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Surface finite viscoelasticity and surface anti-plane waves

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A B S T R A C T

We introduce the surface viscoelasticity under finite deformations. The theory is straightforward generalization of the Gurtin–Murdoch model to materials with fading memory. Surface viscoelasticity may reflect some surface related creep/stress relaxation phenomena observed at small scales. Discussed model could also describe thin inelastic coatings or thin interfacial layers. The constitutive equations for surface stresses are proposed. As an example we discuss propagation shear (anti-plane) waves in media with surface stresses taking into account viscoelastic effects. Here we analysed surface waves in an elastic half-space with viscoelastic coatings. Dispersion relations were derived.

1. Introduction

Waves in solids and fluids constitute a rather important branch of mechanics and physics. Among them it is worth to mention surface/interfacial waves, i.e. waves localized in vicinity of free surfaces or interfaces, see e.g. Achenbach (1973), Strutt (1945), Überall (1973) and Kaplunov and Prikazchikov (2017). Considering real materials one can observe that dissipation phenomena may play a crucial role. For example, Brekhovskikh (1960) noted that the elastic waves theory cannot describe some experimentally observed phenomena. In fact, in the case of Rayleigh waves in viscoelastic media studied by Currie et al. (1977), Currie and O'Leary (1978) and Currie (1979) it was discovered that unlike the elastic case it could be more than one surface wave. For discussion of the number of Rayleigh waves in viscoelastic half-space we refer to Carcione (1992), Chiriţă et al. (2014), Romeo (2001) and Sharma (2020). Similarly, viscoelastic Love waves were analysed by Kiełczyński (2018), Subhash and Gaur (1978). For a current state of the theory of viscoelastic waves we refer to the fundamental book by Borcherdt (2009).

The aim of this paper is to introduce surface viscoelasticity and discuss antiplane surface waves propagation. Surface elasticity model was proposed by Gurtin and Murdoch (1975, 1978) and was generalized by Steigmann and Ogden (1997, 1999). From the physical point of view, these models describe deformations of an elastic solid body with perfectly attached to its surface an elastic membrane or shell, respectively. Nowadays, surface elasticity found various applications at small scales, see e.g. Duan et al. (2008), Eremeyev (2016), Firooz et al. (2021), Jiang et al. (2022), Mogilevskaya et al. (2021), Wang et al. (2011), Zheng et al. (2021) and Kushch and Mogilevskaya (2022). Moreover, it could be also extended to other scales, in particular, as a technique of surface/interface design, see e.g. Aghaei et al. (2021), Halvey et al. (2019). It was shown that within surface elasticity there exist anti-plane surface waves (Eremeyev et al., 2016; Xu et al., 2015), see also Eremeyev (2020), Eremeyev et al. (2019, 2020), Eremeyev and Sharma (2019), Jia et al. (2018), Mikhasev et al. (2021, 2022, 2023), Wu et al. (2020), Zhu et al. (2019) and the reference therein.

Linear model of surface elasticity by Gurtin and Murdoch was extended to viscoelasticity by Ru (2009), where it was used for modelling of nanobeams. Similar one-dimensional model was used by Hasheminejad and Gheshlaghi (2010). Beams with thin

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viscoelastic coatings were studied by Lyu et al. (2020). Two-dimensional surface viscoelasticity was introduced by Altenbach et al. (2012), Hasheminejad and Gheshlaghi (2013) in order to model thin plates and shells. In these papers linear constitutive equations for surface stresses were used.

In our paper we provide a generalization of the Gurtin–Murdoch model towards a finite surface viscoelasticity, i.e. considering finite deformations. In Section 2 we introduce surface stress tensors as a tensor-valued operator dependent of the history of surface deformations. As a result, we get a nonlinear-boundary-value problem taking into account viscoelastic surface stresses. Linearization of the latter problem is also provided. Finally, in order to demonstrate some properties of the model, in Section 3 we consider anti-plane surface waves in an elastic half-space considering viscoelastic surface stresses. Dispersion relations are given, i.e. dependencies of the wave-number and the attenuation coefficient on the frequency.

In the following we use direct tensor calculus as in Lurie (1990), Simmonds (1994) and Eremeyev et al. (2018), so vectors and tensors are shown in bold.

2. Surface viscoelasticity

Let us consider a deformed solid body B which occupies in a reference placement κ a volume $V \subset \mathbb{R}^3$ with a smooth enough boundary $S = \partial V$. Deformations of B can be described as a smooth invertible mapping from κ into a current placement $\chi(t)$ (Truesdell & Noll, 2004)

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \tag{1}$$

where $\mathbf{X}$ and $\mathbf{x}$ are the position vectors in κ and χ, respectively, and t is time.

We introduce the deformation gradient $\mathbf{A}$ and the surface deformation gradient $\mathbf{F}$ as follows

$$\mathbf{A} = \nabla_\kappa \mathbf{x}, \quad \mathbf{F} = \nabla_S \mathbf{x}, \tag{2}$$

where ∇_κ and ∇_S are the Lagrangian 3D and 2D nabla-operators defined in V and on S. They are related to each other through the formula

$$\nabla_S = \mathbf{I}_2 \cdot \nabla_\kappa,$$

where $\mathbf{I}_2 = \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$, $\mathbf{I}$ is the unit tensor, "$\cdot$" stands for the dot product, "$\otimes$" denotes the dyadic product, and $\mathbf{N}$ is the unit outward normal vector to S.

In the following we restrict ourselves to elastic behaviour in the bulk. So there exists a strain energy density W introduced as a function of $\mathbf{A}$ (Ogden, 1997; Truesdell & Noll, 2004)

$$W = W(\mathbf{A}). \tag{3}$$

In the bulk we have the Piola–Kirchhoff stress tensor of the first kind $\mathbf{P}$ and the Cauchy stress tensor $\mathbf{T}$ given by the formulae (Eremeyev et al., 2018; Lurie, 1990)

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{A}}, \quad \mathbf{T} = J\mathbf{A}^{-T} \cdot \mathbf{P}, \tag{4}$$

where $J = \det \mathbf{A}$ and superscript "T" denotes the transposed tensor.

We also introduce the kinetic energy density by the standard formula

$$K = \frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v}, \quad \mathbf{v} = \dot{\mathbf{x}}, \tag{5}$$

where ρ is a mass density in current placement χ and the overdot stands for the derivative with respect to t.

As a result, in the bulk we have the following Eulerian equation of motion

$$\nabla_\chi \cdot \mathbf{T} + \rho \mathbf{f} = \rho \ddot{\mathbf{x}}, \tag{6}$$

where $\mathbf{f}$ is mass force vector and ∇_χ is the 3D nabla-operator in χ and $\mathbf{n}$ is the unit normal to the boundary of B in the current placement.

Following Gurtin and Murdoch (1975, 1978) we introduce the surface Cauchy stress tensor $\mathbf{S}$ and the surface mass density m. So we get the surface kinetic energy density

$$K_s = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}, \tag{7}$$

and on the boundary we have non-trivial boundary condition

$$\mathbf{n} \cdot \mathbf{T} = \nabla_s \cdot \mathbf{S} - m\ddot{\mathbf{x}}, \tag{8}$$

where $\nabla_s = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla_\chi$ is the surface nabla-operator in χ, and $\mathbf{n}$ is the normal to the surface of B in the current placement. Eq. (8) plays a crucial role in dynamics of solids with surface stresses.

In order to take into account surface viscoelasticity we consider $\mathbf{S}$ dependent of history of deformations as follows

$$\mathbf{S} = \mathcal{A}(\mathbf{F}^t(s)), \quad \mathbf{F}^t(s) = \mathbf{F}(t - s), \quad s \geq 0, \tag{9}$$

where $\mathcal{A}$ is an operator describing dependence on the history of deformations $\mathbf{F}^t(s)$. Let us recall that according to the principle of material frame indifference $\mathbf{S}$ should be an indifferent (objective) tensor. This means it has some invariance properties under rigid body motions. So if we consider equivalent motion $\mathbf{x}^*$ as

$$\mathbf{x}^* = \mathbf{a}(t) + \mathbf{x} \cdot \mathbf{O}(t)$$

where vector $\mathbf{a}$ and $\mathbf{O}$ are time-dependent vector and orthogonal tensor, respectively, then we get that $\mathbf{S}^* = \mathbf{O}^T \cdot \mathbf{S} \cdot \mathbf{O}$. Under this transformation we also have $\mathbf{F}^* = \mathbf{F} \cdot \mathbf{O}$. So $\mathcal{A}$ has the following property

$$\mathcal{A}(\mathbf{F}^t(s) \cdot \mathbf{O}(t-s)) = \mathbf{O}^T(t) \cdot \mathcal{A}(\mathbf{F}^t(s)) \cdot \mathbf{O}(t) \quad \forall \mathbf{O}: \ \mathbf{O} \cdot \mathbf{O}^T = \mathbf{I}. \tag{10}$$

For $\mathbf{F}$ we have the polar decomposition in the form

$$\mathbf{F} = \mathbf{U} \cdot \mathbf{Q}, \tag{11}$$

where $\mathbf{Q}$ is an orthogonal tensor and $\mathbf{U}$ is a symmetric non-negative tensor. Note that as $\mathbf{N} \cdot \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a singular tensor. So the standard polar decomposition requires some modifications, see e.g. Eremeyev et al. (2018) for more details. Taking $\mathbf{O} = \mathbf{Q}^T$ we came to another form of the constitutive equation for $\mathbf{S}$

$$\mathbf{S} = \mathbf{Q}^T(t) \cdot \mathcal{B}(\mathbf{U}^t(s)) \cdot \mathbf{Q}(t) \tag{12}$$

with new operator $\mathcal{B}$. Eq. (12) is a general form of surface stresses for simple materials. Nevertheless, some modifications of (12) are possible. Since $\mathbf{U}^2 = \mathbf{C} \equiv \mathbf{F} \cdot \mathbf{F}^T$ we can use the surface Cauchy–Green strain tensor $\mathbf{C}$ instead of its square root $\mathbf{U}$:

$$\mathbf{S} = \mathbf{Q}^T(t) \cdot \mathcal{B}(\mathbf{C}^t(s)) \cdot \mathbf{Q}(t)$$

with another operator $\mathcal{C}$. Moreover, instead of $\mathbf{Q}$ we can use $\mathbf{F}$. Indeed, $\mathbf{Q} = \mathbf{U}^{-1} \cdot \mathbf{F}$, where by the inverse of $\mathbf{U}$ we understood its inversion in the corresponding subspace, see Eremeyev et al. (2018). Finally we came to the form

$$\mathbf{S} = \mathbf{F}^T(t) \cdot \mathcal{S}(\mathbf{C}^t(s)) \cdot \mathbf{F}(t), \tag{13}$$

which is straightforward analogy of the 3D case, see Truesdell (1966, 1991) and Truesdell and Noll (2004).

For further specification of operator $\mathcal{S}(\mathbf{C}^t(s))$ in (13) we can use the concept of fading memory and the relative tensors as in Truesdell (1966, 1991) and Truesdell and Noll (2004). First, we introduce the relative surface gradient tensor $\mathbf{F}_t(\tau)$. To this end we consider $\chi(t)$ as the reference placement and $\chi(\tau)$ as a current one. Here t and τ are some time instants. So $\mathbf{F}_t(\tau)$ is defined through the formula

$$\mathbf{F}_t(\tau) = \nabla_{\chi(t)}\mathbf{x}(\tau),$$

where we have specified the nabla-operator taken in $\chi(t)$. We have the formula related $\mathbf{F}_t(\tau)$ to $\mathbf{F}(\tau)$ and $\mathbf{F}(t)$:

$$\mathbf{F}(\tau) = \mathbf{F}(t) \cdot \mathbf{F}_t(\tau). \tag{14}$$

Obviously, $\mathbf{F}_t(t) = \mathbf{I}$. Using $\mathbf{F}_t(\tau)$ we can introduce the relative surface Cauchy–Green strain tensor $\mathbf{C}_t(\tau)$ by the formula

$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau) \cdot \mathbf{F}_t^T(\tau).$$

Similar to (14) we have that

$$\mathbf{C}(\tau) = \mathbf{F}(t) \cdot \mathbf{C}_t(\tau) \cdot \mathbf{F}^T(t), \tag{15}$$

and $\mathbf{C}_t(t) = \mathbf{I}$. So $\mathbf{C}_t(\tau)$ can be treated as a relative strain measure describing deformations between $\chi(t)$ and $\chi(\tau)$. Introducing the history of the relative Cauchy–Green strain tensor $\mathbf{C}_t^t(s) = \mathbf{C}_t(t-s)$ we came to another form of constitutive equation for $\mathbf{S}$

$$\mathbf{S} = \mathbf{F}^T(t) \cdot \mathbf{S}_e(\mathbf{C}(t)) \cdot \mathbf{F}(t) + \mathbf{F}^T(t) \cdot \mathcal{S}_v\left(\mathbf{G}_t^t(s), \mathbf{C}(t)\right) \cdot \mathbf{F}(t), \tag{16}$$

where $\mathbf{G}_t^t(s) = \mathbf{Q}(t) \cdot \mathbf{C}_t^t(s) \cdot \mathbf{Q}(t)^T - \mathbf{I}$, $\mathbf{S}_e$ is a tensor-valued function of the current value of $\mathbf{C}$, and a history-dependent operator $\mathcal{S}_v$ vanishes when $\mathbf{G} = \mathbf{0}$:

$$\mathcal{S}_v(\mathbf{0}, \mathbf{C}(t)) = \mathbf{0}.$$

Constitutive Eq. (16) is a sum of an "equilibrium term" and a "viscoelastic term" that vanishes when the material was always in the rest.

As an example of constitutive equations we can consider so-called linear finite surface viscoelasticity with $\mathcal{S}_v$ given by

$$\mathcal{S}_v = \int_{-\infty}^{0} \mathbf{K}(\mathbf{C}(t), s) : \mathbf{G}_t^t(s)\, ds, \tag{17}$$

where $\mathbf{K}$ is a fourth-order tensor (kernel) dependent on $\mathbf{C}$ and s, ":" stands for the double-dot product. Note that by the linear finite surface viscoelasticity here we mean linear dependence on history $\mathbf{C}_t^t(s)$. Other nonlinear integral constitutive relations can be introduced similarly to 3D finite viscoelasticity, see e.g. Christensen (1971, 1980), Truesdell and Noll (2004).

If we restrict ourselves to isotropic material behaviour we can represent (16) as follows

$$\mathbf{S} = \mathbf{f}(\mathbf{B}(t)) + \mathcal{F}\left(\mathbf{H}_t^t(s), \mathbf{B}(t)\right), \quad \mathcal{F}(\mathbf{0}, \mathbf{B}(t)) = \mathbf{0}, \tag{18}$$

where $\mathbf{B} = \mathbf{F}^T \cdot \mathbf{F}$ is the left surface Cauchy–Green tensor, $\mathbf{H}_t^t(s) = \mathbf{C}_t^t(s) - \mathbf{I}$, and $\mathbf{f}$ and $\mathcal{F}$ satisfy the isotropy conditions

$$\mathbf{O} \cdot \mathbf{f}(\mathbf{B}) \cdot \mathbf{O}^T = \mathbf{f}(\mathbf{O} \cdot \mathbf{B} \cdot \mathbf{O}^T), \quad \forall \ \mathbf{O} : \mathbf{O} \cdot \mathbf{O}^T = \mathbf{I},$$

$$\mathbf{O}(t) \cdot \mathcal{F}\left(\mathbf{H}_t^t(s), \mathbf{B}(t)\right) \cdot \mathbf{O}^T(t) = \mathcal{F}\left(\mathbf{O}_t^t(s) \cdot \mathbf{H}_t^t(s) \cdot \mathbf{O}_t^t(s)^T, \mathbf{O}(t) \cdot \mathbf{B}(t) \cdot \mathbf{O}^T(t)\right).$$

Another example of constitutive relations can be introduced using surface Rivlin–Ericksen tensors $\mathbf{A}_j$. Using the formal series expansion

$$\mathbf{C}_t^t(s) = \sum_{i=0}^{\infty} \frac{(-1)^i s^i}{i} \mathbf{A}_i(t) \tag{19}$$

and taking only a finite number n of term in (19), we came to the constitutive relations of differential type of order n

$$\mathbf{S} = \mathbf{f}(\mathbf{B}(t)) + \mathbf{L}\left(\mathbf{B}(t), \mathbf{A}_1(t), \dots \mathbf{A}_n(t)\right), \quad \mathbf{L}(\mathbf{B}(t), \mathbf{0}, \dots \mathbf{0}) = \mathbf{0}, \tag{20}$$

where $\mathbf{L}$ is a tensor-valued function. Tensors $\mathbf{A}_j$ can be introduced using the recurrent formulae as follows

$$\mathbf{A}_{i+1} = \dot{\mathbf{A}}_i + \nabla_\chi \mathbf{v} \cdot \mathbf{A}_i + \mathbf{A}_i \cdot (\nabla_\chi \mathbf{v})^T, \quad \mathbf{A}_1 = 2\mathbf{D} \equiv \left(\nabla_\chi \mathbf{v} \cdot \mathbf{I}_2' + \mathbf{I}_2' \cdot (\nabla_\chi \mathbf{v})^T\right). \tag{21}$$

Here $\mathbf{I}_2' = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ and $\mathbf{D}$ is the surface strain rate.

The simplest case is the viscoelastic material of order 1 with constitutive relation

$$\mathbf{S} = \mathbf{f}(\mathbf{B}) + \mathbf{L}(\mathbf{B}, \mathbf{D}), \quad \mathbf{L}(\mathbf{B}, \mathbf{0}) = \mathbf{0}. \tag{22}$$

As $\mathbf{B}$ and $\mathbf{D}$ are symmetric 2D tensors, from isotropy conditions it follows that $\mathbf{f}$ and $\mathbf{L}$ have the form

$$\mathbf{f}(\mathbf{B}) = f_0(I_1, I_2)\mathbf{I}_2' + f_1(I_1, I_2)\mathbf{B}, \tag{23}$$

$$\mathbf{L}(\mathbf{B}, \mathbf{D}) = \ell_0 \mathbf{I}_2' + \ell_1 \mathbf{B} + \ell_1 \mathbf{D}, \tag{24}$$

where scalar coefficients f_0, f_1, and $\ell_i = \ell_i(I_1, \dots, I_5)$ are functions of invariants I_k, $k = 1, \dots, 5$, are given by

$$I_1 = \operatorname{tr} \mathbf{B}, \quad I_2 = \operatorname{tr} \mathbf{B}^2, \quad I_2 = \operatorname{tr} \mathbf{D}, \quad I_4 = \operatorname{tr} \mathbf{D}^2, \quad I_5 = \operatorname{tr} (\mathbf{D} \cdot \mathbf{B}),$$

see e.g. Zubov (1982) for representation of isotropic functions. Restricting ourselves to linear dependence on $\mathbf{D}$ we came to the following representation of the surface stresses

$$\begin{aligned}\mathbf{S} =& f_0(I_1, I_2)\mathbf{I}_2' + f_1(I_1, I_2)\mathbf{B} + g_1(I_1, I_2)(\operatorname{tr} \mathbf{D})\mathbf{I}_2' + g_2(I_1, I_2)(\operatorname{tr} \mathbf{D})\mathbf{B} \\ &+ g_3(I_1, I_2)\operatorname{tr} (\mathbf{D} \cdot \mathbf{B})\mathbf{I}_2' + g_4(I_1, I_2)\operatorname{tr} (\mathbf{D} \cdot \mathbf{B})\mathbf{B} + g_6(I_1, I_2)\mathbf{D}.\end{aligned} \tag{25}$$

with new coefficients g_k. Since here we have 2D tensors, this representation of isotropic linearly viscous material is more simple than its 3D counterpart, see Eq. (41.8) in Truesdell and Noll (2004).

Considering infinitesimal deformations Eqs. (16) with (17) or (25) can be transformed to linear surface viscoelasticity with integral or differential form of governing equations. For example, Eq. (25) became

$$\mathbf{S} = \lambda_s \mathbf{I}_2 \operatorname{tr} \mathbf{e} + 2\mu_s \mathbf{e} + \lambda_v \mathbf{I}_2 \operatorname{tr} \dot{\mathbf{e}} + 2\mu_v \dot{\mathbf{e}}, \tag{26}$$

where $\mathbf{e} = \frac{1}{2}\left(\nabla_\kappa \mathbf{u} \cdot \mathbf{I}_2 + \mathbf{I}_2 \cdot (\nabla_\kappa \mathbf{u})^T\right)$ is the surface strain tensor. Eq. (26) is a surface analogy of the Kelvin–Voigt model of 3D viscoelasticity, see e.g. Christensen (1971). Note that here we have four material parameters that are the surface Lamé moduli λ_s and μ_s, and viscosity moduli λ_v and μ_v.

3. Antiplane surface waves

As an example, let us consider anti-plane surface waves in an elastic half-space with viscoelastic surface stresses. Let it takes the volume $X_2 \le 0$, where X_1, X_2, and X_3 are the Cartesian coordinates with corresponding unit base vectors $\mathbf{i}_k$, so $\mathbf{N} = \mathbf{n} = \mathbf{i}_2$, see Fig. 1. In the following we restrict ourselves to infinitesimal deformations and isotropic material behaviour. So in the bulk we have Hooke's law

$$\mathbf{T} = 2\mu\varepsilon + \lambda \mathbf{I} \operatorname{tr} \varepsilon, \quad \varepsilon = \frac{1}{2}\left(\nabla_\kappa \mathbf{u} + (\nabla_\kappa \mathbf{u})^T\right), \tag{27}$$

where λ and μ are the Lamé moduli, and we use (26) as constitutive equation for the surface stresses.

Let us note that anti-plane shear gives the simplest example of motions (Achenbach, 1973). The displacement vector has the form

$$\mathbf{u} = u(X_1, X_2, t)\mathbf{i}_3. \tag{28}$$

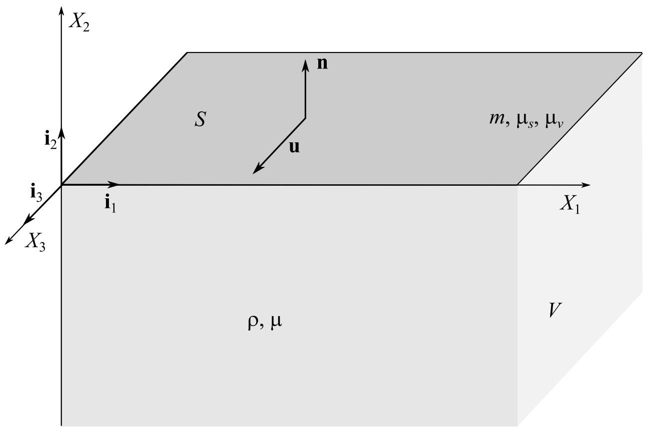

Fig. 1. Elastic half-space with viscoelastic coating.

From (28) it follows that

$$\nabla_\kappa \mathbf{u} = \nabla_\kappa u \otimes \mathbf{i}_3, \quad \nabla_s \mathbf{u} = u,_1 \mathbf{i}_1 \otimes \mathbf{i}_3,$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla_\kappa u \otimes \mathbf{i}_3 + \mathbf{i}_3 \otimes \nabla_\kappa u), \quad \mathbf{e} = \frac{1}{2}\left(u,_1 \mathbf{i}_1 \otimes \mathbf{i}_3 + u,_1 \mathbf{i}_3 \otimes \mathbf{i}_1\right),$$

where $u_{,\eta} = \frac{\partial u}{\partial X_\eta}$, and $\eta = 1, 2$. In this case we have

$$\mathbf{T} = 2\mu\boldsymbol{\varepsilon}, \quad \mathbf{S} = 2\mu_s \mathbf{e} + 2\mu_v \dot{\mathbf{e}}, \tag{29}$$

As a result, equation of motion (6) takes the form of wave equation

$$\mu \Delta u = \rho \ddot{u}, \quad \Delta u = u,_{11} + u,_{22}, \tag{30}$$

whereas the boundary condition (8) transform into

$$\mu u,_2 = \mu_s u,_{11} + \mu_v \dot{u},_{11} - m\ddot{u} \quad \text{at} \quad X_2 = 0. \tag{31}$$

Following Eremeyev et al. (2016) we are looking for a solution in the harmonic form

$$u = U(X_1, X_2)e^{-i\omega t}, \tag{32}$$

where ω is the angular velocity and $i = \sqrt{-1}$. Eqs. (30) and (31) take the form

$$\mu \Delta U = -\rho\omega^2 U, \quad \mu U,_2 = (\mu_s - i\omega\mu_v)U,_{11} + m\omega^2 U, \tag{33}$$

Assuming that U decays with the distance from the half-space surface $X_2 = 0$, we find the solution of (33) in form

$$U = U_0 e^{\sqrt{k^2-\omega^2/c_T^2}X_2} e^{ikX_1}, \tag{34}$$

where U_0 is a complex amplitude, and $c_T = \sqrt{\mu/\rho}$ is the phase velocity of transverse waves (Achenbach, 1973). Note that for a solution relating to a surface wave we assume that the following condition is satisfied

$$\operatorname{Re} \varkappa > 0, \quad \varkappa = \sqrt{k^2 - \omega^2/c_T^2}.$$

Hereinafter Re and Im denote real and imaginary parts of a complex number, respectively. As a result, the solution of (30) takes the form

$$u = U_0 e^{\sqrt{k^2-\omega^2/c_T^2}X_2} e^{i(kX_1-\omega t)}. \tag{35}$$

Note that unlike the elastic case discussed in Eremeyev et al. (2016), here k is a complex wavenumber, in general. So $k = \alpha + i\beta$ where $\alpha = \operatorname{Re}k$, and $\beta = \operatorname{Im}k$ relates to an attenuation of the wave in direction of propagation. Substituting (35) into $(33)_2$ we get the complex dispersion relation

$$\mu\sqrt{k^2 - \omega^2/c_T^2} = -(\mu_s - i\omega\mu_v)k^2 + m\omega^2, \tag{36}$$

that relates α and β with ω. Typical dependencies of α and β on ω are shown in Fig. 2. Here $\bar{\alpha} = \alpha l_d$, $\bar{\beta} = \beta l_d$, $\bar{\omega} = \omega T_s$, where $l_d = m/\rho$ is the dynamic characteristic length and $T_s = l_d/c_s$ is the characteristic time in the Gurtin–Murdoch model, respectively, and $c_s = \sqrt{\mu_s/m}$ is the surface shear wave velocity. Here we used $c_s = c_T/4$.

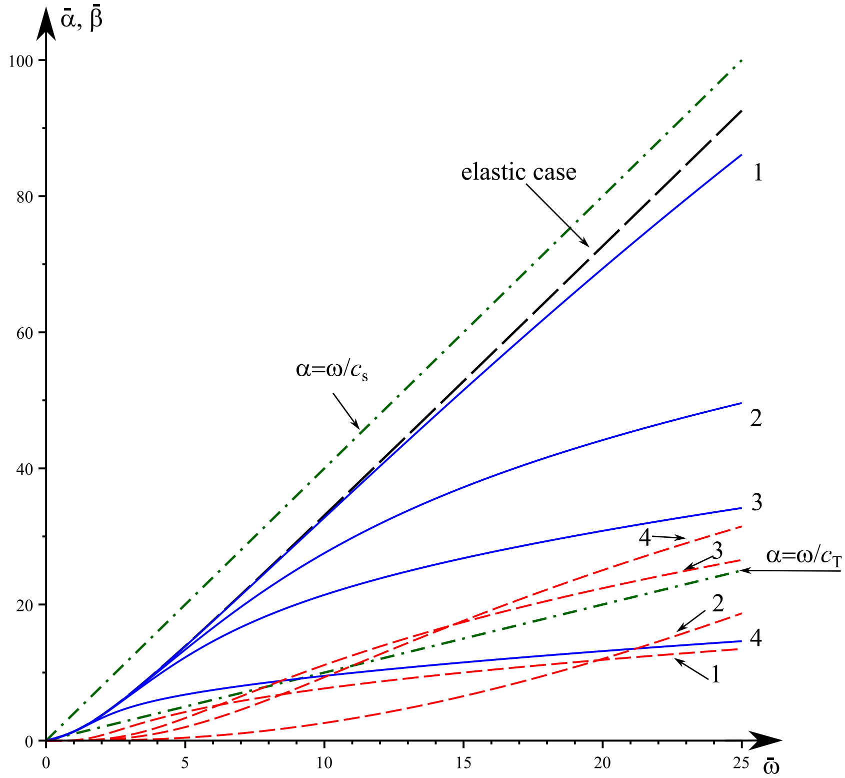

Fig. 2. Dispersion relations. Solid and dashed curves corresponds to $\alpha-\omega$ and $\beta-\omega$ dependencies, respectively. Labels 1, 2, 3, and 4 corresponds to the following values of ratio $\mu_v/\mu_s T_s$: 0.02, 0.1, 0.2, and 1, respectively. Longdashed curve corresponds to the elastic case.

In Fig. 2 the dashed curve corresponds to the elastic case (Eremeyev et al., 2016), i.e. when $\mu_v = 0$. Let us recall that for a pure elastic material the dispersion curve begins at point $(0,0)$ where it has a tangent given by $\alpha = \omega/c_T$. Then it becomes almost parallel to the line $\alpha = \omega/c_s$. So it lies in the sector bounded by these two lines. Note that at $\omega \to \infty$ we have almost non-dispersive waves.

In the case of viscoelastic materials dispersion curves begin again at $(0,0)$ having the same tangent $\alpha = \omega/c_T$. Initially they are following the elastic case. Then they significantly deviate. For a small viscosity dispersion curve almost follows elastic one in a certain relatively large frequency range, see e.g. solid blue curve 1 in Fig. 2. For large viscosity the difference may be significant even for a relatively small ω, see solid blue curve 4 in Fig. 2. Curves 1, 2, 3, and 4 correspond to $\mu_v = q\mu_s T_s$, where $q = 0.02$, 0.1, 0.2, and 1, respectively. Obviously, the attenuation depends on viscosity, see dashed red curves in Fig. 2.

Let us also underline that viscosity changes decay of the solutions with the depth. Indeed, for an elastic material we exponentially decaying solutions, whereas for a viscoelastic material one can see some oscillations with the depth. In Fig. 3 we provide graphs of displacement as a function of X_2. Here

$$\bar{u} = \operatorname{Re} U(0, X_2)/U_0 \equiv e^{\operatorname{Re} \varkappa X_2} \cos\left(\operatorname{Im} \varkappa X_2\right).$$

Depending on ω and μ_v one can see difference in the decay. Curves 1–4 correspond to the same values of μ_v as in Fig. 2. Fig. 3 (a) and (b) relate to $\bar{\omega} = 5$ and $\bar{\omega} = 50$, respectively. Dashed curves corresponds to elastic behaviour. For low viscosity and low frequency the amplitude is almost coincide with the elastic case, see e.g. curve 1 in Fig. 3 (a). For higher frequencies the difference is distinguishable, see again curve 1 in Fig. 3 (b). Unlike the elastic material there values of the depths where the displacement vanishes.

4. Conclusions

We introduced surface finite viscoelasticity with surface stresses dependent on a deformation history. Some particular cases of constitutive relations are presented. The latter include the linear finite surface viscoelasticity and viscoelastic materials of the differential type. In the case of infinitesimal deformations these relations transform into linear surface viscoelasticity. As an example we discussed propagation of anti-plane surface waves in an elastic half-space with a thin coating modelled within the surface Kelvin–Voigt viscoelastic model. We demonstrated that even small dissipation essentially changes the behaviour of dispersion curves and the decay with the depth. This is more pronounced for relatively large values of ω. In a similar way other types of surface or interfacial waves can be studied.

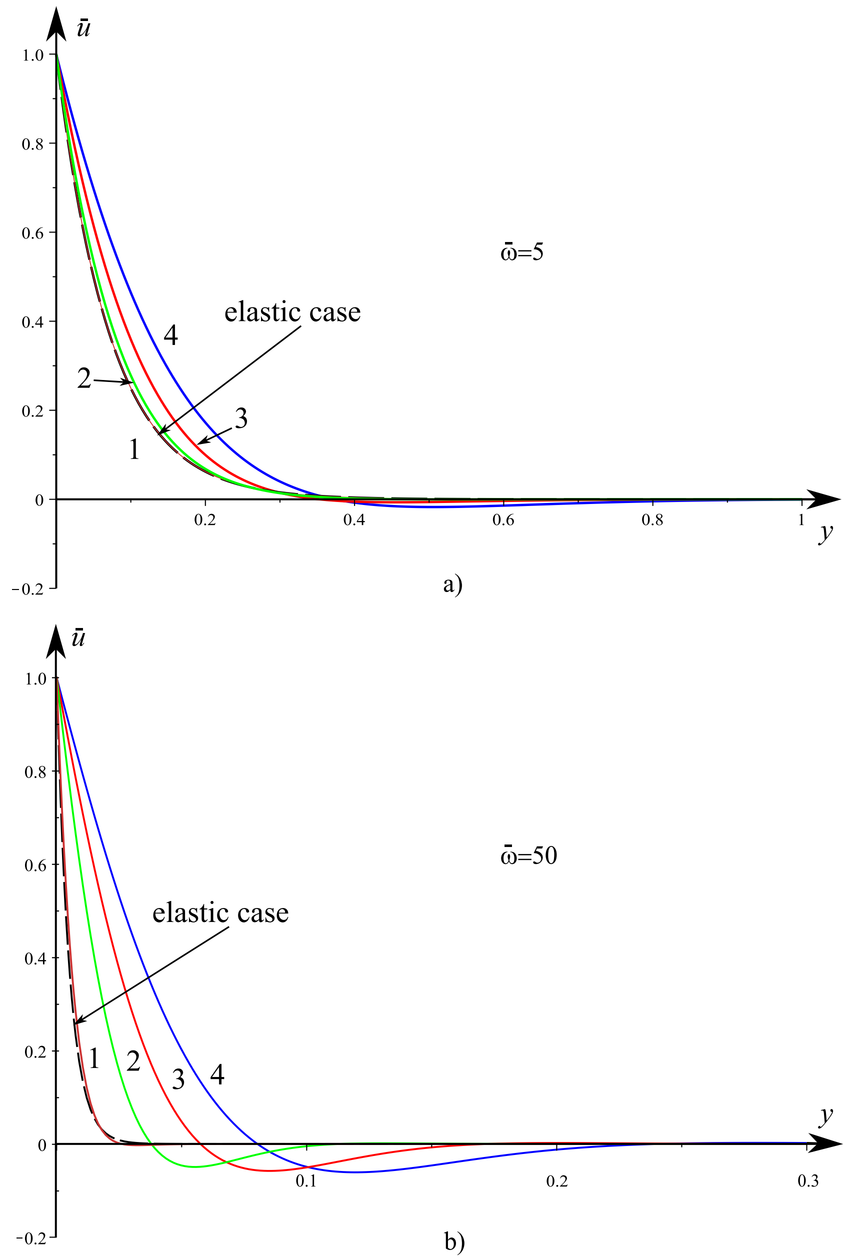

Fig. 3. Displacement $\bar{u}$ *vs* depth $y = |X_2/l_d|$. Other parameters are same as in Fig. 2.

Let us note that surface viscoelasticity model can be also useful for determination of elastic properties of inhomogeneous material at small scales as was done in the case of surface elasticity, see e.g. Dai and Schiavone (2023), Duan et al. (2008), Eremeyev (2016), Firooz et al. (2021), Jiang et al. (2022), Kushch and Mogilevskaya (2022), Mogilevskaya et al. (2021), Shugailo et al. (2023), Wang et al. (2011), Yang et al. (2023), Zheng et al. (2021) and the references therein. In particular, within the finite surface viscoelasticity initial/residual surface stresses can be easily introduced, that can be useful for description of surface stress relaxation phenomena.

In addition we also underline that the presented approach above can be extended towards other models of surface elasticity. In particular, the model by Steigmann and Ogden (1997, 1999) can be extended to material with memory. Following Truesdell and Noll (2004) the Steigmann–Ogden model can be treated as constitutive equation of a 2D material of grade 2. It could be extended towards materials with memory as in Truesdell and Noll (2004). Moreover, for materials of differential type the derived for viscoelastic Kirchhoff–Love shells surface Rivlin–Ericksen tensors can be used, see Zubov (1982).

CRediT authorship contribution statement

Victor A. Eremeyev: Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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