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The complexity of the $L(p, q)$-labeling problem for bipartite planar graphs of small degree

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ABSTRACT

Given a simple graph G, by an $L(p, q)$-labeling of G we mean a function c that assigns nonnegative integers to its vertices in such a way that if two vertices u, v are adjacent then $|c(u) - c(v)| \geq p$, and if they are at distance 2 then $|c(u) - c(v)| \geq q$. The $L(p, q)$-labeling problem can be defined as follows: given a graph G and integer t, determine whether there exists an $L(p, q)$-labeling c of G such that $c(V) \subseteq \{0, 1, \ldots, t\}$. In the paper we show that the problem is $\mathcal{NP}$-complete even when restricted to bipartite planar graphs of small maximum degree and for relatively small values of t. More precisely, we prove that:

(1) if $p < 3q$ then the problem is $\mathcal{NP}$-complete for bipartite planar graphs of maximum degree $\Delta \leq 3$ and $t = p + \max\{2q, p\}$;
(2) if $p = 3q$ then the problem is $\mathcal{NP}$-complete for bipartite planar graphs of maximum degree $\Delta \leq 4$ and $t = 6q$;
(3) if $p > 3q$ then the problem is $\mathcal{NP}$-complete for bipartite planar graphs of maximum degree $\Delta \leq 4$ and $t = p + 5q$.

In particular, these results imply that the $L(2, 1)$-labeling problem in planar graphs is $\mathcal{NP}$-complete for $t = 4$, and that the $L(p, q)$-labeling problem in graphs of maximum degree $\Delta \leq 4$ is $\mathcal{NP}$-complete for all values of p and q, thus answering two well-known open questions.

1. Introduction

The *channel assignment problem* is one of the potential applications for various problems of theoretical computer science. The considered task is to assign channel frequencies (nonnegative integers) to radio transmitters in such a way that each transmitter receives exactly one channel and interference during transmission does not occur. This problem was first formulated in graph-theoretic terms by Hale [9]. In this paper we study a variant of this formulation named *$L(p, q)$-labeling*.

Let $p_1 \geq p_2 \geq \cdots \geq p_k$ ($k \geq 1$) be positive integers and let $G = (V, E)$ be a simple graph. By an *$L(p_1, p_2, \ldots, p_k)$-labeling* of G we mean any function c from the set of vertices V to the set of all nonnegative integers such that $|c(u) - c(v)| \geq p_i$ whenever the distance of u and v is i. The smallest integer t such that labeling c satisfies $c(V) \subseteq \{0, 1, \ldots, t\}$ is called the *span* of c. The minimum value of span, taken over all $L(p_1, p_2, \ldots, p_k)$-labelings of G, is denoted by $\lambda_{p_1,p_2,\ldots,p_k}(G)$. It is easy to see that

$$\min_{i \leq k} p_i \cdot (\chi(G^k) - 1) \leq \lambda_{p_1,p_2,\ldots,p_k}(G) \leq \max_{i \leq k} p_i \cdot (\chi(G^k) - 1) \qquad (1)$$

where G^k is the kth power of graph G (i.e. a graph with $V(G^k) = V(G)$ and edges connecting vertices that are of distance at most k in G) and χ is the chromatic number. Computing of $\chi(G^k)$ is $\mathcal{NP}$-hard for any $k \geq 1$ [10]. Moreover, it is hard to

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approximate $\chi(G^k)$ within a factor of $O(n^{1/2-\varepsilon})$ for any $\varepsilon > 0$ (see [1] for details). This shows that the problem of determining $\lambda_{p_1,p_2,\ldots,p_k}(G)$ is computationally hard.

In this paper we focus on the case $k = 2$, which seems to be the most important. We study the complexity of the following problem, defined for any three of nonnegative integer parameters $p > q$ and t.

The $L(p, q)$-labeling problem (LPQ)	
Instance:	A simple graph G.
Question:	Is there an $L(p, q)$-labeling of G with span at most t, i.e. $\lambda_{p,q}(G) \leq t$?

1.1. Related work and our contribution

The $L(p, q)$-labeling problem has been studied for over 10 years. The first interesting results were given by Chang and Kuo [4], who proved that $\lambda_{2,1}(T)$ can be determined in polynomial time for a tree T. Their algorithm runs in $O(n\Delta^{4.5})$ time (Δ is the maximum degree of a graph and n is the number of its vertices) and after small modifications can be used to solve the $L(p, 1)$-labeling problem on trees, for any $p \geq 2$. On the other hand, recently Fiala et al. [5] have shown that the problem is $\mathcal{NP}$-complete for $q > 1$.

Bodlaender et al. [2] proved that the $L(2, 1)$-labeling problem is $\mathcal{NP}$-complete on planar graphs. They showed that the problem of deciding whether $\lambda_{2,1}(G) \leq 8$ is $\mathcal{NP}$-complete for these graphs; for more results concerning planar graphs see [3,12].

In this paper we obtain – as a direct corollary of some more general considerations – a stronger hardness result for $L(2, 1)$-labeling on planar graphs, involving a smaller bound on the value of span. Namely, we show that for bipartite planar graphs the problem of deciding whether $\lambda_{2,1}(G) \leq 4$ is $\mathcal{NP}$-complete. This establishes the border between $\mathcal{NP}$-complete and polynomial cases for $L(2, 1)$-labeling in planar graphs since it is easy to decide whether a given planar graph admits an $L(2, 1)$-labeling with span at most 3.

Fiala et al. [6] have shown that the $L(p, q)$-labeling problem is $\mathcal{NP}$-complete for every p and q and fixed span $(p+\lceil p/q\rceil q)$. They considered arbitrary graphs, but their results can be formulated in a slightly stronger way: the $L(p, q)$-labeling problem is $\mathcal{NP}$-complete for graphs with maximum degree $\Delta \leq \lceil p/q\rceil + 1$.

Herein we also strengthen these results. More precisely, we show that if $p < 3q$ ($p \geq 3q$) then the problem is $\mathcal{NP}$-complete even for bipartite planar graphs with maximum degree $\Delta \leq 3$ ($\Delta \leq 4$) and for fixed span.

1.2. Outline of the paper

The rest of the paper is organized as follows. In Section 2 we briefly present certain properties of $L(p, q)$-labelings of a graph which are useful in later considerations. In Section 3 we prove that the problem of deciding whether $\lambda_{p,q}(G) \leq p+5q$ is $\mathcal{NP}$-complete for all $p > 3q$ even when restricted to planar bipartite graphs of maximum degree 4. Next, the problem of deciding whether $\lambda_{p,q}(G) \leq p + \max\{2q, p\}$ is shown to be $\mathcal{NP}$-complete for planar bipartite graphs of maximum degree 3 for all $p < 3q$ (in Section 4), and for planar bipartite graphs of maximum degree 4 when $p = 3q$ (in Section 5). Some final conclusions are stated in Section 6.

2. Some properties of $L(p, q)$-labelings

The following lemma will be useful in the remainder of the paper.

Lemma 1 (*[8]*). *Let c be an $L(p, q)$-labeling of graph G. There is an $L(p, q)$-labeling c' of G such that for every vertex $v \in V$ we have*

(1) $c'(v) \leq c(v)$,
(2) $c'(v) = ap + bq$ *for some nonnegative integers a, b.*

As a corollary we obtain that for any graph G, $\lambda_{p,q}(G) = ap + bq$, for some pair of integers $a, b \geq 0$.

3. The case $p > 3q$

In this section we focus on the case $p > 3q$. More precisely, we are going to show that the following subproblem of LPQ is $\mathcal{NP}$-complete.

The $L(p, q)$-labeling problem for $p > 3q$ ($\overline{\mathrm{L}_{p>3q}}$)	
Instance:	A bipartite, planar graph G of maximum degree at most 4.
Question:	Is $\lambda_{p,q}(G) \leq p + 5q$?

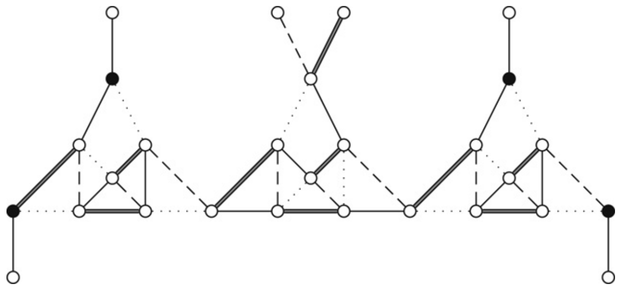

Fig. 1. The gadget H^* and its partial edge 4-coloring (dotted edges receive color 0, non-pendant black color 1, dashed color 2 and thick gray color 3; pendant black edges remain uncolored).

To this aim, we recall some properties of two classic notions: t-colorability and edge t-colorability. Graph G is *t-colorable* (*edge t-colorable*) if and only if there is a function c that maps its vertices (edges) into the set $\{0, 1, \dots, t-1\}$ in such a way that $c(u) \neq c(v)$ $(c(e_1) \neq c(e_2))$ for any two adjacent vertices u, v (edges e_1, e_2). It is known [7] that the 3-colorability problem is $\mathcal{NP}$-complete for planar graphs. We show that it is $\mathcal{NP}$-complete in an even more restricted case. More precisely, we prove that the following problem is $\mathcal{NP}$-complete.

Restricted 3-coloring of planar graphs ($\overline{\text{3CP-1}}$)	
Instance:	An edge 4-colorable planar graph G such that for every vertex $v \in V$ we have $\deg(v) \in \{1, 4\}$.
Question:	Is G 3-colorable?

The $\overline{\text{3CP-1}}$ problem is a variant of the 3-coloring problem considered in [2]; Lemma 2 is a variant of Lemma 33 of [2].

Lemma 2. *The $\overline{\textit{3CP-1}}$ problem is $\mathcal{NP}$-complete.*

Proof. To show that $\overline{\text{3CP-1}}$ is $\mathcal{NP}$-complete, we use a transformation from the following problem (the reader may refer to [7] for a proof of its $\mathcal{NP}$-completeness).

3-coloring of 4-regular planar graphs ($\overline{\text{3CP-2}}$)	
Instance:	A 4-regular planar graph G.
Question:	Is G 3-colorable?

Suppose that $G = (V, E)$ is an instance of $\overline{\text{3CP-2}}$, i.e. G is planar and 4-regular. To get an instance G' of $\overline{\text{3CP-1}}$ from G, we change G in a way similar to that described in [2]. We replace every vertex $v \in V$ by a copy of the gadget H^* shown in Fig. 1. Next, every edge $\{u, v\} \in E$ changes into an edge connecting two black vertices, one in a copy of H^* that replaces u and one in copy of H^* that replaces v. The second step is performed in such a way that the resulting graph G' is planar and all its black vertices are of degree 4.

Clearly, the transformation can be done in polynomial time. Moreover, G' is planar and all its vertices are of degree 4 or 1. To show that G' is edge 4-colorable, it suffices to color edges in copies of H^* in the way shown in Fig. 1 and complete the resulting partial coloring. The completion is achieved by the greedy coloring first, of all uncolored, non-pendant edges, and next, of all pendant edges. In the first step of the completion, the edge currently undergoing coloring has 4 colored neighbors, two of which have identical color 0, so a color from the set $\{1, 2, 3\}$ can always be assigned. In the second step, the edge being colored has 3 neighbors, and therefore it will receive a color from the set $\{0, 1, 2, 3\}$. Therefore the resulting coloring uses 4 colors.

The proof of the fact that G is 3-colorable if and only if G' is 3-colorable is immediate when we observe that in any 3-coloring of gadget H^* all the black vertices must obtain the same color. $\square$

Now we are ready to prove main result of the section.

Theorem 3. *The $\overline{L_{p>3q}}$ problem is $\mathcal{NP}$-complete.*

Proof. We will show how to reduce $\overline{\text{3CP-1}}$ to $\overline{L_{p>3q}}$ in polynomial time.

Let G be an instance of $\overline{\text{3CP-1}}$. Without loss of generality we can assume that $V(G) \cap E(G) = \emptyset$. Let H be the graph resulting from G by inserting a vertex in every edge of G, i.e. $V(H) = V(G) \cup E(G)$ and $E(H) = \{\{u, e\} : u \in V(G)$ is incident with $e \in E(G)\}$. Clearly, H is a bipartite, planar graph of maximum degree 4. To complete the proof, it suffices to show that $\chi(G) \leq 3$ if and only if $\lambda_{p,q}(H) \leq p + 5q$.

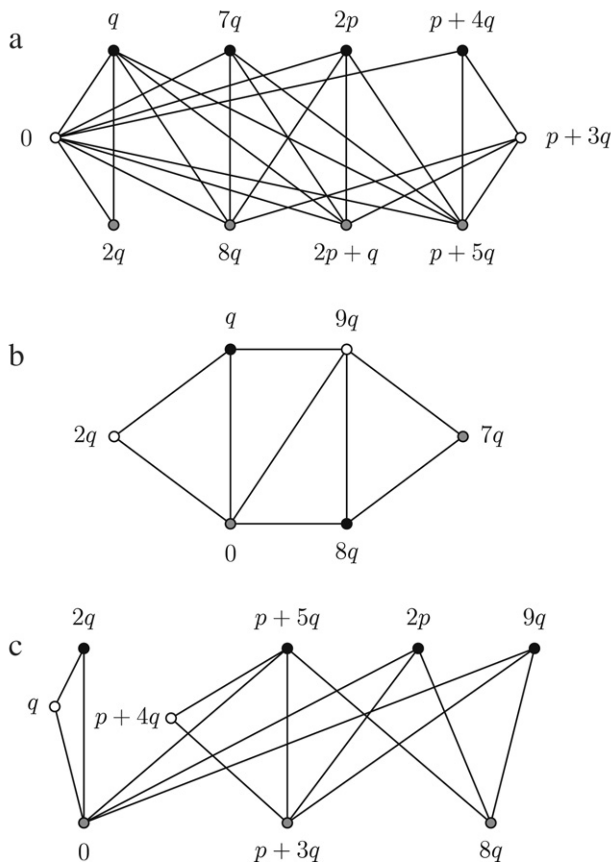

Fig. 2. Label adjacency graphs (cases a–c).

($\Rightarrow$) Let $c\colon V(G) \to \{0, 1, 2\}$ be a 3-coloring of G and $c'\colon E(G) \to \{0, 1, 2, 3\}$ be an edge 4-coloring of G. It is easy to see that function $l\colon V(H) \to \mathbb{N}$ given by

$$l(v) = \begin{cases} qc'(v) & \text{if } v \in E(G) \\ p + 3q + qc(v) & \text{if } v \in V(G) \end{cases} \tag{2}$$

is an $L(p, q)$-labeling of H. Moreover, $\max l(V(H)) = p + 5q$ and therefore $\lambda_{p,q}(H) \leq p + 5q$.

($\Leftarrow$) Let l be an $L(p, q)$-labeling of H such that $\max l(V(H)) \leq p + 5q$. By Lemma 1 we may assume that labels used by l are of the form $ap + bq$, $a, b \geq 0$. Let G_1 be a subgraph of G induced by vertices of degree 4. To complete the proof, it suffices to show that G_1 is 3-colorable. To this aim we focus on labels used by l on vertices of G_1. We will study which of these labels can be adjacent, i.e. may be assigned to adjacent vertices.

First, let us note that every vertex u of G_1 has 4 incident edges in G, say e_1, e_2, e_3 and e_4. Without loss of generality we may assume that $l(e_1) < l(e_2) < l(e_3) < l(e_4)$. There are only three cases to be considered:

(1) $l(u) < l(e_1)$. Then $l(e_4) - l(u) = (l(e_4) - l(e_3)) + (l(e_3) - l(e_2)) + (l(e_2) - l(e_1)) + (l(e_1) - l(u)) \geq p + 3q$. On the other hand $l(e_4) \leq p + 5q$, and thus $l(u) \leq 2q$.

(2) $l(u) > l(e_4)$. Then $l(u) - l(e_1) = (l(u) - l(e_4)) + (l(e_4) - l(e_3)) + (l(e_3) - l(e_2)) + (l(e_2) - l(e_1)) \geq p + 3q$. On the other hand $l(e_1) \geq 0$, and thus $l(u) \geq p + 3q$.

(3) $l(e_1) < l(u) < l(e_4)$. Then $l(e_i) < l(u) < l(e_{i+1})$ for some $i \in \{1, 2, 3\}$. Hence $p + 5q \geq l(e_4) - l(e_1) = (l(e_4) - l(e_3)) + \cdots + (l(e_{i+1}) - l(u)) + (l(u) - l(e_i)) + \cdots + (l(e_2) - l(e_1)) \geq 2p + 2q$, a contradiction.

Summarizing, labels used by l on vertices of G_1 are either not greater than $2q$ or not smaller than $p + 3q$.

For labeling l we define the *label adjacency graph* H_l as the graph with vertices being labels used by l on G_1 and edges connecting pairs of labels which are assigned to adjacent vertices of G_1, i.e. $V(H_l) = l(V(G_1))$ and $E(H_l) = \{\{l(u), l(v)\}\colon u \text{ and } v \text{ are connected by an edge in } G_1\}$. The structure of H_l depends on the ratio p/q. There are six cases to consider:

(a) $3q < p < 4q$. In this case $V(H_l) \subseteq \{0, q, 2q, 7q, 8q, p+3q, p+4q, p+5q, 2p, 2p+q\}$. We will show that H_l is a subgraph of the graph shown in Fig. 2a.

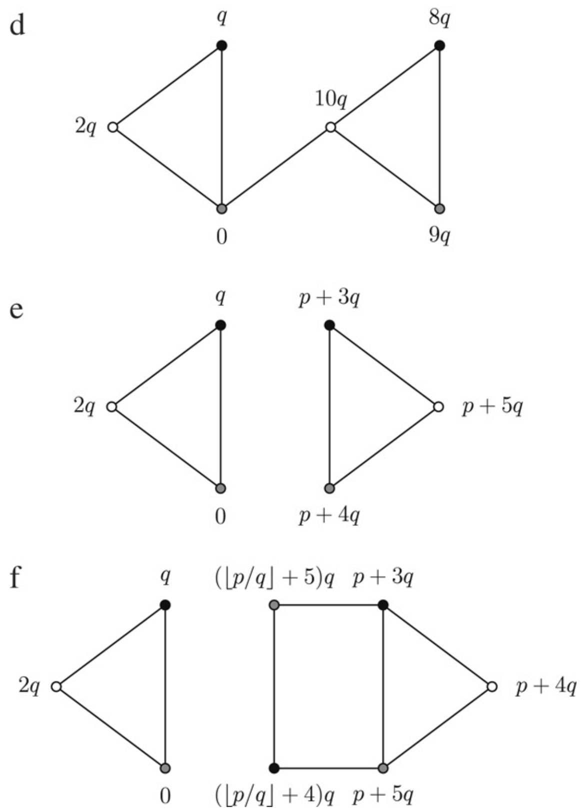

Fig. 3. Label adjacency graphs (cases d–f).

Suppose, contrary to our claim, that there is a pair u, v of adjacent vertices of G_1 such that $\{l(u), l(v)\}$ is not an edge of the graph shown in Fig. 2a. Without loss of generality we can assume that $l(u) < l(v)$. There are two cases to consider:

(1) $\{l(u), l(v)\} \in \{\{7q, 2p\}, \{7q, p + 3q\}, \{7q, p + 4q\}, \{8q, 2p + q\}, \{8q, p + 4q\}, \{8q, p + 5q\}, \{2p, p + 3q\}, \{2p, p + 4q\}, \{2p + q, p + 4q\}, \{2p + q, p + 5q\}\}$. Then $|l(u) - l(v)| < q$, which is impossible due to the fact that u and v are of distance 2 in H.

(2) $\{l(u), l(v)\} \in \{\{0, p+3q\}, \{q, 7q\}, \{q, 2p\}, \{q, p+3q\}, \{q, p+4q\}, \{2q, 7q\}, \{2q, 8q\}, \{2q, 2p\}, \{2q, 2p+q\}, \{2q, p+3q\}, \{2q, p + 4q\}, \{2q, p + 5q\}\}$. Let e be an edge connecting u with v in G_1. Then $l(e) > l(u)$, since otherwise $l(e) \le l(u) - p < 0$ and $l(e) < l(v)$ since otherwise $l(e) \ge l(v) + p > p + 5q$. Therefore $l(u) < l(e) < l(v)$ and $l(u) - l(v) \ge 2p$, a contradiction.

(b) $p = 4q$. In this case $V(H_l) \subseteq \{0, q, 2q, 7q, 8q, 9q\}$. Using the same arguments as above one can show that H_l is a subgraph of the graph shown in Fig. 2b.

(c) $4q < p < 5q$. In this case $V(H_l) \subseteq \{0, q, 2q, 8q, 9q, p+3q, p+4q, p+5q, 2p\}$. Using the same arguments as above one can show that H_l is a subgraph of the graph shown in Fig. 2c.

(d) $p = 5q$. In this case $V(H_l) \subseteq \{0, q, 2q, 8q, 9q, 10q\}$. Using the same arguments as above one can show that H_l is a subgraph of the graph shown in Fig. 3d.

(e) $p > 5q$ and $q|p$. In this case $V(H_l) \subseteq \{0, q, 2q, p+3q, p+4q, p+5q\}$. Using the same arguments as above one can show that H_l is a subgraph of the graph shown in Fig. 3e.

(f) $p > 5q$ and $q \nmid p$. In this case $V(H_l) \subseteq \{0, q, 2q, (\lfloor p/q \rfloor + 4)q, (\lfloor p/q \rfloor + 5)q, p + 3q, p + 4q, p + 5q\}$. Using the same arguments as above one can show that H_l is a subgraph of the graph shown in Fig. 3f.

In each of the above cases H_l is a subgraph of a graph that is 3-colorable (white vertices receive color 0, gray vertices receive color 1, and black vertices receive color 2) and thus it is 3-colorable. This completes the proof. □

4. The case $p < 3q$

Let us now consider the complexity of the $L(p, q)$-labeling problem, restricted to planar subcubic graphs, for values of $p < 3q$.

The $L(p, q)$-labeling problem for $p < 3q$ ($\overline{\mathrm{L}_{p<3q}}$)

Instance: A bipartite, planar graph G of maximum degree at most 3.
Question: Is $\lambda_{p,q}(G) \leq p + \max\{2q, p\}$?

Before we proceed to show the $\mathcal{NP}$-completeness of the above problem, let us present several of its simple properties.

Proposition 4. *A subcubic graph admits an $L(p, q)$-labeling with span at most t ($t = p + \max\{2q, p\}$) only if it admits such a labeling with the additional constraint that vertices of degree 3 can only receive labels 0 or t.*

Proof. Indeed, suppose that the considered graph has an $L(p, q)$-labeling c with span at most t. Observe that any vertex v of degree 3 may only receive label $c(v) \leq t - p - 2q$ or $c(v) \geq p + 2q$, since otherwise it would be impossible to extend the labeling to the neighbors of v (note that $2p + q > t$). Thus, taking into account the condition $p < 3q$, we have $c(v) < q$ or $c(v) > t - q$. Notice that a modification of labeling c such that $c(v) := 0$ in the former case and $c(v) := t$ in the latter case cannot affect its legality, which completes the proof. □

Given a subcubic graph $G = (V_3 \cup V_1, E_G)$ without vertices of degree 2, let $\overline{G} = (V_3 \cup V_2 \cup V_1, E_{\overline{G}})$ denote the subcubic graph formed by inserting exactly two vertices of degree 2 into each edge of G (we assume that $v \in V_d$ if and only if its degree is equal to d, $d \in \{1, 2, 3\}$). Then the following property holds.

Proposition 5. *Graph $\overline{G}$ admits an $L(p, q)$-labeling with span at most t only if it admits such a labeling with the following additional constraints:*

(i) *each vertex from V_3 receives one of the labels 0 or t,*
(ii) *each vertex $v \in V_3$, such that all neighbors of v in G belong to V_3, has the same label as exactly two of these neighbors.*

Proof. If $\overline{G}$ admits an $L(p, q)$-labeling with span at most t, then by Proposition 4 we can define a labeling $\bar{c}$ of $\overline{G}$ whose span is at most t and which fulfills constraint (i). Now, let $v \in V_3$ be a vertex of degree 3 connected in G to vertices $\{v_1, v_2, v_3\} \subseteq V_3$. Let $(vu_{1a}u_{1b}v_1)$, $(vu_{2a}u_{2b}v_2)$, $(vu_{3a}u_{3b}v_3)$ denote the corresponding paths in $\overline{G}$, where $\{u_{1a}, u_{1b}, u_{2a}, u_{2b}, u_{3a}, u_{3b}\} \subseteq V_2$. Without loss of generality we may write $\bar{c}(u_{1a}) < \bar{c}(u_{2a}) < \bar{c}(u_{3a})$. Supposing that $\bar{c}(v) = 0$, we obtain the following conditions by verifying the correctness of $L(p, q)$-labeling $\bar{c}$ along the paths from vertex v to vertices v_i, and taking into account that $q < p < 3q$, $t = p + \max\{2q, p\}$, and $2p + q > t$:

(a)

$$
\begin{aligned}
&|\bar{c}(u_{1a}) - \bar{c}(v)| \geq p \Longrightarrow \bar{c}(u_{1a}) \geq p,\\
&\bar{c}(u_{2a}) - \bar{c}(u_{1a}) \geq q \wedge \bar{c}(u_{3a}) - \bar{c}(u_{2a}) \geq q \Longrightarrow \bar{c}(u_{1a}) \leq t - 2q,\\
&|\bar{c}(u_{1b}) - \bar{c}(v)| \geq q \wedge |\bar{c}(u_{1b}) - \bar{c}(u_{1a})| \geq p \Longrightarrow c(u_{1b}) \geq 2p,\\
&|\bar{c}(v_1) - \bar{c}(u_{1b})| \geq p \wedge \bar{c}(v_1) \in \{0, t\} \Longrightarrow \bar{c}(v_1) = 0;
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\bar{c}(u_{2a}) - \bar{c}(u_{1a}) \geq q \wedge \bar{c}(u_{3a}) - \bar{c}(u_{2a}) \geq q \wedge \bar{c}(u_{1a}) \geq p \Longrightarrow \bar{c}(u_{3a}) \geq p + 2q,\\
&|\bar{c}(v_3) - \bar{c}(u_{3a})| \geq q \wedge \bar{c}(v_3) \in \{0, t\} \Longrightarrow \bar{c}(v_3) = 0;
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\bar{c}(u_{1a}) + q \leq \bar{c}(u_{2a}) \leq \bar{c}(u_{3a}) - q \wedge \bar{c}(u_{1a}) \geq p \Longrightarrow p + q \leq \bar{c}(u_{2a}) \leq t - q,\\
&|\bar{c}(u_{2b}) - \bar{c}(u_{2a})| \geq p \Longrightarrow \bar{c}(u_{2b}) \leq t - p - q,\\
&|\bar{c}(v_2) - \bar{c}(u_{2b})| \geq p \wedge \bar{c}(v_2) \in \{0, t\} \Longrightarrow \bar{c}(v_2) = t.
\end{aligned}
$$

Thus, assuming that $\bar{c}(v) = 0$, we have obtained $\bar{c}(v_1) = \bar{c}(v_3) = 0$ and $\bar{c}(v_2) = t$. Likewise, when $\bar{c}(v) = t$ the same method can be used to prove that $\bar{c}(v_1) = \bar{c}(v_3) = t$ and $\bar{c}(v_2) = 0$. Therefore labeling $\bar{c}$ fulfills both constraint (i) and constraint (ii). □

It is also possible to formulate a somewhat modified version of the converse of the above proposition.

Proposition 6. *Any labeling $c: V_3 \cup V_1 \to \{0, t\}$, such that each vertex $v \in V_3$ has exactly two neighbors with the same label in graph G, can be extended to an $L(p, q)$-labeling $\bar{c}: V_3 \cup V_2 \cup V_1 \to \{0, \ldots, t\}$ of graph $\overline{G}$, preserving $\bar{c}(v) = c(v)$ for all $v \in V_3 \cup V_1$.*

Proof. Since G is subcubic, the subset E_C of its edges for which both end-vertices have the same label, $E_C = \{\{v_1, v_2\} \in E_G: c(v_1) = c(v_2)\}$, forms a set of paths and cycles in G, whereas $E_G \setminus E_C$ is an independent set of edges. We can direct all edges from E_C, obtaining a set of arcs $\vec{E}_C$ such that each vertex $v \in V_3$ is the head of exactly one arc and the tail of exactly one arc from $\vec{E}_C$. Now, consider an $L(p, q)$-labeling procedure in graph $\overline{G}$ which, given a path $(v_1u_1u_2v_2)$ where $\{v_1, v_2\} \subseteq V_3 \cup V_1$ and $\{u_1, u_2\} \subseteq V_2$, determines values of labels $\bar{c}(u_1)$ and $\bar{c}(u_2)$ according to the following set of rules (illustrated in Fig. 4):

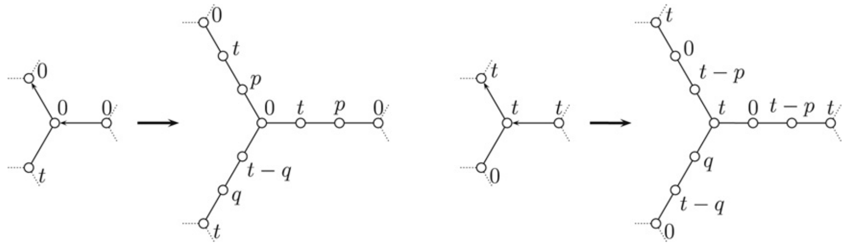

Fig. 4. Illustration of the procedure for converting a labeling of graph G into a labeling of graph $\overline{G}$. Directed edges denote arcs from $\vec{E}_C$.

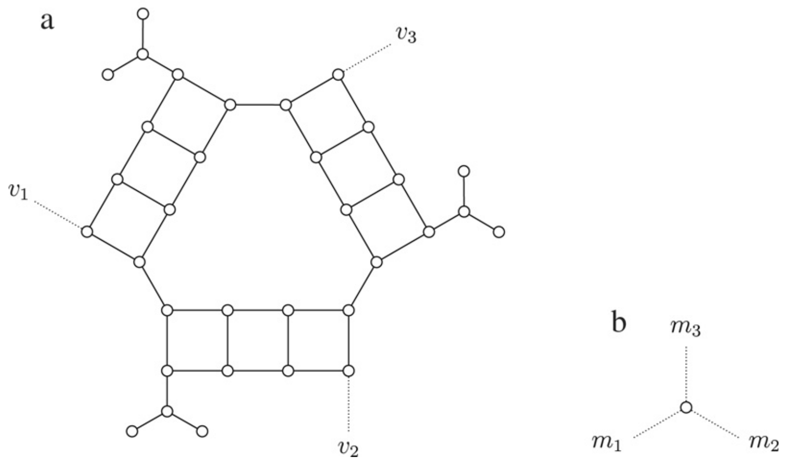

Fig. 5. Auxiliary gadgets: (a) $H_1(m)$, (b) $H_2(v)$.

(a) if $c(v_1) = c(v_2) = 0$ and $(v_1, v_2) \in \vec{E}_C$, then $\overline{c}(u_1) := p$ and $\overline{c}(u_2) := t$,
(b) if $c(v_1) = c(v_2) = t$ and $(v_1, v_2) \in \vec{E}_C$, then $\overline{c}(u_1) := t - p$ and $\overline{c}(u_2) := 0$,
(c) if $c(v_1) = 0$ and $c(v_2) = t$, then $\overline{c}(u_1) := t - q$ and $\overline{c}(u_2) := q$.

By applying the above rules we determine label values $\overline{c}(u)$ for all vertices $u \in V_2$, and the correctness of the obtained labeling is easy to verify. □

The proof of $\mathcal{NP}$-completeness of $\overline{\mathsf{L}_{p<3q}}$ proceeds by reduction from the problem of exact cover by 3-sets, restricted to cubic planar instances. This problem, referred to as $\overline{\mathsf{X3C}}$, can be formulated in the way described below and is known to be $\mathcal{NP}$-complete [11].

Exact Cover by 3-Sets ($\overline{\mathsf{X3C}}$)	
Instance:	A cubic bipartite planar graph $H = (V \cup M, E)$ having bipartite partitions V and M, such that $\lvert V\rvert = \lvert M\rvert = 3s$.
Question:	Is there a subset $M' \subseteq M$ of cardinality $\lvert M'\rvert = s$ covering all vertices in V?

Theorem 7. *The $\overline{\mathsf{L}_{p<3q}}$ problem is $\mathcal{NP}$-complete.*

Proof. In order to prove the claim we will show that for all values $p < 3q$ there exists a polynomial time algorithm which, given an $\overline{\mathsf{X3C}}$ instance H, determines a subcubic bipartite planar graph $\overline{G}$ such that problem $\overline{\mathsf{X3C}}$ has a positive solution for H if and only if $\lambda_{p,q}(\overline{G}) \leq t$.

Consider the transformation of $\overline{\mathsf{X3C}}$ instance $H = (V \cup M, E)$ into graph G, defined by the following two operations:

(1) each vertex $m \in M$, adjacent to some three vertices $\{v_1, v_2, v_3\} \subseteq V$, is replaced by the gadget $H_1(m)$ shown in Fig. 5a,
(2) each vertex $v \in V$, adjacent to some three vertices $\{m_1, m_2, m_3\} \subseteq M$, remains unchanged as the gadget $H_2(v)$ (Fig. 5c).

It is easy to observe that the resulting graph G is subcubic, bipartite and planar. Furthermore, G has no vertices of degree 2. Therefore we can create graph $\overline{G}$ from G in such a way that Propositions 5 and 6 hold; the entire construction of $\overline{G}$ from H is evidently polynomial. Since graph $\overline{G}$ is also subcubic, bipartite and planar, it now suffices to show that problem $\overline{\mathsf{X3C}}$ has a positive answer for graph H if and only if $\lambda_{p,q}(\overline{G}) \leq t$.

($\Leftarrow$) Suppose that $\lambda_{p,q}(\overline{G}) \leq t$. Then by Proposition 5 it is possible to define a labeling of vertices of G with integers 0 and t, such that each vertex of degree 3 in G with three neighbors in G of degree 3 has a label different from exactly one

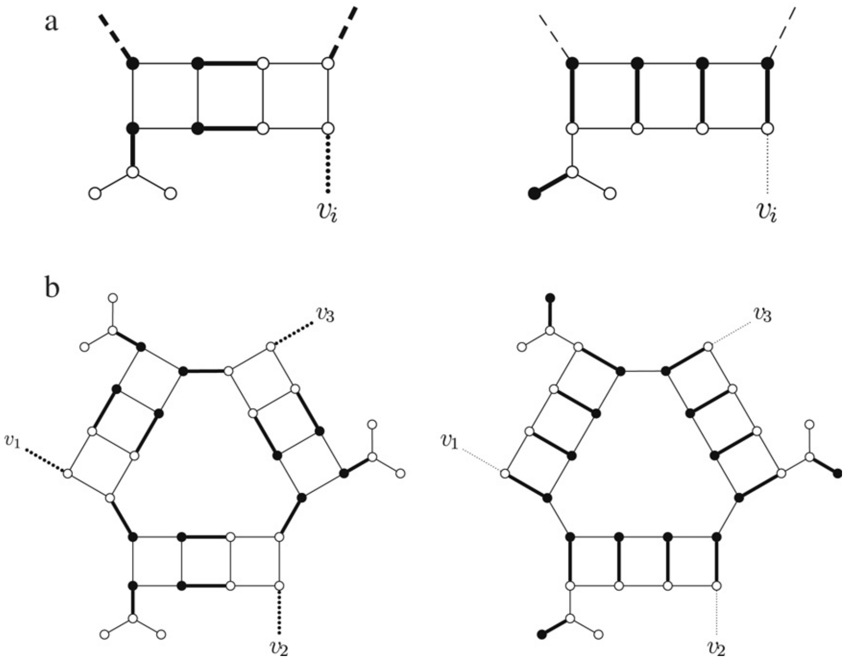

Fig. 6. Configurations of selected edges (marked in bold): (a) both possible configurations in a fragment of gadget $H_1(m)$, for $i = 1, 2, 3$, (b) corresponding configurations in gadget $H_1(m)$. In the considered labeling all black nodes have the same labels (one of $\{0, t\}$); all white nodes have the other label.

of those neighbors; an edge in G connecting such two vertices having different labels will be called *selected*. Let $m \in M$ be arbitrarily chosen. Let us consider all possible arrangements of selected edges in a fragment of gadget $H_1(m) \subseteq G$, as shown in Fig. 6a; observe that for the labeling to satisfy the imposed constraints, the number of selected edges in each cycle of the graph must be even. We easily conclude that there are exactly two possible configurations of outgoing edges of the gadget $H_1(m)$: either all outgoing edges of $H_1(m)$ are selected, as shown in the left-hand configuration in Fig. 6b, or none of them are selected, as presented in the right-hand configuration. The sought solution to the $\overline{\mathsf{X3C}}$ problem is now constructed as the set of those $m \in M$ for which all outgoing edges in $H_1(m)$ are selected. All that remains to be observed is that such a solution does indeed cover every vertex $v \in V$. This is true, since by applying Proposition 5 to the corresponding gadget $H_2(v) \subseteq G$ we immediately obtain that for each of these gadgets exactly one outgoing edge must always be selected.

($\Rightarrow$) Suppose that problem $\overline{\mathsf{X3C}}$ has a positive answer for graph H. Consider a labeling of vertices of graph G given by the following procedure:

(1) For all $v \in V$, assign label 0 to all vertices belonging to gadgets $H_2(v)$.
(2) For all $H_1(m)$, $m \in M$, adopt the left-hand configuration of selected edges (shown in Fig. 6) for those m which belong to the cover in the solution to $\overline{\mathsf{X3C}}$ for H, and the right-hand configuration for all other m. Next, label vertices of degree 3 belonging to gadget $H_1(m)$ using labels 0 and t in such a way that the end-vertices of an edge receive different labels if and only if the edge is selected.
(3) Assign labels 0 and t to vertices of degree 1 in G so that each vertex of degree 3 is adjacent to exactly two vertices with the same label.

It is easy to verify that such a labeling fulfills the assumptions of Proposition 6. In consequence, this labeling can be extended to an $L(p, q)$-labeling of graph $\overline{G}$ using labels $\{0, \dots, t\}$, therefore $\lambda_{p,q}(\overline{G}) \le t$, which completes the proof. $\square$

Putting $p = 2$ and $q = 1$ in the definition of problem $\overline{\mathsf{L}_{p<3q}}$, from the above theorem we obtain a corollary for the $L(2, 1)$-labeling problem which was mentioned in the introduction.

Corollary 8. *The problem of deciding whether a bipartite planar graph G fulfills $\lambda_{2,1}(G) \le 4$ is $\mathcal{NP}$-complete.*

5. The case $p = 3q$

In order to perform the proof for the boundary case $p = 3q$, we will apply a slight modification of the method from Section 4 to show the $\mathcal{NP}$-completeness of the following problem.

The $L(p, q)$-labeling problem for $p = 3q$ ($\overline{\mathsf{L}_{p=3q}}$)	
Instance:	A bipartite, planar graph G of maximum degree at most 4.
Question:	Is $\lambda_{p,q}(G) \le 6q$?

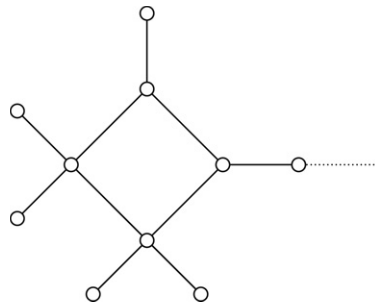

Fig. 7. The gadget H_3 used to construct graph $\tilde{G}$ from graph $\overline{G}$.

The question posed in problem $\overline{\mathsf{L}_{p=3q}}$ is in fact equivalent to that in the previously considered $\overline{\mathsf{L}_{p<3q}}$ problem, since now $t = p + \max\{2q, p\} = 3q + \max\{2q, 3q\} = 6q$. However, when $p = 3q$, Propositions 4–6 no longer hold for the subcubic graph $\overline{G}$ constructed from graph $G = (V_1 \cup V_3, E_G)$. In order to eliminate this problem, we define graph $\tilde{G}$ as the graph formed by attaching the gadget H_3 shown in Fig. 7 by an edge to each vertex of degree 3 in $\overline{G}$ (obviously, $\Delta(\tilde{G}) \le 4$). By Lemma 1, if $\lambda_{p,q}(\tilde{G}) \le 6q$, then there exists an optimal labeling of $\tilde{G}$ using only the labels $\{0, q, 2q, 3q, 4q, 5q, 6q\}$.

We easily obtain the following analogues of Propositions 4–6 for the considered case.

Proposition 9. *Graph $\tilde{G}$ admits an $L(p, q)$-labeling using labels from the range $\{0, \ldots, 6q\}$ only if it admits such a labeling with the additional constraint that vertices of degree 4 can only receive labels 0 or $6q$.*

Proof. Indeed, it is easy to verify that for a vertex v of degree 4, assignment of one of the labels $\{q, 2q, 3q, 4q, 5q\}$ is not possible, since then the $L(p, q)$-labeling cannot be extended to the neighbors of v. □

Proposition 10. *Graph $\tilde{G}$ admits an $L(p, q)$-labeling using labels from the range $\{0, \ldots, 6q\}$ only if it admits such a labeling with the following additional constraints:*

(i) *each vertex from V_3 receives one of the labels 0 or $6q$,*
(ii) *each vertex $v \in V_3$, such that all neighbors of v in G belong to V_3, has the same label as exactly two of these neighbors.*

Proof. The proof proceeds by analogy to that of Proposition 5. Note that due to the properties of labelings of gadget H_3, none of the vertices from V_2 can receive label $2q$ or label $4q$. □

Proposition 11. *Any labeling $c: V_3 \cup V_1 \to \{0, 6q\}$, such that each vertex $v \in V_3$ has exactly two neighbors with the same label in graph G, can be extended to an $L(p, q)$-labeling $\tilde{c}: V(\tilde{G}) \to \{0, \ldots, 6q\}$ of graph $\tilde{G}$, preserving $\tilde{c}(v) = c(v)$ for all $v \in V_3 \cup V_1$.*

Proof. For graph G, the proof proceeds by a construction identical to that used in the proof of Proposition 6 (using values $p = 3q$ and $t = 6q$). The obtained labeling of G can then be easily extended to a labeling of $\tilde{G}$ by labeling all the attached gadgets H_3. □

By introducing the above propositions in the proof of Theorem 7, we obtain a polynomial time algorithm which, given an $\overline{\mathsf{X3C}}$ instance H, determines a bipartite planar graph $\tilde{G}$, $\Delta(\tilde{G}) \le 4$, such that problem $\overline{\mathsf{X3C}}$ has a positive solution for H if and only if $\lambda_{p,q}(\tilde{G}) \le 6q$, where $p = 3q$. Thus, we have the following statement.

Theorem 12. *The $\overline{L_{p=3q}}$ problem is $\mathcal{NP}$-complete.*

6. Final remarks

We have just proved that the problem of computing $\lambda_{p,q}(G)$ is $\mathcal{NP}$-hard for subcubic graphs, in the case $p < 3q$, and graphs with maximum degree $\Delta \le 4$, in the case $p \ge 3q$; the obtained results hold even when considerations are restricted to bipartite planar graphs. On the other hand, it is clear that the problem of computing $\lambda_{p,q}(G)$ is polynomially solvable for all graphs with maximum degree at most 2. Taking this into account, we may say that our results almost completely determine the border line that separates $\mathcal{NP}$-hard cases from polynomial ones. Only one case remains unsolved: subcubic graphs in the case $p \ge 3q$. Our methods fail in that case; we conjecture it to be $\mathcal{NP}$-hard.

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References

[1] G. Agnarsson, R. Greenlaw, M.M. Halldórsson, On powers of chordal graphs and their colorings, Congressus Numer. 144 (2000) 41–65.
[2] H.L. Bodlaender, T. Kloks, J. van Leeuwen, R.B. Tan, Approximations for λ-coloring of graphs, The Comput. J. 47 (2004) 193–204.
[3] T. Calamoneri, The $L(h, k)$-labelling problem: An annotated bibliography, The Comput. J. 49 (2006) 585–608.
[4] G.J. Chang, D. Kuo, The $L(2, 1)$-labeling problem on graphs, SIAM J. Discrete Math. 9 (1996) 309–316.
[5] J. Fiala, P. Golovach, J. Kratochvil, Computational complexity of the distance constrained labelling problem for trees, in: Proc. ICALP 2008, in: LNCS, vol. 5125, Springer, 2008, pp. 294–305.
[6] J. Fiala, T. Kloks, J. Kratochvil, Fixed-parameter complexity of λ-labelings, Discrete Appl. Math. 113 (2001) 59–72.
[7] M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified $\mathcal{NP}$-complete graph problems, Theoret. Comput. Sci. 1 (1976) 237–267.
[8] J.P. Georges, D.W. Mauro, Generalized vertex labelings with a condition at distance two, Congressus Numer. 109 (1995) 141–159.
[9] W.K. Hale, Frequency assignment: Theory and applications, Proc. IEEE 68 (1980) 1497–1514.
[10] S.T. McCormick, Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem, Math. Program. 26 (1983) 153–171.
[11] C. Moore, J.M. Robson, Hard tiling problems with simple tiles, Discrete Computat. Geom. 26 (2001) 573–590.
[12] R.K. Yeh, A survey on labeling graphs with a condition at distance two, Discrete Math. 306 (2006) 1217–1231.