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System information propagation for composite structures

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We study in detail the decoherence process of a quantum register, coupled to a composite environment. We use recently developed methods of information transfer study in open quantum systems to analyze information flow between the register and its environment. We show that there are regimes when not only the register decoheres effectively to a classical bit string, but this bit string is redundantly encoded in the environment, making it available to multiple observers. This process is more subtle and in general qualitatively different than in the case of a single qubit due to the presence of various protected subspaces: decoherence-free subspaces and so-called orthogonalization-free subspaces. We show that this leads to a rich structure of coherence loss or protection in the asymptotic state of the register and a part of its environment. We formulate a series of examples illustrating these structures.

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I. INTRODUCTION

In our previous paper [1] we investigated the process of formation of the so-called spectrum broadcasting structure (SBS)[1] in the spin-spin model [2–5]. Our main result was the description of how the interaction of such a form leads to the objectivization of the information about the central system. The aim of this paper is to develop the ideas sketched previously: generalize them and apply them in particular to a quantum register. The latter, leads to qualitatively different results with respect to the previous study [1] due to the presence of various protected spaces, e.g., decoherence-free subspaces. We also consider more explicit examples of the objectivization process.

The problem of how knowledge, being a *classical* (not quantum) resource, is protruding out of the quantum regime has been deeply investigated by Żurek *et al.* in a series of works [2,3,6–11] (see [12,13] for an overview), leading to the concept of *quantum Darwinism* [14,15]. The essence of quantum Darwinism is the statement that this information about a measured system, which is being efficiently proliferated into different parts of the environment and in consequence *redundantly imprinted and stored* in them, becomes objective. Each part contains almost complete classical information about the system. This redundancy is crucial for the objectivity [14–20], as parts may be accessed independently by many observers gaining the same or similar knowledge. See Refs. [21] and [22] for experimental demonstrations of the effects of quantum Darwinism.

The problem of how some information is being distributed in many copies by an intrinsically quantum mechanism is highly nontrivial due to the no-cloning theorem [23,24] forbidding the direct copying of a state. Even a weaker form of copying of quantum states, the so-called state broadcasting, is not always possible [25,26].

Nonetheless, the broadcast information becomes accessible to many independent perceiving persons, or *observers*. Note that although the term *objectivity* is widely used in this context, the word *intersubjectivity*, in the meaning of Ajdukiewicz [27,28], is more precise.

It has been shown recently [19,29] that the emergence of the classical and objective properties in this spirit is due to a form of information broadcasting (similar to the so-called spectrum broadcasting [30]) leading to the creation of a specific quantum state structure between the system and a part of its environment, viz., the mentioned SBS. The SBS ensures different observers a perfect access to some property of the observed system. This *intersubjectivization* is an interplay between the *decoherence* and *orthogonalization* process, as explained below.

Throughout this paper we consider in particular how the intersubjectivity of the state of the central system emerges in a model of a quantum register [31,32] interacting with a spin bath. Quantum registers are collections of qubits and are analogous to registers of CPUs of classical computers. They constitute a basic building block for quantum computers [33,34]. Using the quantum measurement limit we get a rigorous description of decoherence and orthogonalization within this scenario.

One of the major problems in development of quantum computers is how to deal with errors caused by the decoherence induced by the environment, e.g., using error correcting codes [35,36]. On the other hand, in order to be able to store

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[1] See Eq. (6) below for the definition.

and process information, quantum registers should be able to distinguish between their states requiring some form of their orthogonality. This emphasizes the importance of a careful study of the two parts of intersubjectivization.

The paper is organized as follows. In Sec. II we introduce basic terms and overview the framework of SBS. Then, in Sec. III we analyze the role of coarse-graining of the environment, consisting of a large number of subsystems grouped into macroscopic parts to form composite systems, the main topic of the paper. Using the weak law of large numbers (LLN), this toolbox allows us to investigate the asymptotic behavior of the orthogonalization and decoherence processes in such models. In Sec. IV we generalize the results of [1] to the case of quantum registers using the tools developed in Sec. III. We define and calculate the so-called orthogonalization and decoherence factors for this model. We develop some tools which are useful for the analysis of the quasiperiodic functions often occurring in similar models [1,3,6,32,37]. In Sec. V we consider possible types of interaction of a quantum register within a spin environment. Finally, in Sec. VI, we give examples of orthogonalization-free and decoherence-free setups.

II. FRAMEWORK OF SPECTRUM BROADCAST STRUCTURES

We briefly describe the formalism of SBS for completeness and to set the notation. Let us consider a central system S interacting with M-partite environment. We assume the quantum measurement limit, meaning that the central interaction Hamiltonian dominates the dynamics. The assumption that the environmental subsystem does not interact simplifies the analysis significantly and is a common practice. Thus the evolution is governed by a Hamiltonian of the generalized von Neumann measurement form:

$$H_{\text{int}} = \sum_{k=1}^{K}\left(X_k \otimes \sum_{m=1}^{M} Y_{km}\right), \tag{1}$$

where $\{X_k\}$ commute and diagonalize in a basis $\{|\boldsymbol{\epsilon}\rangle\}$. The resulting evolution is given by the unitary operator:

$$U(t) \equiv \exp\left(-iH_{\text{int}}t\right) = \sum_{\boldsymbol{\epsilon}} |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}| \otimes \bigotimes_{m=1}^{M} U_{\boldsymbol{\epsilon}}^{(m)}(t), \tag{2}$$

where $U_{\boldsymbol{\epsilon}}^{(m)}(t) \equiv \exp\left(-i\sum_k \langle\boldsymbol{\epsilon}|X_k|\boldsymbol{\epsilon}\rangle Y_{km}t\right)$. We refer to a possible value of $\boldsymbol{\epsilon}$, i.e., observed (or broadcast) value of the state, as *qualitas*.[2]

Within the context of the quantum measurement limit we assume that the initial state of the central system together with the environment is in a product form $\varrho(0) = \varrho_S(0) \otimes \bigotimes_{m=1}^{M}\varrho^{(m)}(0)$. The evolved state is $\varrho(t) = U(t)\varrho(0)U^{\dagger}(t)$.

Assume that the first fM parts are under observation, $f \in (0,1)$, and the rest of them remain unobserved. The latter can be modeled by [38] by a partial trace operation, leading to the following reduced density matrix:

$$\varrho^{(fM)}(t) \equiv \mathrm{Tr}_{(1-f)M}\varrho(t) = \sum_{\boldsymbol{\epsilon}} \sigma_{\boldsymbol{\epsilon}}|\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}| \otimes \bigotimes_{m=1}^{fM}\varrho_{\boldsymbol{\epsilon}}^{(m)}(t) + \sum_{\boldsymbol{\epsilon}\neq\boldsymbol{\epsilon}'}\left[\sigma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}\prod_{m=fM+1}^{M}\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t)\right] \times |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}'| \otimes \bigotimes_{m=1}^{fM}\varrho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t), \tag{3}$$

with $\sigma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \equiv \langle\boldsymbol{\epsilon}|\varrho(0)|\boldsymbol{\epsilon}'\rangle$, $\sigma_{\boldsymbol{\epsilon}} \equiv \sigma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}$,

$$\varrho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t) \equiv U_{\boldsymbol{\epsilon}}^{(m)}(t)\varrho^{(m)}(0)U_{\boldsymbol{\epsilon}'}^{(m)\dagger}(t), \tag{4a}$$

$$\varrho_{\boldsymbol{\epsilon}}^{(m)}(t) \equiv \varrho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}^{(m)}(t), \tag{4b}$$

and $\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t) \equiv \mathrm{Tr}\left(\varrho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t)\right)$. One often defines products called the decoherence factors:

$$\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) \equiv \prod_{m=fM+1}^{M}\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t). \tag{5}$$

We say that the joint state of S and fM parts of the environment (3) constitutes an SBS if it is of the form

$$\sum_{\boldsymbol{\epsilon}} p_{\boldsymbol{\epsilon}}|\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}| \otimes \bigotimes_{m=1}^{fM}\varrho_{\boldsymbol{\epsilon}}^{(m)}(t), \tag{6}$$

with $\varrho_{\boldsymbol{\epsilon}}^{(m)}(t)$ and $\varrho_{\boldsymbol{\epsilon}'}^{(m)}(t)$ for $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'$ having orthogonal supports (meaning perfect distinguishability with a single measurement). By measuring the state $\varrho_{\boldsymbol{\epsilon}}^{(m)}(t)$, any of the local observers extracts the same information about *qualitas*, i.e., the index $\boldsymbol{\epsilon}$, without disturbing it (after forgetting the results).

A convenient measure of orthogonality of a pair of states, $\varrho_{\boldsymbol{\epsilon}}$ and $\varrho_{\boldsymbol{\epsilon}'}$, is the so-called generalized fidelity, or overlap [39], $B(\varrho_{\boldsymbol{\epsilon}},\varrho_{\boldsymbol{\epsilon}'}) \equiv \mathrm{Tr}\sqrt{\sqrt{\varrho_{\boldsymbol{\epsilon}}}\varrho_{\boldsymbol{\epsilon}'}\sqrt{\varrho_{\boldsymbol{\epsilon}}}}$. This function is multiplicative with respect to the tensor product. It can be shown [39] that $B(\varrho_{\boldsymbol{\epsilon}},\varrho_{\boldsymbol{\epsilon}'}) = 1$ if and only if (iff) $\varrho_{\boldsymbol{\epsilon}} = \varrho_{\boldsymbol{\epsilon}'}$. Let

$$B_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t) \equiv B\left(\varrho_{\boldsymbol{\epsilon}}^{(m)}(t),\varrho_{\boldsymbol{\epsilon}'}^{(m)}(t)\right). \tag{7}$$

From (2) and (4) we see that $B_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t) = 1$ for all t iff

$$\Omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)} \equiv \sum_k\left(\langle\boldsymbol{\epsilon}|X_k|\boldsymbol{\epsilon}\rangle - \langle\boldsymbol{\epsilon}'|X_k|\boldsymbol{\epsilon}'\rangle\right)Y_{km} = \mathbf{0}, \tag{8}$$

with $\mathbf{0}$ being a zero operator. What is more, if (8) holds for all unobserved ms, then $\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = 1$.

Note that (8) does not mean that the eigenvalue of the Hamiltonian (1) is degenerated. Still, this property is clearly similar to the notion of degeneracy. For this reason we say that a set Q of *qualitas* is *nondegenerate* iff for any $\boldsymbol{\epsilon},\boldsymbol{\epsilon}' \in Q$, $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'$, the value of $||\Omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{m}||_1$ averaged over ms is positive: $\langle||\Omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{m}||_1\rangle_m > 0$. This means that the nondegenerate *qualitas* can be distinguished by a statistically significant part of observers and environment. The *qualitas* contained in Q are called nondegenerate and other *qualitas* are called degenerate.

More particularly, *qualitas* $\boldsymbol{\epsilon}$ is *degenerated for the observer* m iff there exist $\boldsymbol{\epsilon}'$, $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'$ such that (8) holds. Similarly, the *qualitas* is *degenerated for the environment* if (8) holds for all unobserved ms.

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[2] The word *qualitas* is a Latin transcription by Cicero of the Greek word *ποιοτης* introduced in a similar context by Socrates in Plato's dialogue *Theaetetus*. In philosophy it means a quality, i.e., a qualitative property. This notion is deeply analyzed by phenomenology.

The importance of decoherence and overlap factors stems from the following crucial result [1]: The optimal trace-norm distance ϵ_{SBS} of the actual partially traced state (3) to an ideal SBS is bounded by

$$\epsilon_{\text{SBS}} \leqslant \sum_{\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'} \left[|\sigma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)| + \sqrt{\sigma_{\boldsymbol{\epsilon}} \sigma_{\boldsymbol{\epsilon}'}} \sum_{m=1}^{fM} B_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t) \right]. \tag{9}$$

III. COARSE GRAINING OF THE ENVIRONMENT

One of the crucial aspects of the quantum-to-classical transition is the question what does it mean to be macroscopic [40] (see also [29]). In other words, if the intersubjectivization is an effect of influence of environment large enough on large enough observers, the natural question about the meaning of "enough" arises. While we do not provide an explicit answer to the question, we try to deepen its understanding in Proposition 1.

Let us consider an N-partite Hamiltonian (1), $N \gg M$, and the initial state in a product form $\varrho(0) = \varrho_S(0) \otimes \bigotimes_{j=1}^{N} \rho^{(j)}(0)$. We divide the N environmental subsystems into M arbitrary disjoint parts, $mac_1, \ldots, mac_M$. The mth part is identified with a subset of indices $mac_m \subseteq \{1, \ldots, N\}$ and is called a macrofraction if $N_{mac}^{(m)} \equiv |mac_m|$ scales with N [29], thus introducing an environmental coarse-graining. Without loss of generality we assume that the macrofractions contain consecutive indices, with the first fM macrofractions being observed, and denote $Obs \equiv \bigcup_{m=1}^{fM} mac_m$, $N_{\text{dis}} \equiv N - |Obs|$. We can relate Y_{km} in (1) with $\sum_{j \in mac_m} Z_{kj}$ for some Z_{kj}.

$\rho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)}(t)$ and $\rho_{\boldsymbol{\epsilon}}^{(j)}(t)$ are defined analogously as in (4). For a particular mth macrofraction, mac_m, and for a given *qualitas* $\boldsymbol{\epsilon}$ we define the macrofraction states, cf. (4):

$$\varrho_{\boldsymbol{\epsilon}}^{(m)}(t) \equiv \bigotimes_{j \in mac_m} \rho_{\boldsymbol{\epsilon}}^{(j)}(t). \tag{10}$$

We now assume that the macrofractions are large enough so that we can use the law of large numbers (LLN) [41], stating that a sample average converges to the expected value.[3]

Let us denote $B^{(m)}(t) \equiv \max_{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in Q} B_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t)$ and $\gamma(t) \equiv \max_{\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in Q} \gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$, where the maxima are taken over nondegenerate *qualitas*. We show in Appendix A that if the size of macrofractions is large enough, then with high probability $B^{(m)}(t)$ and $\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$ are small, viz.:

Proposition 1. For a given observer m, any $\delta, t > 0$ and for $N_{mac}^{(m)}$ and N_{dis} large enough with probability at least $1 - \delta$ we have $B^{(m)}(t) \leqslant \delta$ and $|\gamma(t)|^2 \leqslant \delta$.

Thus from (9) the partially traced state (3) approaches in the trace norm the SBS form. We illustrate this with examples for quantum registers in Appendix C.

IV. SPECTRUM BROADCAST STRUCTURES IN QUANTUM REGISTERS

Interaction (1) has been only briefly analyzed from the point of view of information transfer in terms of SBS [1]. We provide such an analysis for a K-qubit register:

$$H_{\text{int}} = \frac{1}{2} \sum_{k=1}^{K} \left(\tilde{\sigma}_z^{(k)} \bigotimes \sum_{j=1}^{N} g_{kj} \sigma_z^{(j)} \right), \tag{11}$$

where we use $\tilde{\sigma}_z^{(k)}$ to denote the σ_z matrix acting on the kth register and $\sigma_z^{(j)}$ acts on jth of the environmental spins. The interaction strength is controlled by the coupling matrix $G = [g_{kj}]$. For $K = 1$ we get the spin-spin model, one of the canonical models of decoherence [2–4,6,42]. The interaction (11) can be rewritten in the following way:

$$\frac{1}{2} \left(\sum_{k=1}^{K} \tilde{\sigma}_z^{(k)} \right) \otimes \left[\sum_{m=1}^{fM} \left(\sum_{j \in mac_m} g_{kj} \tilde{\sigma}_z^{(j)} \right) + \sum_{j \notin Obs} g_{kj} \sigma_z^{(j)} \right]. \tag{12}$$

Derivation of the evolution operator, cf. (2), from (11) is straightforward. Now $|\boldsymbol{\epsilon}\rangle \equiv |\epsilon_1, \ldots, \epsilon_K\rangle$ is a product of $|\pm\rangle$ states of qubits of length K, $\boldsymbol{\epsilon} \in \{\pm 1\}^K$, and $U_{\boldsymbol{\epsilon}}^{(j)}(t) = \exp[-\frac{i}{2} t g_{\boldsymbol{\epsilon}}^{(j)} \sigma_z^{(j)}]$, where $g_{\boldsymbol{\epsilon}}^{(j)} \equiv \sum_{k=1}^{K} \epsilon_k g_{kj}$.

Calculation of the partially traced state $\varrho_{S:fM}(t)$, cf. (3), gives

$$\varrho_{S:fM}(t) = \sum_{\boldsymbol{\epsilon}} \sigma_{\boldsymbol{\epsilon}} |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}| \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}}^{(j)}(t)$$
$$+ \sum_{\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'} \sigma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}'| \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)}(t). \tag{13}$$

The notation $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}'$ means that there exists at least one index k such that $\epsilon_k \neq \epsilon'_k$. As one sees from (13), the candidate for the pointer basis is now the product basis $\{|\boldsymbol{\epsilon}\rangle\}$, describing a classical K-bit register.

A. Decoherence and overlap factors

Further we use the SU(2) Euler angles parametrization [43] $\rho^{(j)}(0) = R^{(j)} D^{(j)} R^{(j)\dagger}$ for the jth qubit, where

$$D^{(j)} \equiv \text{diag}(\lambda^{(j)}, 1 - \lambda^{(j)}), \tag{14a}$$

$$R^{(j)} \equiv \exp\left(i \frac{\alpha^{(j)}}{2} \sigma_z\right) \exp\left(i \frac{\beta^{(j)}}{2} \sigma_y\right) \exp\left(i \frac{\gamma^{(j)}}{2} \sigma_z\right)$$
$$= \begin{bmatrix} e^{\frac{i}{2}(\alpha^{(j)} + \gamma^{(j)})} \cos \frac{\beta^{(j)}}{2} & e^{\frac{i}{2}(\alpha^{(j)} - \gamma^{(j)})} \sin \frac{\beta^{(j)}}{2} \\ -e^{-\frac{i}{2}(\alpha^{(j)} - \gamma^{(j)})} \sin \frac{\beta^{(j)}}{2} & e^{-\frac{i}{2}(\alpha^{(j)} + \gamma^{(j)})} \cos \frac{\beta^{(j)}}{2} \end{bmatrix}. \tag{14b}$$

For a particular qubit j let

$$\vartheta^{(j)} \equiv -\tfrac{1}{2}(2\lambda^{(j)} - 1) \sin \beta^{(j)}, \tag{15a}$$

$$\zeta^{(j)} \equiv (2\lambda^{(j)} - 1) \cos \beta^{(j)}. \tag{15b}$$

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[3] Compare this approach with [3], where the central limit theorem was used in the analysis of the decoherence factors, owing to the fact that the latter can be represented as a Fourier transform of a probability measure on sums of independent random variables.

Let us introduce the following frequency for a jth qubit:

$$\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \equiv \frac{1}{2}\sum_{k=1}^{K}(\epsilon_k - \epsilon_k')g_{kj} = \frac{g^{(j)}_{\boldsymbol{\epsilon}} - g^{(j)}_{\boldsymbol{\epsilon}'}}{2}, \tag{16}$$

cf. (8) for a macrofraction.[4] From (4) we have

$$\rho^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = \frac{1}{2}\begin{bmatrix} (1+\zeta^{(j)})e^{-it\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}} & \vartheta^{(j)}e^{-\frac{it}{2}\left(g^{(j)}_{\boldsymbol{\epsilon}}+g^{(j)}_{\boldsymbol{\epsilon}'}\right)} \\ \vartheta^{(j)*}e^{\frac{it}{2}\left(g^{(j)}_{\boldsymbol{\epsilon}}+g^{(j)}_{\boldsymbol{\epsilon}'}\right)} & (1-\zeta^{(j)})e^{it\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}} \end{bmatrix}, \tag{17a}$$

$$\rho^{(j)}_{\boldsymbol{\epsilon}}(t) = \frac{1}{2}\begin{bmatrix} 1+\zeta^{(j)} & e^{-ig^{(j)}_{\boldsymbol{\epsilon}}t}\vartheta^{(j)} \\ e^{ig^{(j)}_{\boldsymbol{\epsilon}}t}\vartheta^{(j)*} & 1-\zeta^{(j)} \end{bmatrix}. \tag{17b}$$

The decoherence factor, cf. (5), for $K = 1$ is well known [3] and has the form

$$\gamma_{-+}(t) = \prod_{j\notin Obs}[\cos(g_j t) + i\zeta^{(j)}\sin(g_j t)]. \tag{18}$$

In general, decoherence factors are given by

$$\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = \prod_{j\notin Obs}\left[\cos\left(\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}t\right) - i\zeta^{(j)}\sin\left(\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}t\right)\right]. \tag{19}$$

In Appendix B we calculate the overlap functions for the mth macrofraction

$$B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = \prod_{j\in mac_m}\sqrt{1 - 4\vartheta^{(j)2}\sin^2\left(\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}t\right)}. \tag{20}$$

The functions within products in (19) and (20) are periodic in time with the frequency $\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}$. The coarse-graining of the environment into macrofractions together with random couplings turns the above functions into quasiperiodic ones. We give a more involved analysis of such functions in Appendix D.

Note that the decay of the factors at a specific moment of time by no means guarantees that the functions cannot revive. In a finite-dimensional setting they will in fact revive, but by increasing the environment size one can make the revivals highly unlikely as per Proposition 1.

B. Asymptotic behavior of $B(t)$ and $\gamma(t)$

Above we considered formulas for orthogonalization and decoherence factors for sufficiently large environments at a given moment of time. In contrast, we can also ask about long time averages of $B^{(m)}(t)$ and $\gamma(t)$ for a given finite size of the environment, as investigated in Proposition 2 below (see Appendix D for the proof):

Proposition 2. If the coupling constants $\{g_{kj}\}$ are independent and continuously distributed, then for a given m the long time averages,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)|^2dt \text{ and} \tag{21a}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)^2dt, \tag{21b}$$

do not depend on $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ and are given by the formulas

$$\overline{|\gamma|^2} \equiv \prod_{j\notin Obs}\frac{1+\zeta^{(j)2}}{2} \text{ and} \tag{22a}$$

$$\overline{B^{(m)2}} \equiv \prod_{j\in mac_m}[1 - 2\vartheta^{(j)}], \tag{22b}$$

respectively.

From Proposition 2 it follows directly that the long time averages of $B^{(m)2}(t)$ and $|\gamma(t)|^2$ do not depend on the distributions of the coupling constants. Another consequence is that under the stated assumptions all possible *qualitas* are being “seen” with the same accuracy.

In order to get some intuition about Proposition 2, let us note that the time average of a product of periodic functions in (19) and (20), all with different periods, almost surely (assuming a nondegenerate distribution of the coupling constants) is equal to the product of time averages of each function, if the time is long enough. On the other hand, for a particular term in the product the value of the coupling constant influences only the period, not the average value.

V. THE INTERSUBJECTIVIZATION PROCESS OF A QUANTUM REGISTER

Comparing (8) with (16), we see that $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are degenerate for the mth macrofraction if $\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} = 0$ for all $j \in mac_m$. In this section we discuss situations when this can occur.

As shown, e.g., in [32], the quantum register model exhibits a dynamical structure with the existence of so-called decoherence-free subspaces (DFS). We also introduce and investigate here the notion of orthogonalization-free subspaces (OFS). The former means the degeneracy of some *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ for the environment and the latter the degeneracy for *all* observers.

We say that a subspace $S \subseteq \{\pm\}^K$ exhibits a strong DFS property if

$$\forall_{t\in\mathbb{R}_+}\forall_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'\in S}\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = 1 \tag{23}$$

and a weak DFS if

$$\forall_{t\in\mathbb{R}_+}\forall_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'\in S}|\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)| = 1. \tag{24}$$

From (19) it is easy to see that the strong DFS holds iff

$$\forall_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'\in S}\forall_{j\notin Obs}\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} = 0, \tag{25}$$

and the weak DFS occurs if we have

$$\forall_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'\in S}\forall_{j\notin Obs}\left[\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} = 0 \vee \zeta^{(j)} = \pm 1\right]. \tag{26}$$

The strong DFS means that the register state remains invariant under time evolution rather than being unitarily rotated inside a DFS. This is a much more desired property from an experimentalist point of view, as such rotation could lead to the system being uncontrollable [32].

Similarly, we define OFS to occur in the case when

$$\forall_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'\in S}\forall_{m=1,\cdots fM}B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = 1, \tag{27}$$

which by (20) holds if

$$\forall_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'\in S}\forall_{j\in Obs}\left[\omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} = 0 \vee \vartheta^{(j)} = 0\right]. \tag{28}$$

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[4] For the Hamiltonian (11) we have $\Omega^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} = \omega^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}\sigma^{(j)}_z$.

It is easy to see that a difference between the definitions of the strong DFS (25) and OFS (28) includes the scope of one of the universal quantificators, covering all spins outside or inside the observed part, respectively. The coupling matrix can be divided in the following way:

$$G = \left[\begin{array}{ccc|ccc} g_{11} & \cdots & g_{1,|Obs|} & g_{1,|Obs|+1} & \cdots & g_{1,N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{K,1} & \cdots & g_{K,|Obs|} & g_{K,|Obs|+1} & \cdots & g_{K,N} \end{array}\right]. \tag{29}$$

$$\underbrace{\qquad\qquad}_{\text{observed}} \qquad \underbrace{\qquad\qquad}_{\text{unobserved}}$$

We see that the matrix G has a block structure $G = [G_1|G_2]$, where $G_1,\ G_2$ describes the interaction with the observed and unobserved part of the environment, respectively. From (16) it is obvious that (25) and (28) hold if $\boldsymbol{\epsilon} - \boldsymbol{\epsilon}'$ is in the kernel of G_2^T and G_1^T, respectively. Now let us deal in more detail with DFSs and OFSs.

A. Decoherence and orthogonalization

In order to have both decoherence and orthogonalization we need $\omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)} \neq 0$ to hold for all pairs $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ and for a statistically significant part of js, meaning *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are nondegenerate.

As given in Proposition 1, it is a usual situation that the partially traced state approaches the SBS form, meaning that the quantum register has decohered in the classical register basis $|\boldsymbol{\epsilon}\rangle$ and the information about this register is redundantly stored in the environment.

B. No decoherence and no orthogonalization

Now let us consider the situation when for some $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ and all j we have $\omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)} = 0$. This means that the *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are degenerate. We assume there are only two strings of bits $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ with such a property; if there are more the analysis is analogous. This is the OFS and strong DFS case.

From (19) one sees that the coherence between the states $|\boldsymbol{\epsilon}\rangle$ and $|\boldsymbol{\epsilon}'\rangle$ is preserved by the evolution. From (16) we also have, cf. (8),

$$U_{\boldsymbol{\epsilon}}^{(j)}(t) = U_{\boldsymbol{\epsilon}'}^{(j)}(t) \tag{30}$$

for all environments j. Thus $U(t)$, cf. (2), is given by

$$\Pi_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \otimes \bigotimes_{j=1}^{N} U_{\boldsymbol{\epsilon}}^{(j)}(t) + \sum_{\boldsymbol{\epsilon}'' \neq \boldsymbol{\epsilon}, \boldsymbol{\epsilon}'} |\boldsymbol{\epsilon}''\rangle\langle\boldsymbol{\epsilon}''| \otimes \bigotimes_{j=1}^{N} U_{\boldsymbol{\epsilon}''}^{(j)}(t), \tag{31}$$

where $\Pi_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \equiv |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}| + |\boldsymbol{\epsilon}'\rangle\langle\boldsymbol{\epsilon}'|$ is the projector on the register subspace spanned by $\{|\boldsymbol{\epsilon}\rangle, |\boldsymbol{\epsilon}'\rangle\}$. In particular, $B_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t) = \gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = 1$ for all m.

If there are no other DFSs and the conditions for formation of the broadcast state are met apart from the pair $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}'$, i.e., all the decoherence and orthogonalization factors apart from $B_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(m)}(t)$ and $\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$ disappear, the partially traced state approaches the following SBS:

$$\varrho_{S:fM}(t) = \Pi_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \varrho_S(0) \Pi_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}}^{(j)}(t) + \sum_{\boldsymbol{\epsilon}'' \neq \boldsymbol{\epsilon}, \boldsymbol{\epsilon}'} \sigma_{\boldsymbol{\epsilon}''} |\boldsymbol{\epsilon}''\rangle\langle\boldsymbol{\epsilon}''| \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}''}^{(j)}(t). \tag{32}$$

Information that leaked into the environment about the register's state $|\boldsymbol{\epsilon}\rangle$ is not complete, viz. it is impossible to tell if the register is in the state $|\boldsymbol{\epsilon}\rangle$ or $|\boldsymbol{\epsilon}'\rangle$ by observing the environment, and this holds no matter how big the macrofractions are. The information is simply not in the environment. Moreover, the $|\boldsymbol{\epsilon}\rangle$ and $|\boldsymbol{\epsilon}'\rangle$ block of the initial state $\varrho_S(0)$ is fully preserved by the dynamics in this case.

C. Decoherence without orthogonalization

Next we consider the case when for some $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ it holds that $\omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)} = 0$ for all $j \in Obs$ and $\omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)} \neq 0$ for all $j \notin Obs$. This means that *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are degenerate for all observers but not for the environment.

From (19) and (20) it follows that the decoherence takes place but the orthogonalization does not. This is a relatively common situation in real life, when the environment is unable to store faithfully information about the decohering system (e.g., due to too high intrinsic noise as compared to the interaction strength).

The property (30) holds again here for all j in all observed macrofractions, which implies that $\rho_{\boldsymbol{\epsilon}}^{(j)}(t) = \rho_{\boldsymbol{\epsilon}'}^{(j)}(t)$. Thus the resulting asymptotic state is similar to (32) but with destroyed coherences:

$$\varrho_{S:fM}(t) = (\sigma_{\boldsymbol{\epsilon}} |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}| + \sigma_{\boldsymbol{\epsilon}'} |\boldsymbol{\epsilon}'\rangle\langle\boldsymbol{\epsilon}'|) \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}}^{(j)}(t) + \sum_{\boldsymbol{\epsilon}'' \neq \boldsymbol{\epsilon}, \boldsymbol{\epsilon}'} \sigma_{\boldsymbol{\epsilon}''} |\boldsymbol{\epsilon}''\rangle\langle\boldsymbol{\epsilon}''| \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}''}^{(j)}(t). \tag{33}$$

Observing the string of bits $\boldsymbol{\epsilon}$ in the environment only tells us that the system is with probability $\frac{\sigma_{\boldsymbol{\epsilon}}}{\sigma_{\boldsymbol{\epsilon}} + \sigma_{\boldsymbol{\epsilon}'}}$ in the state $|\boldsymbol{\epsilon}\rangle$ and with probability $\frac{\sigma_{\boldsymbol{\epsilon}'}}{\sigma_{\boldsymbol{\epsilon}} + \sigma_{\boldsymbol{\epsilon}'}}$ in the state $|\boldsymbol{\epsilon}'\rangle$.

D. Orthogonalization without decoherence

The last possible situation is when for some $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ we have $\omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)} \neq 0$ for all $j \in Obs$ and $\omega_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)} = 0$ for all $j \notin Obs$. This is a reversed situation to the one above: the decoherence does not take place, but the orthogonalization does. The meaning is that *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are degenerate only for the environment.

This situation is quite peculiar and only possible because orthogonalization and decoherence are driven by different parts of the environment: the observed and the unobserved, respectively. Otherwise, one can prove that for the same portion of the environment it always holds[5] $|\gamma(t)| \leqslant B(t)$. The asymptotic state is in this case

$$\varrho_{S:fM}(t) = \sum_{\tilde{\boldsymbol{\epsilon}}} \sigma_{\tilde{\boldsymbol{\epsilon}}} |\tilde{\boldsymbol{\epsilon}}\rangle\langle\tilde{\boldsymbol{\epsilon}}| \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}}^{(j)}(t) + \sigma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) |\boldsymbol{\epsilon}\rangle\langle\boldsymbol{\epsilon}'| \otimes \bigotimes_{j \in Obs} \rho_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}^{(j)}(t) + \text{H.c.}, \tag{34}$$

---

[5] Let $\gamma(t) = \mathrm{tr}(\varrho U_i^\dagger U_j) \equiv \mathrm{tr}(\sqrt{\varrho} V \sqrt{\varrho}) \equiv \mathrm{tr} X$, where we introduced $V \equiv U_i^\dagger U_j$ and $X \equiv \sqrt{\varrho} V \sqrt{\varrho}$. On the other hand, $B(t) = \mathrm{tr}(\sqrt{U_i \sqrt{\varrho} U_i^\dagger U_j \varrho U_j^\dagger U_i \sqrt{\varrho} U_i^\dagger}) = \mathrm{tr}(\sqrt{\sqrt{\varrho} V \sqrt{\varrho} \sqrt{\varrho} V^\dagger \sqrt{\varrho}}) = \mathrm{tr} \sqrt{X X^\dagger}$. It is now an easy consequence of the polar decomposition that for any X, $|\mathrm{tr} X| \leqslant \mathrm{tr} \sqrt{X X^\dagger}$, from which $|\gamma(t)| \leqslant B(t)$.

where, for each j, $\varrho^{(j)}_{\tilde{\boldsymbol{\epsilon}}}$ are fully distinguishable for all $\tilde{\boldsymbol{\epsilon}}$, including $\varrho^{(j)}_{\boldsymbol{\epsilon}}$ and $\varrho^{(j)}_{\boldsymbol{\epsilon}'}$.

The state (34) possesses what one may call a genuine multipartite coherence (which can include entanglement). Indeed, let us trace out one, say mth, of the observed macrofractions. This produces an extra decoherence factor, $\gamma^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = \prod_{j\in mac_m} \mathrm{Tr}\,(U^{(j)}_{\boldsymbol{\epsilon}}\rho^{(j)}(0)U^{(j)\dagger}_{\boldsymbol{\epsilon}'})$. By the quoted property $|\gamma^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)| \leqslant B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$, so the assumed vanishing of the latter implies destruction of the coherences. Thus, tracing out a single macrofraction for the asymptotic state destroys all the remaining coherences and brings the state to an SBS form:

$$\mathrm{Tr}_{mac}\,\varrho_{S:fM} = \sum_{\tilde{\boldsymbol{\epsilon}}} \sigma_{\tilde{\boldsymbol{\epsilon}}} |\tilde{\epsilon}\rangle\langle\tilde{\epsilon}| \otimes \varrho^{(1)}_{\tilde{\boldsymbol{\epsilon}}} \otimes \cdots \xcancel{\varrho^{(m)}_{\tilde{\boldsymbol{\epsilon}}}} \cdots \otimes \varrho^{(fM)}_{\tilde{\boldsymbol{\epsilon}}}. \tag{35}$$

We postpone a further investigation of those states, especially in a relation to cryptographic protocols, to a subsequent publication.

We provide theoretical examples of setups in which the above cases occur below in Sec. VI.

VI. EXAMPLES OF DECOHERENCE AND ORTHOGONALIZATION FOR QUANTUM REGISTERS

Now, as an illustration, we consider several specific choices of coupling constants g_{kj} such that register spins interact nontrivially with environmental ones.

A. Collective decoherence

Here the coefficients g_{kj} are k independent, which leads to an exceptionally rich family of DFSs. If the coupling coefficients depend solely on the distance between the register and environmental spin, then such a choice can be obtained by placing the system as shown in Fig. 1. This setup can be achieved, e.g., in crystalline solid with help of a scanning tunneling microscope.

Obviously, by (16), if $\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k) = 0$ (i.e., the number of "+1" entries is the same for both $\vec{\boldsymbol{\epsilon}}$ and $\vec{\boldsymbol{\epsilon}'}$), then no broadcasting occurs, as no information about the system is transferred into the environment whatsoever. Thus by (25) and (28) both register states belong to the same DFS and OFS for *all* j, and so $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are degenerated.

B. Cylindrical symmetry

Let us now consider a generalization of the above example with the register and environmental spins organized as shown in Fig. 2. The register spins are located on a straight line and the environment is composed of a collection of $M = K$ circles, each containing L spins. The geometry is chosen such that the mth register spin is coplanar with the mth circle.

The locations of environmental spins are most easily expressed by the number of the circle to which it belongs and its location on the circle. Due to the geometry of the composite system, the coupling constants are independent of the latter. We assume that they depend on the distance between kth register spin and lth spin on the mth circle as

$$g_{klm} = g_{km} = \frac{g_0}{r^3_{km}} \approx \frac{g_0}{r^3_0}\left[1 - \frac{3}{2}(k-m)^2\frac{d^2_0}{r^2_0}\right], \tag{36}$$

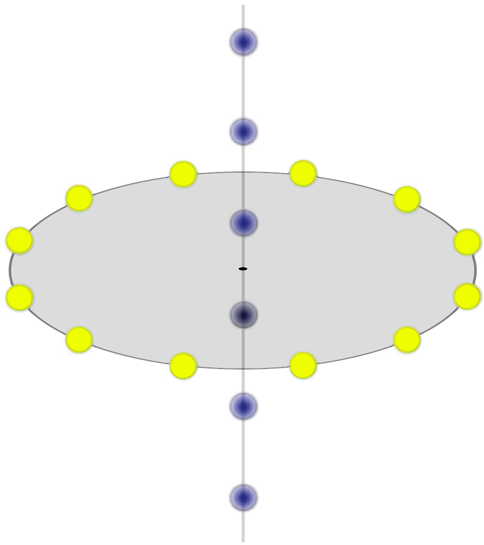

FIG. 1. The geometry of the composite system leading to the collective decoherence. The register spins (yellow) are placed on a circle through the center of which passes a straight line on which the environmental spins are located (blue). The position of the environmental spins on the line is symmetrical with respect to the circle.

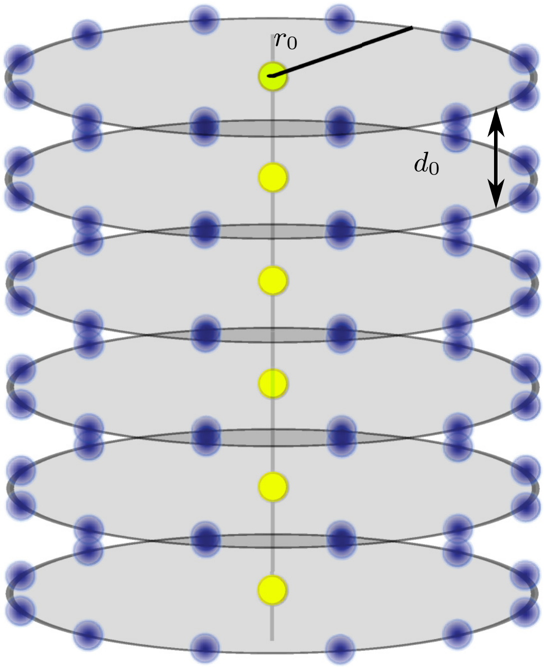

FIG. 2. A different geometry with a linear register (yellow) and a collection of environmental spins (blue) located on M circles of L spins. The radius of each circle is r_0, while the distance between neighboring circles is d_0.

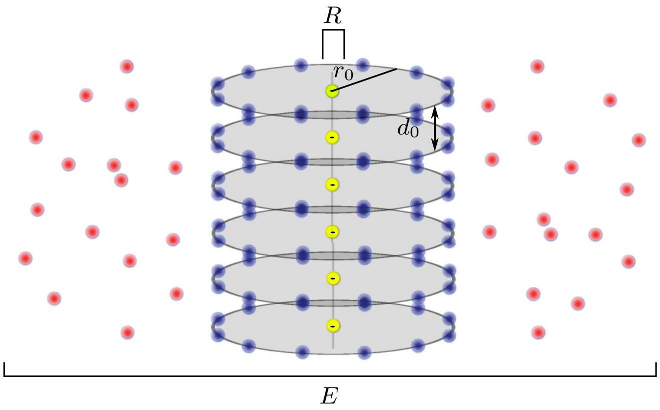

FIG. 3. The geometry of the extended register model. The environment E now consists not only of the cylindrically placed spins (blue) but also of additional spins placed randomly in the space (red).

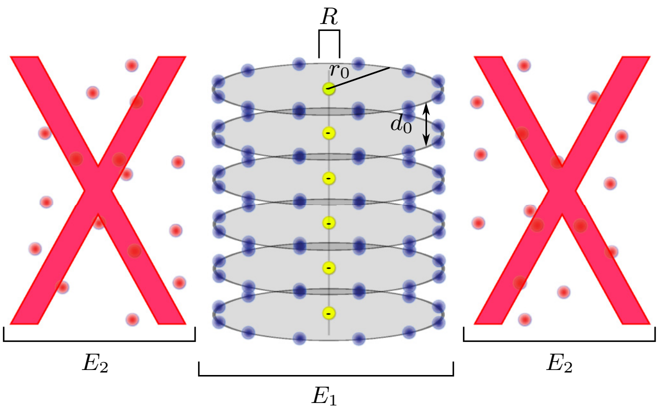

FIG. 4. A geometry that allows for case with decoherence but no orthogonalization to occur. Frequencies $\omega^{(m)}_{\boldsymbol{\epsilon\epsilon'}}$ can vanish for spins in E_1 but are generally nonzero for the randomly located ones from E_2.

where for the approximation we used the Taylor expansion and assumed that $r_0 \gg M d_0$. The meaning of the symbols is explained conceptually in Fig. 2 (which is rescaled in width to improve readability).

The frequencies $\tilde{\omega}^{(m)}_{\boldsymbol{\epsilon\epsilon'}} \equiv \omega^{(ml)}_{\boldsymbol{\epsilon\epsilon'}}$ by (16) are

$$\begin{aligned}\tilde{\omega}^{(m)}_{\boldsymbol{\epsilon\epsilon'}} &= \frac{1}{2}\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)g_{km}\\ &= -\frac{3}{4}\frac{g_0 d_0^2}{r_0^5}\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)\left[-\frac{2}{3}\frac{r_0^2}{d_0^2} + (k-m)^2\right]. \quad (37)\end{aligned}$$

Now, as discussed before, the no-decoherence criterion demands that for each m the frequency should be equal to 0. Observe that $\tilde{\omega}^{(m)}_{\boldsymbol{\epsilon\epsilon'}}$ can be regarded as a quadratic function of m of the form $am^2 + bm + c$, where $a = \sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)$, $b = \sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)k$, and $c = \sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)(k^2 - \frac{2}{3}\cdot\frac{r_0^2}{d_0^2})$. For (25) to be met one must have $a = b = c = 0$, or equivalently, $\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k) = 0$, $\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)k = 0$, and $\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)k^2 = 0$. Note that the difference $\epsilon_k - \epsilon'_k$ can take values of either 0 or ±2. One can see that there exists an infinite number of systems of the above equations with nontrivial solutions, i.e., there exist an infinite number of setups of this kind with DFS or OFS.[6] An example of pairs of *qualitas* $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon'}$ is defined by sets $\{1, 5, 6\}$ and $\{2, 3, 7\}$ for $K = 7$; these sets specify which elements of vectors $\vec{\boldsymbol{\epsilon}}$ and $\vec{\boldsymbol{\epsilon}}'$, respectively, have a $+1$ value, with -1 at other places.

C. Decoherence-free and orthogonalization-free processes

We apply the result of Sec. V to the extended linear register model, with the geometry given in Fig. 3, and show examples of setups leading to the cases with decoherence and no orthogonalization (Sec. V C) and with orthogonalization and no decoherence (Sec. V D).

The environment consists of two parts: the first contains a collection of cylindrically placed spins (blue) as in the original model, while the spins of the second part (red) are randomly distributed in space. Let us now analyze the two cases separately.

1. Decoherence but no orthogonalization

Let the observable environment macrofraction (E_1) be the "blue" spins (see Fig. 4) while we discard the "random" ones. As it was shown before, within this model it is possible to choose two *qualitas*, $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon'}$, for which the frequencies corresponding to E_1 vanish and therefore no orthogonalization occurs. The *qualitas* are thus degenerated for an observer contained in the environment E_1.

However, location of "red" spins means that G_2 is a random matrix, and thus the decoherence factor computed from it is a quasiperiodic function with generally a very big recurrence time so that the system is effectively a decohering one. Thus the *qualitas* are nondegenerate for the environment E_2.

2. Orthogonalization but no decoherence

This situation is actually the opposite to the previously considered one with the roles of E_1 and E_2 exchanged (see Fig. 5). Now, if the states $|\boldsymbol{\epsilon}\rangle$ and $|\boldsymbol{\epsilon'}\rangle$ are chosen properly, they do not decohere, but the orthogonalization still takes place as the frequencies pertaining to the observable macrofraction do not vanish in general.

As mentioned, this degeneracy for the environment (E_2) but not for the observer (E_1) is somehow peculiar, without a clear philosophical meaning. It may be also interesting from the point of view of data processing in quantum registers.

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[6] For a given $K \in \mathbb{N}_+$ let us denote $[K] = \{1, \dots, K\}$. The number of possible sizes of subsets of $[K]$ is $K + 1$, the number of possible sums of their elements is at most of order K^2, and the number of possible sums of squares of their elements is at most of the order K^3. So, there is at most $(K+1)K^2K^3 = (K+1)K^5$ different values of the triple of size, sum of elements, and sum of squares of elements for subsets of $[K]$. On the other hand, there exist 2^K different subsets of the set $[K]$. Thus, when $2^K > (K+1)K^5$, there must exist at least two subsets with equal number of elements, their sum, and the sum of squares.

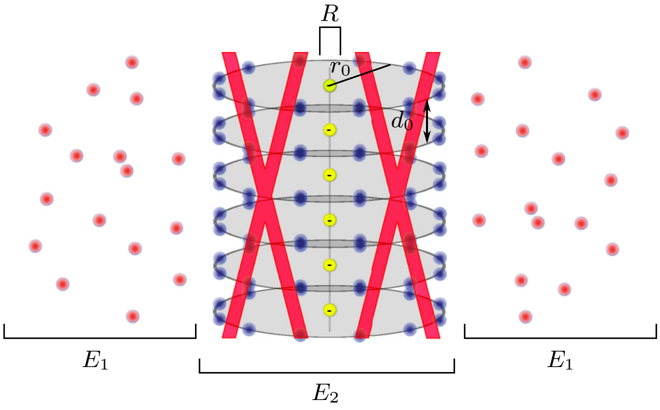

FIG. 5. A geometry that allows for case 4 to occur (orthogonalization but no decoherence). Here the situation is reversed as compared with Fig. 4.

D. Small overlap of interactions

Suppose that the coupling coefficients in G are given by the following function:

$$g_{kj} = f_{\mu_k,\sigma}(j), \tag{38}$$

where $f_{\mu,\sigma} : \mathbb{R} \to \mathbb{R}$ is a C^0 function describing a normalized "saw pulse" centered around μ and of width σ, given by

$$f_{\mu,\sigma}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, \mu - \frac{\sigma}{2}), \\ \frac{2}{\sigma}x + (1 - \frac{2\mu}{\sigma}) & \text{for } x \in [\mu - \frac{\sigma}{2}, \mu), \\ -\frac{2}{\sigma}x + (1 + \frac{2\mu}{\sigma}) & \text{for } x \in [\mu, \mu + \frac{\sigma}{2}), \\ 0 & \text{for } x \in [\mu + \frac{\sigma}{2}, \infty). \end{cases} \tag{39}$$

Furthermore, we assume that

$$\mu_k = \frac{N}{K}(k-1) + 1, k = 1, \dots, K. \tag{40}$$

Such a choice states that entries of successive rows of G are given by such "saw pulses" with equidistant maxima and the same width (see Fig. 6).

Moreover, we assume that the overlap between any two consecutive rows of G is small, i.e.,

$$\frac{\sigma}{2} \lesssim \frac{N}{K}, \tag{41}$$

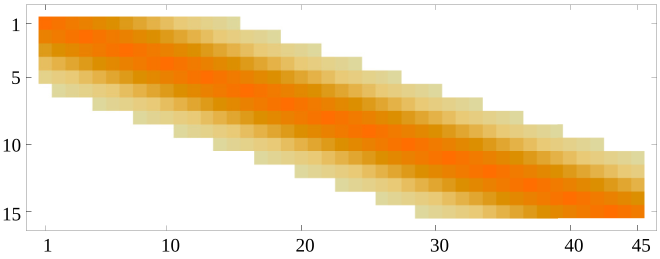

FIG. 6. The structure of the coupling matrix G with the values chosen as in (38) with $\sigma = 30$. The vertical and horizontal axes correspond to the row and column of G, respectively. The values of the coupling coefficients are expressed using a coloring technique (higher intensity represents higher value).

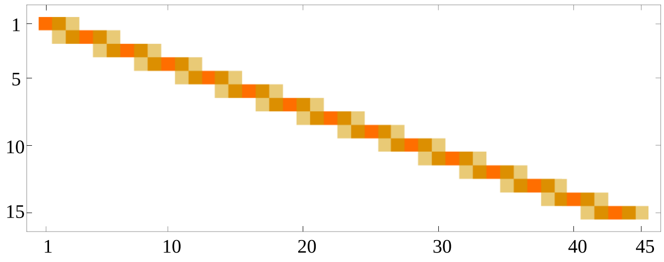

FIG. 7. An example of a coupling matrix G leading to the small overlap case. Note that the overlap of an arbitrary kth row with its neighboring rows is such that they vanish in the vicinity of its maximum.

which simply means that the separation between the maxima of two successive rows is greater than the width σ of the function $f_{\mu,\sigma}$ (see Fig. 7).

From (25) it follows that given such assumptions the strong decoherence condition cannot be fulfilled. We see that the coupling coefficient matrix G is of full rank, and therefore the decoherence and orthogonalization occur. Indeed, consider the kth row of G. If G was not of full rank, then it would be possible to express this row as a linear combination of the remaining ones. But due to assumption (41) this is impossible, as in the region close to the peak of the kth row the entries of all other ones are equal to zero.

The meaning of this setup is that the evolution of each spin is governed mainly by a single spin from the register, and thus the information it enquires is unambiguous. In consequence, all *qualitas* of the register are nondegenerate.

VII. CONCLUSIONS

The results of this paper concern the cases when the evolution of a quantum system composed of many subsystems is dominated by interaction with a central element, which is a model of an object being observed by its surroundings. For such a model we were able to show that if the environment is large then in a generic situation both of the intersubjectivization elements, orthogonalization and decoherence, occur.

We studied in detail the decoherence process of a quantum register, coupled to a spin environment through a ZZ interaction. Our main interest was in the information transfer from the quantum register to the environment. This required a departure from the standard approach as not all of the environment could be traced out.

Following our earlier research, the main object of the study was the so-called partially traced state, obtained from the full system-environment state by tracing out only a fraction of the environment. In particular, we were interested if the partially traced state approaches what we call a spectrum broadcast structure. It implies a certain objectivization of a (decohered) state of the register: a classical bit-string, labeling decohered states of the register, is present in the environment in many copies and can be read out without any disturbance (on average).

Exploiting certain properties of quasiperiodic functions, we formulated and studied conditions when SBSs are formed asymptotically. Due to the presence of decoherence-free subspaces and what we call orthogonalization-free subspaces, possible structures that can appear are much richer than in the case of a single central spin. In particular, we reported a new kind of SBS where some of the coherences are preserved but still some information is objective. We also presented a series of theoretical examples illustrating how different forms of the coupling matrix can lead to different patters of information proliferation. The next possible step would be to design a concrete experimental proposal around the presented examples.

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APPENDIX A: PROOF OF PROPOSITION 1

We now prove Proposition 1.

Let us recall that a sequence of random variables $(X_N)_N$ satisfies LLN if for $S_N \equiv X_1 + \cdots X_N$ we have

$$\frac{1}{N} S_N \longrightarrow [N \to \infty] \frac{1}{N} \langle S_N \rangle, \tag{A1}$$

where the convergence is in probability. One can show [41] that LLN holds if[7] $(X_N)_N$ are independent and identically distributed (i.i.d.) and $\langle |X_1|\rangle < \infty$. Then (A1) means

$$\frac{1}{N} S_N \longrightarrow [N \to \infty] \langle X_1 \rangle \tag{A2}$$

in probability, i.e.,

$$\forall_{\delta_1,\delta_2>0} \exists_{N_0} \forall_{N \geqslant N_0} P\left(\left| \frac{S_N}{N} - \langle X_1 \rangle \right| \leqslant \delta_1 \right) \geqslant 1 - \delta_2. \tag{A3}$$

From (5), (7), and (10) it follows that

$$\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = \prod_{j \notin Obs} \mathrm{Tr}\, \rho^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t), \tag{A4a}$$

$$B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) = \prod_{j \in mac_m} B\big(\rho^{(j)}_{\boldsymbol{\epsilon}}(t), \rho^{(j)}_{j}(t)\big). \tag{A4b}$$

Now let us define

$$\chi^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) \equiv 1 - \left[\mathrm{Tr}\left(\rho^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)\right)\right]^2, \tag{A5a}$$

$$\kappa^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) \equiv 1 - \left[B\left(\rho^{(j)}_{\boldsymbol{\epsilon}}(t), \rho^{(j)}_{\boldsymbol{\epsilon}'}(t)\right)\right]^2. \tag{A5b}$$

Using the inequality $\log x \leqslant -(1-x)$ we get

$$|\gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)|^2 \leqslant \exp\left[-\sum_{j \notin Obs} \chi^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)\right], \tag{A6a}$$

$$B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t) \leqslant \exp\left[-\frac{1}{2}\sum_{j \in mac_m} \kappa^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)\right]. \tag{A6b}$$

In order to avoid cumbersome notation, we concentrate on a particular pair of *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ and omit their indices. In particular, we write $B^{(m)}(t) = B^{(m)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$; $\gamma(t) = \gamma_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$; $\kappa^{(j)}(t) = \kappa^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$; and $\chi^{(j)}(t) = \chi^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'}(t)$.

From (A5) (and nondegeneracy of $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$) it follows that $\langle|\kappa^{(j)}(t)|\rangle > 0$ and $\langle|\chi^{(j)}(t)|\rangle > 0$ are finite for any probability distributions, and if we assume they are i.i.d., we can apply LLN for the right-hand sides of (A6).

Let us fix some $\delta > 0$ and $t > 0$. From LLN it follows that there exist $N^t_{0,mac}$ and $N^t_{0,\mathrm{dis}}$ such that for all $N^{(m)}_{mac} \geqslant N^t_{0,mac}$ and $N_{\mathrm{dis}} \geqslant N^t_{0,\mathrm{dis}}$ with probability at least $1 - \delta$,

$$\left| \left(\sum_{j \notin Obs} \chi^{(j)}(t) \right) - N_{\mathrm{dis}} \langle \chi(t) \rangle \right| \leqslant \delta N_{\mathrm{dis}} \langle \chi(t) \rangle, \tag{A7a}$$

$$\left| \left(\sum_{j \in mac_m} \kappa^{(j)}(t) \right) - N^{(m)}_{mac} \langle \kappa(t) \rangle \right| \leqslant \delta N^{(m)}_{mac} \langle \kappa(t) \rangle, \tag{A7b}$$

where we denote $\langle \kappa(t)\rangle = \langle \kappa^{(j_0)}(t)\rangle$ and $\langle \chi(t)\rangle = \langle \chi^{(j_0)}(t)\rangle$ for arbitrary $j_0 \in mac_m$. To see this, in (A3) we take $\delta_2 = \delta$, and $\delta_1 = \delta\langle\kappa(t)\rangle$ for (A7b), and $\delta_1 = \delta\langle\chi(t)\rangle$ for (A7a). Thus from (A6) and (A7) we get

$$|\gamma(t)|^2 \leqslant \exp\left[-(1-\delta) N_{\mathrm{dis}} \langle \chi(t) \rangle\right], \tag{A8a}$$

$$B^{(m)}(t) \leqslant \exp\left[-\frac{1-\delta}{2} N^{(m)}_{mac} \langle \kappa(t) \rangle\right]. \tag{A8b}$$

If we take $N^{(m)}_{mac}$ and N_{dis} satisfying the conditions

$$N_{\mathrm{dis}} \geqslant \max\left[N^t_{0,\mathrm{dis}}, \frac{\log \frac{1}{\delta}}{(1-\delta)\langle \chi(t) \rangle} \right], \tag{A9a}$$

$$N^{(m)}_{mac} \geqslant \max\left[N^t_{0,mac}, \frac{2\log \frac{1}{\delta}}{(1-\delta)\langle \kappa(t) \rangle} \right], \tag{A9b}$$

we get the thesis of the proposition for the assumed *qualitas* $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$. In order to finish the proof for all *qualitas* we take the maximal values of $N^{(m)}_{mac}$ and N_{dis}.

APPENDIX B: CALCULATION OF OVERLAP FACTORS

We now calculate the decoherence and overlap factors using the notation of Sec. IV A. Let us define

$$\mathbf{M}^{(j)}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}'} \equiv \sqrt{D} R^{\dagger} U^{\dagger}_{\boldsymbol{\epsilon}}(t) U_{\boldsymbol{\epsilon}'}(t) R D R^{\dagger} U^{\dagger}_{\boldsymbol{\epsilon}'}(t) U_{\boldsymbol{\epsilon}}(t) R \sqrt{D}, \tag{B1}$$

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[7] In fact, the stated condition is sufficient even for the strong law of large numbers to hold, where the convergence is almost sure.

where we have omitted the upper indexes (j) on the right-hand side to improve readability. After pulling some of the unitary operators out of the square roots and using the cyclic property of the trace we use (B1) to write the formula

$$B\big(\rho_{\epsilon}^{(j)}(t), \rho_{\epsilon'}^{(j)}(t)\big) = \mathrm{Tr}\sqrt{\mathbf{M}_{\epsilon\epsilon'}^{(j)}}. \tag{B2}$$

For a 2×2 matrix $\mathbf{M}_{\epsilon\epsilon'}^{(j)}$ the eigenvalues $M_{\epsilon\epsilon'\pm}^{(j)}$ satisfy

$$M_{\epsilon\epsilon'\pm}^{(j)} = \frac{1}{2}\left[\mathrm{Tr}\,\mathbf{M}_{\epsilon\epsilon'}^{(j)} \pm \sqrt{\big(\mathrm{Tr}\,\mathbf{M}_{\epsilon\epsilon'}^{(j)}\big)^2 - 4\det\mathbf{M}_{\epsilon\epsilon'}^{(j)}}\right]. \tag{B3}$$

Straightforward calculations show that for $\mathbf{M}_{\epsilon\epsilon'}^{(j)}$ given by (B1) we have

$$\begin{aligned}\mathrm{Tr}\,\mathbf{M}_{\epsilon\epsilon'}^{(j)} &= \lambda^{(j)2} + (1 - \lambda^{(j)2})\\ &\quad - (2\lambda^{(j)} - 1)^2 \sin^2\beta^{(j)} \sin^2\big(\omega_{\epsilon\epsilon'}^{(j)} t\big),\end{aligned} \tag{B4}$$

and $\det\mathbf{M}_{\epsilon\epsilon'}^{(j)} = \lambda^{(j)2}(1 - \lambda^{(j)2})$. From (B3) it follows that $M_{\epsilon\epsilon'+}^{(j)}(t)M_{\epsilon\epsilon'-}^{(j)}(t) = \det\mathbf{M}_{\epsilon\epsilon'}^{(j)}$, and thus

$$\begin{aligned}B\big(\rho_{\epsilon}^{(j)}(t), \rho_{\epsilon'}^{(j)}(t)\big) &= \Big|\sqrt{M_{\epsilon\epsilon'+}^{(j)}(t)} + \sqrt{M_{\epsilon\epsilon'-}^{(j)}(t)}\Big|\\ &= \Big(M_{\epsilon\epsilon'+}^{(j)}(t) + M_{\epsilon\epsilon'-}^{(j)}(t) + 2\sqrt{M_{\epsilon\epsilon'+}^{(j)}(t)M_{\epsilon\epsilon'-}^{(j)}(t)}\Big)^{\frac{1}{2}}\\ &= \sqrt{\mathrm{Tr}\,\mathbf{M}_{\epsilon\epsilon'}^{(j)}(t) + 2\sqrt{\det\mathbf{M}_{\epsilon\epsilon'}^{(j)}(t)}}\\ &= \sqrt{1 - (2\lambda^{(j)} - 1)^2 \sin^2\beta^{(j)} \sin^2\big(\omega_{\epsilon\epsilon'}^{(j)} t\big)}.\end{aligned} \tag{B5}$$

Using iteratively the multiplicativity of the overlap, from (B5) we obtain [1]

$$B_{\epsilon\epsilon'}^{(m)}(t) = \prod_{j \in mac_m} \sqrt{1 - (2\lambda^{(j)} - 1)^2 \sin^2\beta^{(j)} \sin^2(\omega_{\epsilon\epsilon'}^{(j)} t)}. \tag{B6}$$

APPENDIX C: SHORT TIME BEHAVIOR OF SBS FORMATION

We use here the results and notation of Sec. IV A. From (A5), (19), and (20) we have

$$0 \leqslant \chi_{\epsilon\epsilon'}^{(j)}(t) = \sin^2(\omega_{\epsilon\epsilon'}^{(j)} t)[1 - (2\lambda^{(j)} - 1)^2 \cos^2\beta^{(j)}], \tag{C1a}$$

$$0 \leqslant \kappa_{\epsilon\epsilon'}^{(j)}(t) = (2\lambda^{(j)} - 1)^2 \sin^2\beta^{(j)} \sin^2(\omega_{\epsilon\epsilon'}^{(j)} t). \tag{C1b}$$

We choose the following probability distributions:

(i) The initial state eigenvalue $\lambda^{(j)}$ (14a) is distributed according to the eigenvalue part of the Hilbert-Schmidt measure [44]:

$$P_{HS}(\lambda^{(j)}) \equiv 3(2\lambda^{(j)} - 1)^2. \tag{C2}$$

(ii) The initial state Euler angles $(\alpha^{(j)}, \beta^{(j)}, \gamma^{(j)})$ (14b) are distributed with the SU(2) Haar measure.

One can then easily calculate the following averages over these distributions: $\langle\sin^2\beta^{(j)}\rangle = \frac{2}{3}$, $\langle\cos^2\beta^{(j)}\rangle = \frac{1}{3}$, $\langle(2\lambda^{(j)} - 1)^2\rangle = \frac{3}{5}$. From this for $t > 0$ we have

$$\begin{aligned}\big\langle\chi_{\epsilon\epsilon'}^{(j)}(t)\big\rangle &= (1 - \langle(2\lambda^{(j)} - 1)^2\rangle\langle\cos^2\beta^{(j)}\rangle)\\ &\quad \times\big\langle\sin^2\big(\omega_{\epsilon\epsilon'}^{(j)} t\big)\big\rangle = \frac{4}{5}\big\langle\sin^2(\omega_{\epsilon\epsilon'}^{(j)} t)\big\rangle > 0,\end{aligned} \tag{C3a}$$

$$\begin{aligned}\big\langle\kappa_{\epsilon\epsilon'}^{(j)}(t)\big\rangle &= \langle(2\lambda^{(j)} - 1)^2\rangle\langle\sin^2\beta^{(j)}\rangle\big\langle\sin^2(\omega_{\epsilon\epsilon'}^{(j)} t)\big\rangle\\ &= \frac{2}{5}\big\langle\sin^2(\omega_{\epsilon\epsilon'}^{(j)} t)\big\rangle > 0.\end{aligned} \tag{C3b}$$

The following calculations are performed in the short time regimes, $\omega_{\epsilon\epsilon'}^{(j)} t \ll 1$. We consider the case when the coupling constants g_{kj} are distributed with any continuous measure of a nonzero and finite second moment, $\overline{g^2} \equiv \langle g_{kj}^2\rangle > 0$. Using the Taylor expansion and (16) we get

$$\begin{aligned}\big\langle\sin^2\big(\omega_{\epsilon\epsilon'}^{(j)} t\big)\big\rangle &\approx t^2\big\langle\big(\omega_{\epsilon\epsilon'}^{(j)}\big)^2\big\rangle\\ &= \frac{1}{2}t^2\left\langle\sum_{k=1}^{K}(\epsilon_k - \epsilon'_k)g_{kj}\right\rangle \leqslant t^2\sum_{k=1}^{K}\langle|g_{kj}|\rangle.\end{aligned} \tag{C4}$$

From (C3) and (C4) we get

$$\big\langle\chi_{\epsilon\epsilon'}^{(j)}(t)\big\rangle \leqslant \frac{4}{5}t^2 K\overline{g^2}, \tag{C5a}$$

$$\big\langle\kappa_{\epsilon\epsilon'}^{(j)}(t)\big\rangle = \frac{2}{5}\big\langle\sin^2\big(\omega_{\epsilon\epsilon'}^{(j)} t\big)\big\rangle \leqslant \frac{2}{5}t^2 K\overline{g^2}. \tag{C5b}$$

From (A9) we can infer that when we want to assure that $B(t_B) \leqslant \delta$ we take[8] at least

$$N_{mac}^{(m)} \geqslant \frac{5\log\frac{1}{\delta}}{t^2 K\overline{g^2}}, \tag{C6}$$

and similarly for $|\gamma(t_D)|^2 \leqslant \delta$ we need at least

$$N_{\mathrm{dis}} \geqslant \frac{5\log\frac{1}{\delta}}{4t^2 K\overline{g^2}}. \tag{C7}$$

APPENDIX D: PROOF OF PROPOSITION 2

Here we prove Proposition 2.

We say that a set of numbers $\{\alpha_i\}_{i=1}^N \subset \mathbb{R}$ is impartitionable iff for any $\varsigma \in \{\pm 1\}^N$ we have $\sum_{i=1}^N \varsigma_i\alpha_i \neq 0$. The quantity

$$\boldsymbol{\delta}_{\boldsymbol{\Sigma}}(\vartheta) \equiv \min_{\varsigma \in \{\pm 1\}^N}\left|\sum_{i=1}^{N}\varsigma_i\alpha_i\right| \tag{D1}$$

is called the minimal discrepancy [45], and the numbers in the set ϑ are impartitionable iff $\boldsymbol{\delta}_{\boldsymbol{\Sigma}}(\vartheta) > 0$. In other words, a set of real numbers ϑ is impartitionable iff one cannot find a partition of the set ϑ into two subsets summing to the same value.

It is easy to see that if $\{\alpha_i\}_{i=1}^N$ are independent and continuously distributed random variables, then for any $\varsigma \in \{\pm 1\}^N$

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[8] We use here the Puiseux expansion $\frac{\log\delta}{1-\delta} = \log\delta + \delta\log\delta + O(\delta^2)$ and $\lim_{\delta\to 0_+}\delta\log\delta = 0$. This shows the importance of a careful choice of the value of δ_1 in (A3) in derivation of (A7).

and $t > 0$ we have $P(\sum_{i=1}^{N} \varsigma_i \alpha_i t = 0) = 0$, so the coupling constants in Proposition 2 are impartitionable almost surely.

We start with the following:

Lemma 1. If real numbers $\{\alpha_i\}_{i=1}^N$ are impartitionable, then

(1)

$$\prod_{i=1}^{N} \cos \alpha_i = \frac{1}{2^N} \sum_{\upsilon \in \{\pm 1\}^N} \cos\left(\sum_{i=1}^{N} \upsilon_i \alpha_i\right), \tag{D2}$$

(2)

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\prod_{i=1}^{N} \cos(\alpha_i t)\right] dt = 0. \tag{D3}$$

Proof.

(1) For $N = 1$ we obviously have

$$\cos \alpha_1 = \frac{1}{2} \sum_{\upsilon \in \{\pm\}} \cos(\upsilon \alpha_1). \tag{D4}$$

Let us assume that (D2) holds for some N, and let $\{\alpha_i\}_{i=1}^{N+1}$ be impartitionable. Then

$$\begin{aligned}
\prod_{i=1}^{N+1} \cos \alpha_i &= \frac{1}{2^N} \sum_{\upsilon \in \{\pm 1\}^N} \cos\left(\sum_{i=1}^{N} \upsilon_i \alpha_i\right) \cos \alpha_{N+1} \\
&= \frac{1}{2^N} \sum_{\upsilon \in \{\pm 1\}^N} \frac{1}{2}\left[\cos\left(\sum_{i=1}^{N} \upsilon_i \alpha_i + \alpha_{N+1}\right)\right. \\
&\quad \left. + \cos\left(\sum_{i=1}^{N} \upsilon_i \alpha_i - \alpha_{N+1}\right)\right] \\
&= \frac{1}{2^{N+1}} \sum_{\upsilon \in \{\pm 1\}^{N+1}} \cos\left(\sum_{i=1}^{N+1} \upsilon_i \alpha_i\right).
\end{aligned} \tag{D5}$$

(2) First, let us note that for any α and $T > 0$,

$$\int_0^T \cos(\alpha t) dt \leqslant \int_0^{\frac{\pi}{\alpha}} \cos(\alpha t) dt = \frac{2}{\alpha}, \tag{D6}$$

and similarly, $-\frac{2}{\alpha} \leqslant \int_0^T \cos(\alpha t) dt$. It is easy to see that if the numbers in $\{\alpha_i\}_{i=1}^N$ are impartitionable, then for any $t > 0$ the numbers in $\{t\alpha_i\}_{i=1}^N$ are also impartitionable. From (D2) we have

$$\begin{aligned}
\int_0^T &\left[\prod_{i=1}^{N} \cos(\alpha_i t)\right] dt \\
&= \int_0^T \left[\frac{1}{2^N} \sum_{\upsilon \in \{\pm 1\}^N} \cos\left(\sum_{i=1}^{N} \upsilon_i \alpha_i t\right)\right] dt \\
&= \frac{1}{2^N} \sum_{\upsilon \in \{\pm 1\}^N} \int_0^T \cos\left(\sum_{i=1}^{N} \upsilon_i \alpha_i t\right) dt \equiv \varpi(T).
\end{aligned} \tag{D7}$$

Let $\iota \equiv \boldsymbol{\delta}_{\boldsymbol{\Sigma}}(\{\alpha_i\}_{i=1}^N)$. Now, using (D6) we have

$$\varpi(T) \leqslant \frac{1}{2^N} \sum_{\upsilon \in \{\pm 1\}^N} \frac{2}{\sum_{i=1}^{N} \upsilon_i \alpha_i} \leqslant \frac{2}{\iota}, \tag{D8}$$

and similarly, $-\frac{2}{\iota} \leqslant \varpi(T)$. Since $\frac{2}{\iota}$ is constant, we have $\lim_{T\to\infty} \frac{\varpi(T)}{T} = 0$, and thus (D3) holds.

From Lemma 1 we immediately get:

Corollary 1. If the real numbers in $\{\alpha_i\}_{i=1}^N$ are impartitionable, then for any $\{c_i\}_{i=1}^N$,

1.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \prod_{i=1}^{N} [\cos^2(\alpha_i t) + c_i^2 \sin^2(\alpha_i t)] dt = \prod_{i=1}^{N} \frac{1 + c_i^2}{2}, \tag{D9}$$

2.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \prod_{i=1}^{N} [1 - c_i^2 \sin^2(\alpha_i t)] dt = \prod_{i=1}^{N} \left(1 - \frac{c_i^2}{2}\right). \tag{D10}$$

Proof. Let us first note that if the numbers in $\{\alpha_i\}_{i=1}^N$ are impartitionable, then also the numbers in $\{2\alpha_i\}_{i=1}^N$ are impartitionable.

We have $\cos^2 \alpha + c^2 \sin^2 \alpha = \frac{1}{2}(1 + c^2) + (c^2 - 1) \cos(2\alpha)$. Thus the product within the integral in (D9) is equal to

$$\sum_{\sigma \subseteq \{1,\ldots,N\}} \left[\left(\prod_{i\in\sigma} \frac{1 + c_i^2}{2}\right)\left(\prod_{i\notin\sigma} (c_i^2 - 1) \cos(2\alpha_i t)\right)\right]. \tag{D11}$$

We can apply (D3) to see that the only term in (D11) which does not vanish in the limit is the one with $\sigma = \{1, \ldots, N\}$, thus giving (D9).

Similarly, using $1 - c^2 \sin^2 \alpha = 1 - \frac{c^2}{2} - \frac{c^2}{2} \cos(2\alpha)$ we get (D10).

In order to complete the proof of Proposition 2 using Corollary 1, from the definition of $\gamma(t)$ we take $c_i^2 = \zeta_i^2$ and apply (D9) to get (22a). Similarly, to get (22b) we take $c_i^2 = (2\lambda_i - 1)^2 \sin^2 \beta_i$ in (D10).

In fact, we have shown an even stronger result, viz. that it is enough for $\{\alpha_i\}_{i=1}^N$ to be impartitionable, not necessarily continuously distributed, but this statement seems to have a limited physical meaning.

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[1] P. Mironowicz, J. K. Korbicz, and P. Horodecki, Monitoring of the Process of System Information Broadcasting in Time, Phys. Rev. Lett. **118**, 150501 (2017).

[2] W. H. Żurek, Environment-induced superselection rules, Phys. Rev. D **26**, 1862 (1982).

[3] F. M. Cucchietti, J. P. Paz, and W. H. Żurek, Decoherence from spin environments, Phys. Rev. A **72**, 052113 (2005).

[4] M. Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Springer, Berlin, 2007).
[5] M. Zwolak, C. J. Riedel, and W. H. Zurek, Amplification, Decoherence, and the Acquisition of Information by Spin Environments, Sci. Rep. **6**, 25277 (2016).
[6] W. H. Żurek, Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?, Phys. Rev. D **24**, 1516 (1981).
[7] W. H. Żurek, S. Habib, and J. P. Paz, Coherent States via Decoherence, Phys. Rev. Lett. **70**, 1187 (1993).
[8] J. P. Paz and W. H. Żurek, Quantum Limit of Decoherence: Environment Induced Superselection of Energy Eigenstates, Phys. Rev. Lett. **82**, 5181 (1999).
[9] D. A. R. Dalvit, J. Dziarmaga, and W. H. Żurek, Unconditional Pointer States from Conditional Master Equations, Phys. Rev. Lett. **86**, 373 (2001).
[10] W. H. Żurek, Environment-Assisted Invariance, Entanglement, and Probabilities in Quantum Physics, Phys. Rev. Lett. **90**, 120404 (2003).
[11] H. Ollivier, D. Poulin, and W. H. Żurek, Objective Properties from Subjective Quantum States: Environment as a Witness, Phys. Rev. Lett. **93**, 220401 (2004).
[12] W. H. Żurek, Decoherence, eigenselection, and the quantum origins of the classical, Rev. Mod. Phys. **75**, 715 (2003).
[13] M. Schlosshauer, Decoherence, the measurement problem, and interpretations of quantum mechanics, Rev. Mod. Phys. **76**, 1267 (2005).
[14] R. Blume-Kohout and W. H. Żurek, Quantum Darwinism: Entanglement, branches, and the emergent classicality of redundantly stored quantum information, Phys. Rev. A **73**, 062310 (2006).
[15] W. H. Żurek, Quantum Darwinism, Nat. Phys. **5**, 181 (2009).
[16] M. Zwolak, H. T. Quan, and W. H. Żurek, Quantum Darwinism in a Mixed Environment, Phys. Rev. Lett. **103**, 110402 (2009).
[17] M. Zwolak, H. T. Quan, and W. H. Żurek, Redundant imprinting of information in nonideal environments: Objective reality via a noisy channel, Phys. Rev. A **81**, 062110 (2010).
[18] M. Zwolak, C. J. Riedel, and W. H. Żurek, Amplification, Redundancy, and Quantum Chernoff Information, Phys. Rev. Lett. **112**, 140406 (2014).
[19] R. Horodecki, J. K. Korbicz, and P. Horodecki, Quantum origins of objectivity, Phys. Rev. A **91**, 032122 (2015).
[20] F. G. S. L. Brandao, M. Piani, and P. Horodecki, Generic emergence of classical features in quantum Darwinism, Nat. Commun. **6**, 7908 (2015).
[21] R. Brunner, R. Akis, D. K. Ferry, F. Kuchar, and R. Meisels, Coupling-Induced Bipartite Pointer States in Arrays of Electron Billiards: Quantum Darwinism in Action?, Phys. Rev. Lett. **101**, 024102 (2008).
[22] A. M. Burke, R. Akis, T. E. Day, G. Speyer, D. K. Ferry, and B. R. Bennett, Periodic Scarred States in Open Quantum Dots as Evidence of Quantum Darwinism, Phys. Rev. Lett. **104**, 176801 (2010).
[23] W. K. Wootters and W. H. Żurek, A single quantum cannot be cloned, Nature (London) **299**, 802 (1982).
[24] D. G. B. J. Dieks, Communication by EPR devices, Phys. Lett. A **92**, 271 (1982).
[25] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Noncommuting Mixed States Cannot be Broadcast, Phys. Rev. Lett. **76**, 2818 (1996).
[26] M. Piani, P. Horodecki, and R. Horodecki, No-Local-Broadcasting Theorem for Multipartite Quantum Correlations, Phys. Rev. Lett. **100**, 090502 (2008).
[27] K. Ajdukiewicz, *Zagadnienia i kierunki filozofii: teoria poznania, metafizyka* (Warszawa, Czytelnik, 1949) [in Polish].
[28] K. Ajdukiewicz, *The Scientific World-Perspective and Other Essays 1931-1963* (Dordrecht, Reidel, 1977).
[29] J. K. Korbicz, P. Horodecki, and R. Horodecki, Objectivity in a Noisy Photonic Environment through Quantum State Information Broadcasting, Phys. Rev. Lett. **112**, 120402 (2014).
[30] J. K. Korbicz, P. Horodecki, and R. Horodecki, Quantum-correlation breaking channels, broadcasting scenarios, and finite Markov chains, Phys. Rev. A **86**, 042319 (2012).
[31] M. V. Gurudev Dutt, L. Childress, L. Jiang, E. Togan, J. Maze, F. Jelezko, A. S. Zibrov, P. R. Hemmer, and M. D. Lukin, Quantum Register Based on Individual Electronic and Nuclear Spin Qubits in Diamond, Science **316**, 1312 (2007).
[32] P. Należyty and D. Chruściński, Decoherence-free subspaces for a quantum register interacting with a spin environment, Open Syst. Inf. Dyn. **20**, 1350014 (2013).
[33] G. Werth, V. N. Gheorghe, and F. G. Major, Quantum computing with trapped charged particles, in *Charged Particle Traps II*, Springer Series on Atomic, Optical, and Plasma Physics (Springer, Berlin, 2009), Vol. 54.
[34] A. Ekert, P. M. Hayden, and H. Inamori, Basic concepts in quantum computation, in *Les Houches - Ecole d'Ete de Physique Theorique*, edited by R. Kaiser, C. Westbrook, F. David (Springer, Berlin, 2001), Vol. 72.
[35] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, Phys. Rev. A **55**, 900 (1997).
[36] J. Kempe, D. Bacon, D. A. Lidar, and K. B. Whaley, Theory of decoherence-free fault-tolerant universal quantum computation, Phys. Rev. A **63**, 042307 (2001).
[37] J. K. Korbicz and J. Tuziemski, Information transfer during the universal gravitational decoherence, Gen. Relativ. Gravitation **49**, 152 (2017).
[38] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 2002).
[39] C. A. Fuchs and J. van de Graaf, Cryptographic distinguishability measures for quantum-mechanical states, IEEE Trans. Inf. Theory **45**, 1216 (1999).
[40] G. Jaeger, What in the (quantum) world is macroscopic?, Am. J. Phys. **82**, 896 (2014).
[41] J. Jakubowski and R. Sztencel, *Wstęp do teorii prawdopodobieństwa* (Script, Warsaw, 2001) [in Polish].
[42] R. Blume-Kohout and W. H. Żurek, A simple example of "Quantum Darwinism": Redundant information storage in many-spin environments, Found. Phys. **35**, 1857 (2005).
[43] T. Tilma and E. C. G. Sudarshan, Generalized Euler angle parametrization for SU(N), J. Phys. A **35**, 10467 (2002).
[44] H.-J. Sommers and K. Życzkowski, Statistical properties of random density matrices, J. Phys. A **37**, 8457 (2004).
[45] A. G. Percus, G. Istrate, and C. Moore (eds.), *Computational Complexity and Statistical Physics* (Oxford University Press, New York, 2004).