Discrete Applied Mathematics 184 (2015) 237–242

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

The computational complexity of the backbone coloring problem for planar graphs with connected backbones ☆

Robert Janczewski *, Krzysztof Turowski

Department of Algorithms and Systems Modelling, Gdańsk University of Technology, Narutowicza 11/12, Gdańsk, Poland

ARTICLE INFO

Article history:
Received 29 January 2014
Received in revised form 8 August 2014
Accepted 28 October 2014
Available online 13 November 2014

Keywords:
Planar graphs
Cacti
Backbone chromatic number

ABSTRACT

In the paper we study the computational complexity of the backbone coloring problem for planar graphs with connected backbones. For every possible value of integer parameters $\lambda \geq 2$ and $k \geq 1$ we show that the following problem:

Instance: A simple planar graph G, its connected spanning subgraph (backbone) H.
Question: Is there a λ-backbone coloring c of G with backbone H such that $\max c(V(G)) \leq k$?

is either NP-complete or polynomially solvable (by algorithms that run in constant, linear or quadratic time). As a result of these considerations we obtain a complete classification of the computational complexity with respect to the values of λ and k.

We also study the problem of computing the backbone chromatic number for two special classes of planar graphs: cacti and thorny graphs. We construct an algorithm that runs in $O(n^3)$ time and solves this problem for cacti and another polynomial algorithm that is 1-absolute approximate for thorny graphs.

1. Introduction

One of the main algorithmic issues in area of radio network design is the frequency assignment problem. Consider, for example, the radio network with given topology and assume existence of a certain substructure in the network (called the backbone) with higher requirements concerning the level of interferences. Such connections could be recognized as the ones with high traffic loads, crucial for the reliability of communication. In the case of backbone we should assign to the adjacent base stations the channels separated by a certain frequency gap, while for the rest of the network it suffices not to assign the same channel to the adjacent stations. The solution of this problem is to minimize the total frequency bandwidth required by the network while keeping the acceptable level of interference between signals.

This problem is closely related to the general framework for graph coloring problems: given a radio network, we can model its topology as a graph and the assignment of the frequency channels to the transmitters as a color assignment. In this model the base stations (transmitters, receivers) and possible interferences between them are represented respectively as the vertices and the edges of the graph. We define two vertices as adjacent if their frequency bands are close enough that their signals interfere.

Formally, in this model of radio networks, introduced by Broersma in [1], we consider the so-called λ-backbone colorings of a simple graph G with backbone (spanning subgraph) H, i.e. functions $c\colon V(G) \to \mathbb{N}_+$ which satisfy $|c(u) - c(v)| \geq \lambda$

☆ This project has been partially supported by Narodowe Centrum Nauki under contract DEC-2011/02/A/ST6/00201.
* Corresponding author.
E-mail addresses: skalar@eti.pg.gda.pl (R. Janczewski), Krzysztof.Turowski@eti.pg.gda.pl (K. Turowski).

http://dx.doi.org/10.1016/j.dam.2014.10.028
0166-218X/

for each edge $uv \in E(H)$ and $c(u) \neq c(v)$ for each edge $uv \in E(G)$, where $\lambda \geq 2$ is an integer. The λ-backbone coloring problem (BBC) is to find a λ-backbone coloring function c which minimizes the total span or, equivalently, $\max c(V(G))$. The smallest integer k such that there exists a λ-backbone coloring c with $\max c(V(G)) = k$ is called the λ-backbone chromatic number and denoted by $BBC_\lambda(G, H)$. A λ-backbone coloring function c is optimal if and only if $\max c(V(G)) = BBC_\lambda(G, H)$.

The first tight lower and upper bounds on $BBC_\lambda(G, H)$ depending on the chromatic number $\chi(G)$ of G were presented by Broersma in [1]:

$$\chi(G) \leq BBC_\lambda(G, H) \leq \lambda(\chi(G) - 1) + 1. \tag{1}$$

Another bounds for general graphs that depend on $\chi(H)$ and the number n of vertices of G, were presented in [9]:

$$\lambda(\chi(H) - 1) + 1 \leq BBC_\lambda(G, H) \leq \lambda(\chi(H) - 1) + n - \chi(H) + 1. \tag{2}$$

Clearly the backbone coloring problem is an extension of the classical vertex coloring problem so computing the exact value of the backbone chromatic number in general case is NP-hard. Furthermore, deciding whether for a given number k the inequality $BBC_\lambda(G, H) \leq k$ holds is NP-complete for any $k \geq \lambda + 2$ even in a case when H is restricted to a matching [3]. There were some works concerning the backbone coloring problem for planar graphs, see e.g. [2,8], but none of them gave a complete classification of its computational complexity. In the paper we deal with it.

The remainder of the paper is organized as follows. Section 2 contains preliminary results. In Section 3 we present our main result: the classification of the computational complexity of the following problem for all possible values λ and k:

Instance: A simple planar graph G, its connected spanning subgraph (backbone) H.
Question: Is there a λ-backbone coloring c of G with backbone H such that $\max c(V(G)) \leq k$?

The last section contains an algorithm for optimal coloring of cacti with connected backbones and 1-absolute approximate algorithm for coloring of thorny graphs. Both algorithms are polynomial.

2. Preliminaries

Theorem 1. *Let G be a graph and H be its spanning bipartite subgraph. Then*

$$BBC_\lambda(G, H) \leq \lambda + 2\chi(G) - 2. \tag{3}$$

Proof. It is an easy consequence of Proposition 13 of [8], which states that $BBC_\lambda(G, H) \leq (\chi(G) + \lambda - 2)\chi(H) - \lambda + 2$ provided that G is a graph, H is its subgraph and $\lambda \geq 2$. □

Lemma 2. *Let $1 \leq x \leq \lambda$. If H is a spanning subgraph of a nonempty graph G and $c : V \to \mathbb{N}_+$ is a λ-backbone coloring of graph G with backbone H such that $\max c(V) \leq \lambda + x$ then:*

(1) *the vertices colored with $1, 2, \ldots, x$ form an independent set in H,*
(2) *the vertices colored with $x + 1, x + 2, \ldots, \lambda$ are isolated in H,*
(3) *the vertices colored with $\lambda + 1, \lambda + 2, \ldots, \lambda + x$ form an independent set in H,*
(4) *H is bipartite and, provided it is connected, its bipartition is $c^{-1}(\{1, 2, \ldots, x\})$ and $c^{-1}(\{\lambda + 1, \ldots, \lambda + x\})$.*

Proof. (1), (3) Obvious.

(2) Easy consequence of the fact that $\min\{x + 1, x + 2, \ldots, \lambda\} + \lambda > \lambda + x$ and $\max\{x + 1, x + 2, \ldots, \lambda\} - \lambda \leq 0$.

(4) Follows from (1)–(3) and the fact that H has no isolated vertices. □

3. Main results

In this section we present the computational complexity of the backbone coloring problem for general planar graphs with connected backbones.

Theorem 3. *If G is planar and H is a connected spanning subgraph of G then the problem $BBC_\lambda(G, H) \leq k$ is decidable in $O(1)$ time for $k \leq \lambda$.*

Proof. If H is nonempty, then $BBC_\lambda(G, H) \geq \lambda + 1$. If H is both empty and connected, $H = G = K_1$ and $BBC_\lambda(G, H) = 1$. □

Theorem 4. *If G is planar and H is a connected spanning subgraph of G then the problem $BBC_\lambda(G, H) \leq \lambda + 1$ is decidable in $O(n)$ time.*

Proof. We prove that this problem is equivalent to the problem of deciding whether G is bipartite.

($\Rightarrow$) Let $BBC_\lambda(G, H) \leq \lambda + 1$. From Lemma 2 we know that H is bipartite and the only colors used are 1 and $\lambda + 1$. G must be also bipartite, otherwise the coloring would contain at least one vertex with a color outside of $\{1, \lambda + 1\}$.

($\Leftarrow$) Let G be bipartite. Then G with any backbone H can be colored using the colors 1 and $\lambda + 1$ assigned to the vertices in the first and second partition of G, respectively. □

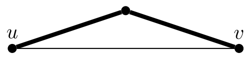


Fig. 1. Triangle that replaces the edge uv.

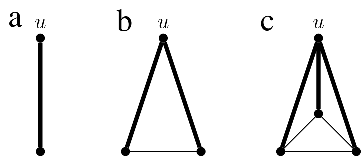


Fig. 2. Gadgets that replaces the vertex u: (a) the case $k = \lambda + 3$; (b) the case $k = \lambda + 4$; (c) the case $k = \lambda + 5$.

The above result can be strengthened for $\lambda \geq 5$: $BBC_\lambda(G, H) \leq \lambda + 1$ is decidable in $O(n)$ time for planar G and any backbone H. Indeed, $BBC_\lambda(G, H) \leq \lambda + 1$ implies that both G and H are bipartite. If they are bipartite, then we color G with colors $1, \lambda + 1$ on all non-isolated vertices in H and use the colors $\{2, 3, 4, 5\}$ to color all remaining vertices (this is possible due to the famous Four Color Theorem).

Theorem 5. *If G is planar and H is a connected spanning subgraph of G then the problem $BBC_\lambda(G, H) \leq \lambda + 2$ is decidable in $O(n^2)$ time.*

Proof. It was proved in [8] (see Theorem 17) that the problem is polynomially solvable even for nonplanar graphs. The proof is based on a reduction to the 2-SAT problem. The reduction can be done in $O(n^2)$ time and the 2-SAT is solvable in $O(n^2)$ time [5], so our claim holds. □

The requirement that H is connected is necessary. Otherwise, it was proved in [3] that for instances with matching backbone the problem $BBC_\lambda(G, H) \leq \lambda + 2$ is NP-complete. In fact, the proof implies (although it is not stated explicitly) that NP-completeness holds even when the problem is restricted to planar graphs with matching backbones.

Theorem 6. *If G is planar and H is a connected spanning subgraph of G, then the problem $BBC_\lambda(G, H) \leq k$ is NP-complete:*

(1) *for* $k = \lambda + 3$ *if* $\lambda \geq 3$,
(2) *for* $k = \lambda + 4$ *if* $\lambda \geq 4$,
(3) *for* $k = \lambda + 5$ *if* $\lambda \geq 5$.

Proof. We pick an arbitrary spanning tree T of G and replace every edge uv of T by a triangle shown in Fig. 1. Next, for each vertex u of G, we replace v by a gadget shown in Fig. 2 (in both cases bold edges belong to the backbone T'). We claim that the resulting graph G' with the backbone spanning tree T' has a λ-backbone coloring that uses colors from 1 to k if and only if G is 3-colorable.

If G is 3-colorable, we simply expand the 3-coloring to all new vertices by coloring them as follows: vertices of degree 2 in T' receive color $\lambda + 3$ and all other pendant vertices connected with vertex in G of color x receive colors $x + \lambda, x + \lambda + 1, \dots, x + k - 3$. This gives the desired λ-backbone coloring.

Now let c' be a λ-backbone coloring of G' with backbone T' such that $\max c'(V(G')) \leq k$. All vertices of the original graph G lie in the same partition of T', so, due to Lemma 2, they are either colored with $1, 2, \dots, k - \lambda$ or $\lambda + 1, \lambda + 2, \dots, k$. Without loss of generality we assume that the first possibility holds. Let v be a vertex of G. Let u_1, u_2 be its pendant (in T') neighbors with maximum and minimum color, respectively. v has exactly $k - \lambda - 2$ pendant neighbors in T' and all of them have different colors. Therefore $c'(v) \leq c'(u_1) - \lambda \leq c'(u_2) - (k - \lambda - 3) - \lambda = c'(u_2) - k + 3 \leq 3$ which proves that G is 3-colorable.

Since it is well known that 3-coloring of planar graphs is NP-complete (even for graphs with degree 4, [4]), the problem $BBC_\lambda(G, H) \leq k$ for planar graphs and connected backbones is also NP-complete. □

Theorem 7. *If G is planar and H is a connected spanning subgraph of G, then the problem $BBC_\lambda(G, H) \leq k$ is decidable in $O(n)$ time for every fixed $\lambda + 6 \leq k \leq 2\lambda$.*

Proof. If $\chi(H) \geq 3$ then $BBC_\lambda(G, H) \geq 2\lambda + 1$—hence H is necessarily bipartite. It turns out that it is also a sufficient condition, since from Theorem 1 we have $BBC_\lambda(G, H) \leq \lambda + 2\chi(G) - 2 \leq \lambda + 6$. □

Theorem 8. *If G is planar and H is a connected spanning subgraph of G, then the problem $BBC_\lambda(G, H) \leq k$ is NP-complete for every fixed $2\lambda + 1 \leq k \leq 3\lambda$.*

Proof. It is known (see the right-hand side of the inequality (1) and the left-hand side of the inequality (2)) that $BBC_\lambda(G, G) = \lambda(\chi(G) - 1) + 1$. Therefore $BBC_\lambda(G, G) \leq k$ if and only if $\chi(G) \leq 3$ for all $2\lambda + 1 \leq k \leq 3\lambda$. To complete the proof it suffices to recall that the 3-coloring problem for planar graphs is NP-complete [4]. □

Table 1
The complexity of the backbone coloring problem: G—planar, H—connected.

$BBC_2(G,H) \leq k$			$BBC_3(G,H) \leq k$		
$k \leq 2$	$O(1)$	Theorem 3	$k \leq 3$	$O(1)$	Theorem 3
$k = 3$	$O(n)$	Theorem 4	$k = 4$	$O(n)$	Theorem 4
$k = 4$	$O(n^2)$	Theorem 5	$k = 5$	$O(n^2)$	Theorem 5
$5 \leq k \leq 6$	NPC	Theorem 8	$6 \leq k \leq 9$	NPC	Theorem 6, 8
$k \geq 7$	$O(1)$	Theorem 9	$k \geq 10$	$O(1)$	Theorem 9

$BBC_4(G,H) \leq k$			$BBC_5(G,H) \leq k$		
$k \leq 4$	$O(1)$	Theorem 3	$k \leq 5$	$O(1)$	Theorem 3
$k = 5$	$O(n)$	Theorem 4	$k = 6$	$O(n)$	Theorem 4
$k = 6$	$O(n^2)$	Theorem 5	$k = 7$	$O(n^2)$	Theorem 5
$7 \leq k \leq 12$	NPC	Theorem 6, 8	$8 \leq k \leq 15$	NPC	Theorem 6, 8
$k \geq 13$	$O(1)$	Theorem 9	$k \geq 16$	$O(1)$	Theorem 9

$BBC_\lambda(G,H) \leq k$ $(\lambda \geq 6)$		
$k \leq \lambda$	$O(1)$	Theorem 3
$k = \lambda + 1$	$O(n)$	Theorem 4
$k = \lambda + 2$	$O(n^2)$	Theorem 5
$\lambda + 3 \leq k \leq \lambda + 5$	NPC	Theorem 6
$\lambda + 6 \leq k \leq 2\lambda$	$O(n)$	Theorem 7
$2\lambda + 1 \leq k \leq 3\lambda$	NPC	Theorem 8
$k \geq 3\lambda + 1$	$O(1)$	Theorem 9

Theorem 9. *If G is planar and H is a connected spanning subgraph of G, then the problem $BBC_\lambda(G, H) \leq k$ is trivial for $k \geq 3\lambda+1$.*

Proof. Every planar graph is 4-colorable, so from the upper bound from inequality (1) we obtain $BBC_\lambda(G, H) \leq 3\lambda + 1$. □

We may sum up all presented results in Table 1, containing the complete classification of the computational complexity with respect to the values of λ and k. Similar results, but not complete, for backbones being trees are presented in Table 2.

4. Backbone coloring of cacti and thorny graphs

In this section, we present algorithms for solving the backbone coloring problem for special classes of planar graphs: cacti and thorny graphs. Cacti, introduced first in literature under name Husimi trees [7], are defined as follows:

Definition 1. A connected graph is a cactus if and only if every edge of it belongs to at most one cycle.

It can be shown that every cactus can be constructed from a single vertex using a sequence of two operations: addition of a new pendant vertex or attachment of a cycle to one of the vertices of the graph. Clearly, every cactus is outerplanar and therefore also planar. The chromatic number of the cacti is at most 3. Cacti can be recognized in linear time. In [6], there was introduced the class of thorny graphs:

Definition 2. A connected graph G is thorny if and only if it has at least one decomposition into subgraphs $G_1, G_2, \ldots, G_k$ so the following conditions are fulfilled:

(1) all graphs G_i, $1 \leq i \leq k$, are cycles or paths,
(2) graph G is a union of the graphs $G_1, G_2, \ldots, G_k$,
(3) for every $2 \leq i \leq k$, the union of graphs $G_1, G_2, \ldots, G_{i-1}$ intersects with G_i only in a vertex or an edge.

As with the cacti, we can define thorny graphs as results of a sequence of three operations, starting from the K_1 graph: addition of a new pendant vertex, attachment of a cycle to one of the vertices of the graph or attachment of a cycle to one of the edges of the graph. Hence, every cactus is thorny and every thorny graph is planar and connected. Furthermore, it was shown in [6] that outerplanar graphs form a subclass of thorny graphs. In the same paper, there was presented algorithm that recognizes thorny graphs in $O(n^3)$ time.

Since thorny graphs are 3-colorable [6], we know from inequalities (1) and (2) that if H is not bipartite then $BBC_\lambda(G, H) = 2\lambda + 1$. Furthermore, we obtain the following corollary from Theorem 1:

Corollary 10. *If G is a thorny graph and H is a connected bipartite subgraph of G then $BBC_\lambda(G, H) \leq \lambda + 4$.* □

Table 2
The complexity of backbone coloring problem: G—planar, T—spanning tree.

$BBC_2(G,T) \le k$		
$k \le 2$	$O(1)$	Theorem 3
$k = 3$	$O(n)$	Theorem 4
$k = 4$	$O(n^2)$	Theorem 5
$5 \le k \le 6$	open	
$k \ge 7$	$O(1)$	Theorem 9

$BBC_3(G,T) \le k$		
$k \le 3$	$O(1)$	Theorem 3
$k = 4$	$O(n)$	Theorem 4
$k = 5$	$O(n^2)$	Theorem 5
$k = 6$	NPC	Theorem 6
$7 \le k \le 8$	open	
$k \ge 9$	$O(1)$	Theorem 1, 9

$BBC_4(G,T) \le k$		
$k \le 4$	$O(1)$	Theorem 3
$k = 5$	$O(n)$	Theorem 4
$k = 6$	$O(n^2)$	Theorem 5
$7 \le k \le 8$	NPC	Theorem 6
$k = 9$	open	
$k \ge 10$	$O(1)$	Theorem 1

$BBC_5(G,T) \le k$		
$k \le 5$	$O(1)$	Theorem 3
$k = 6$	$O(n)$	Theorem 4
$k = 7$	$O(n^2)$	Theorem 5
$8 \le k \le 10$	NPC	Theorem 6
$k \ge 11$	$O(1)$	Theorem 1

$BBC_\lambda(G,T) \le k$ $(\lambda \ge 6)$		
$k \le \lambda$	$O(1)$	Theorem 3
$k = \lambda + 1$	$O(n)$	Theorem 4
$k = \lambda + 2$	$O(n^2)$	Theorem 5
$\lambda + 3 \le k \le \lambda + 5$	NPC	Theorem 6
$k \ge \lambda + 6$	$O(1)$	Theorem 1

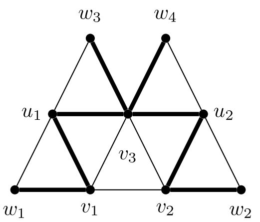


Fig. 3. Thorny graph G with backbone H, $BBC_\lambda(G, H) = \lambda + 4$.

This bound is tight, at least for $\lambda \ge 3$. An example is given in Fig. 3. Suppose that this graph has a λ-backbone coloring c such that $\max c(V(G)) \le \lambda + 3$. Then, by Lemma 2, the non-backbone triangle $v_1v_2v_3$ is colored either with $\{1, 2, 3\}$ or $\{\lambda+1, \lambda+2, \lambda+3\}$. Without loss of generality we assume that the first possibility holds. Then, at least one of the vertices u_i will be assigned $\lambda + 3$ as its color since all edges u_iv_j are in H. Hence, there would be a vertex w_i such that for some v_j and u_k we will have $w_iv_j \in E(H)$, $c(v_j) = 3$ and $w_iu_k \in E(G)$, $c(u_k) = \lambda + 3$. Therefore, such w_i cannot be colored using any color less than $\lambda + 4$—a contradiction.

However, in case of cacti graphs we may tighten the upper bound:

Theorem 11. *If G is a cactus and H is a connected bipartite spanning subgraph of G then $BBC_\lambda(G, H) \le \lambda + 3$. The bound is tight.*

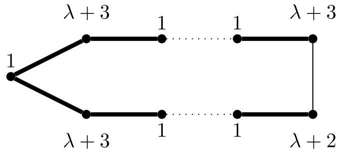
Proof. We claim that we can find a λ-backbone coloring using only colors from the set $\{1, 2, \lambda+2, \lambda+3\}$. G can be constructed from a single vertex using a sequence two operations: addition of a new pendant vertex or attachment of a cycle to one of the vertices of the graph. A single vertex can be easily colored with 1. To complete the proof, it suffices to show that, given a partial coloring, we can extend it without recoloring in both mentioned operations.

If we attach a pendant vertex v to a vertex u already colored with $c(u)$, we may assign $c(v) = \lambda + 4 - c(u)$. If we attach a pendant even cycle, we may assign colors $\lambda + 4 - c(u)$ and $c(u)$ alternately to the vertices on the cycle, thus obtaining a valid coloring.

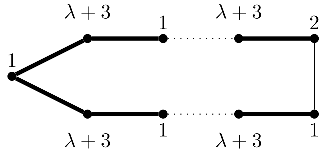
The only remaining case is thus the attachment of an odd cycle to a vertex v. Notice that such cycle contains one edge outside of H, since otherwise $\chi(H) > 2$. Let us denote this edge as u_1u_2. Then, either both paths: from v to u_1 and from v to u_2 have odd or even length. In the first case (left example in Fig. 4) we assign the colors $\lambda + 4 - c(v)$ and $c(v)$ alternately along both paths without u_1 and color u_1 with $\lambda + 3 - c(v)$.

In the second case (right example in Fig. 4) it is possible that either $v = u_1$ or $v = u_2$ (but not both since $u_1 \ne u_2$). Hence, we assign the colors $\lambda + 4 - c(v)$ and $c(v)$ along both paths and finally recolor an arbitrary $u_i \ne v$ with $\lambda + 3 - c(v)$.

Finally, the graph in Fig. 5 demonstrates that this bound cannot be improved, even if we restrict backbones to trees. Suppose on the contrary that it has λ-backbone coloring that uses colors $1, 2, \ldots, \lambda + 2$. Then, one of the vertices of the



(a) An example of odd paths.



(b) An example of even paths.

Fig. 4. The attachment of an odd cycle to a vertex v.

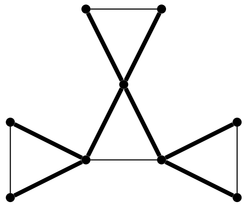


Fig. 5. Cactus G with backbone H, $BBC_\lambda(G, H) = \lambda + 3$.

internal triangle is colored with a color from the set $\{2, 3, \dots, \lambda+1\}$. But in this case, we cannot assign to the two uncolored its neighbors different colors less than $\lambda + 3$, which would satisfy the backbone coloring conditions. □

This result, together with Theorems 3–5, gives us a complete algorithm for finding an optimal coloring of a given cactus G with connected spanning backbone H. Since cacti can be decomposed into the sequence of two operations: attachment of a single vertex and attachment of a cycle (even or odd), starting from a single vertex in $O(n^3)$, the whole algorithm runs in $O(n^3)$. In case of thorny graphs we obtain 1-absolute approximate algorithm by combining the results from Theorems 3–5 with Corollary 10.

References

[1] H. Broersma, A general framework for coloring problems: old results, new results, and open problems, in: Combinatorial Geometry and Graph Theory, Springer Berlin, Heidelberg, 2003, pp. 65–79.
[2] H. Broersma, F.V. Fomin, P.A. Golovach, G.J. Woeginger, Backbone colorings for networks, in: Proceedings of the 29th International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2003, in: LNCS, vol. 2880, 2003, pp. 131–142.
[3] H. Broersma, J. Fujisawa, B. Marchal, D. Paulusma, A.N.M. Salman, K. Yoshimoto, λ-backbone colorings along pairwise disjoint stars and matchings, Discrete Math. 309 (18) (2009) 5596–5609.
[4] D.P. Dailey, Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete, Discrete Math. 30 (3) (1980) 289–293.
[5] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman & Co, 1979.
[6] K. Giaro, R. Janczewski, M. Małafiejski, A polynomial algorithm for finding T-span of generalized cacti, Discrete Appl. Math. 129 (2003) 371–382.
[7] F. Harary, G. Uhlenbeck, On the number of Husimi trees. I., Proc. Natl. Acad. Sci. 39 (4) (1953) 315–322.
[8] F. Havet, A.D. King, M. Liedloff, I. Todinca, (Circular) backbone colouring: Forest backbones in planar graphs, Discrete Appl. Math. 169 (2014) 119–134.
[9] R. Janczewski, K. Turowski, The backbone coloring problem for bipartite backbones, Graphs Combin. (in press).