|+\rangle$ in this setup under realistic phonon-induced pure dephasing is plotted in Fig. 3 for different temperatures. As is to be expected, effective disentanglement occurs faster than physical disentanglement. Furthermore, sudden death of entanglement appears for sufficiently high temperatures (e.g., when the dephasing is strong enough). For a limited range of temperatures, sudden birth of entanglement is also observed. The second phenomenon is due to the enhancement of coherence which occurs when wave packets from the two QD's meet due to positive interference between them; this

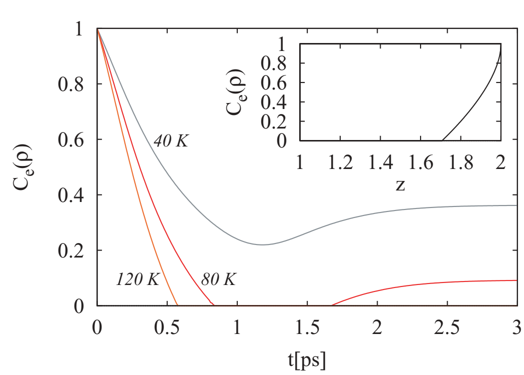

FIG. 3. (Color online) The time evolution of effective entanglement for different temperatures. Inset: Effective entanglement as a function of the measured parameter z.

mechanism does not lead to the sudden birth of physical entanglement [21].

V. CONCLUSION AND DISCUSSION

We have considered the evolution of entanglement in a DQD system under phonon-induced pure dephasing minimized over a set of attainable data (the considered measurement schemes are based on SET's). In all of the considered measurement setups a reduction of effective (minimized) entanglement compared to the physical one was observed. Furthermore, in the setups where the knowledge of QD occupations is limited, sudden death was found in situations when the physical entanglement lives for arbitrarily long times and its sudden death is not possible. For a limited range of temperatures (coupling strengths) even the sudden birth of entanglement occurs (due to a mechanism that does not cause sudden birth of physical entanglement). Hence, we have shown that the analysis of entanglement dynamics in systems with difficult measurement access can lead to qualitative differences in measured entanglement.

The present analysis may provide a starting point to analogous research in general dynamics including quantum process tomography ([27,28]). For instance, one may want to probe the entangling power [29] of a sequence of quantum gates in time in a possibly cheap way, when only some of the measurements are not costly. Then the principles of the previous analysis may be applied with the help of the Choi-Jamiołkowski isomorphism [30,31]. Fundamental parameters of the dynamics, including the fidelity of a quantum process, can be then estimated, that is, by minimization in time under a restricted set of data, especially when coarse-graining information about the dynamics is needed.

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