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The convex domination subdivision number of a graph

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Abstract: Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a dominating set of G if every vertex in $V \setminus D$ has at least one neighbor in D. The distance $d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path in G. An (u, v)-path of length $d_G(u, v)$ is called an (u, v)-geodesic. A set $X \subseteq V$ is convex in G if vertices from all (a, b)-geodesics belong to X for any two vertices $a, b \in X$. A set X is a convex dominating set if it is convex and dominating set. The *convex domination number* $\gamma_{\text{con}}(G)$ of a graph G equals the minimum cardinality of a convex dominating set in G. *The convex domination subdivision number* $\text{sd}_{\gamma_{\text{con}}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the convex domination number. In this paper we initiate the study of convex domination subdivision number and we establish upper bounds for it.

Keywords: convex dominating set, convex domination number, convex domination subdivision number.

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1. Introduction

Throughout this paper, G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed* *neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The *degree* of a vertex v is $\deg_G(v) = |N_G(v)|$. A *leaf* is a vertex of degree one and a *universal vertex* is a vertex of degree $|V(G)| - 1$. We denote the number of leaves in a graph G by $\ell(G)$. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The private neighborhood of a vertex u with respect to a set $D \subseteq V$, where $u \in D$, is the set $PN_G[u, D] = N_G[u] - N_G[D - \{u\}]$. If $v \in PN_G[u, D]$, then we say that v is a private neighbor of u with respect to D. For a set S of vertices of G we denote by $G[S]$ the subgraph induced by S in G. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)–path in G. A (u, v)–path of length $d_G(u, v)$ is called (u, v)–*geodesic*. The greatest distance between any pair of vertices u, v in G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle. The girth of a graph with no cycle is defined ∞. The *edge-connectivity* $\kappa'(G)$ of G is the minimum number of edges whose removal results in a disconnected graph. Clearly for every graph G, $\kappa'(G) \leq \delta(G)$. Consult [14] for the notation and terminology which are not defined here.

A set $A \subset V(G)$ is a *dominating set* of G if $N_G[A] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a $\gamma(G)$–set. A set X is *weakly convex* in G if for any two vertices $a, b \in X$ there exists an (a, b)–geodesic such that all of its vertices belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* if it is weakly convex and dominating. The *weakly convex domination number* $\gamma_{\mathrm{wcon}}(G)$ of a graph G equals the minimum cardinality of a weakly convex dominating set in G.

A set $X \subset V(G)$ is convex in G if vertices from all (a, b)-geodesics belong to X for any two vertices $a, b \in X$. A set X is a convex dominating set if it is convex and dominating. The *convex domination number* of a graph G, denoted by $\gamma_{\mathrm{con}}(G)$, equals the minimum cardinality of a convex dominating set in G and a convex dominating set of minimum cardinality is called a $\gamma_{\mathrm{con}}(G)$–set. The (weakly) convex domination number was first investigated in [15], and since then has been studied by several authors [4, 16, 17].

Let us denote by G_{uv} or G_e the graph obtained from a graph G by subdividing an edge $e = uv \in E(G)$. The following result was proved in [7].

Proposition 1. The difference between $\gamma_{\mathrm{con}}(G)$ and $\gamma_{\mathrm{con}}(G_{uv})$ and between $\gamma_{\mathrm{con}}(G_{uv})$ and $\gamma_{\mathrm{con}}(G)$ can be arbitrarily large.

It means that subdividing an edge can arbitrarily increase or decrease the convex domination number.

The *(weakly convex, convex) domination subdivision number* $\mathrm{sd}_{\gamma}(G)$ ($\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G)$, $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G)$) of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the (weakly convex, convex) domination number. (An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a *subdivision vertex*). Since the (weakly convex, convex) domination number of the graph K_2 does not change when its only edge is subdivided, we will always assume that when we discuss $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G)$ all graphs involved are connected with $\Delta(G) \geq 2$.
The domination subdivision number, defined in Velammal's thesis [18], has been studied be several authors (see for instance [1, 9, 11, 13]). A similar concept related to connected domination in [10], to Roman domination in [2], to rainbow domination in [5, 8], and to 2–domination in [3].
The purpose of this paper is to initialize the study of the convex domination subdivision number $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G)$. Since subdividing an edge may decrease the convex domination number (Proposition 1), it may not be immediately obvious that the convex domination subdivision number is defined for all connected graphs with $\Delta(G) \geq 2$. We will show this shortly.
We make use of the following results in this paper.

Proposition 2. [15] If G is a connected graph of order n, then $\gamma_{\mathrm{wcon}}(G) \leq \gamma_{\mathrm{con}}(G)$.

Proposition 3. [4] If $G \neq K_n$ and D is a $\gamma_{\mathrm{con}}(G)$-set, then every cut-vertex belongs to D.

2. Basic properties of convex domination subdivision number

In this section, we investigate the basic properties of the convex domination subdivision number of a graph.

Theorem 1. Let G be a connected graph on at least three vertices, let E_S be a set of edges of G, let H be obtained from G by subdividing the edges in E_S, and let S be the set of subdivision vertices. If D_H is a convex dominating set of H, but $D := D_H - S$ is not a convex dominating set of G, then there exists a cycle of length at most 4 in G through some vertex of D.

Proof. We first show that D is a dominating set of G. If v is an arbitrary vertex of G, then either (i) $v \in D$, or (ii) $v \in N_H(w)$ for some vertex $w \in D_H - S$, or (iii) $v \in N_H(w)$ for some vertex $w \in S$. In case (i) or (ii) it is immediate the v is dominated by D, and in case (iii) w is the subdivision vertex

of an edge e of H that is incident with v, and that it follows by the convexity of D_H that the other end of e is contained in D_H and thus is D, so v is also dominated by D.
Since D is a dominating set of G, it follows that D is not convex in G. Let $a, b \in D$ be two vertices of G such that there exists an (a, b)-geodesic P in G containing vertices of $V(G) - D$. We assume that a and b have been chosen so that $d(a, b)$ is minimal with this property. Then

$$V(P) \cap D = \{a, b\}. \tag{1}$$

Let $P = a, a_1, a_2, \ldots, a_k$, where $a_k = b$. Clearly, $k \geq 2$. Now P corresponds to an (a, b)-path P_H in H. None of the edges of P, except possibly aa_1 and $a_{k-1}b$, are in E_S, since otherwise P would contain vertices of D in its interior, contradicting (1). Since D_H is convex in H, it follows that P_H is not an (a, b)-geodesic in H. Hence P_H is longer than P, so at least one of the edges of P, without loss of generality aa_1, is in E_S. Let u be the subdivision vertex of aa_1. Now a_1 is dominated in H by some vertex $b_1 \in D_H$ (possibly $b_1 = b$). We claim that $b_1 \neq u$. Suppose, to the contrary, that $b_1 = u$. Since D_H is convex and since $a_1 \notin D_H$, we conclude that every (u, b)-geodesic in H passing through a. Hence, uP_Hb is a (u, b)-path in H of length at most $k + 1$ which is not a (u, b)-geodesic. Let P'_H be a (u, b)-geodesic in H. Clearly, the length of P'_H is at most k. Now aP'_Hb corresponds to an (a, b)-path P' in G of length at most $k - 1$ which contradicts $d(a, b) = k$. Thus $b_1 \neq u$. Since b_1, a_1, u, a is a path joining two vertices in D_H that contains vertices not in D_H, it follows by the convexity of D_H that there exists a (b_1, a)-path Q in H of length at most two. The paths a, u, a_1, b_1 and Q form a cycle of length at most five in H, which corresponds to a cycle of length at most four in G containing a, as desired. $\square$

A closer look at the proof of Theorem 1 leads to the next result.

Corollary 1. Let G be a connected graph on at least three vertices, let E_S be a set of edges of G, let H be obtained from G by subdividing the edges in E_S, and let S be the set of subdivision vertices. If D_H is a convex dominating set of H, then $D := D_H - S$ is a dominating set of G.

Theorem 2. For any connected graph G of order $n \geq 3$ and size m, $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq m$.

Proof. Let H be the graph obtained from G by subdividing all edges of G, let T be the set of all subdivision vertices and let D_H be a convex dominating set of H. Clearly, H is a bipartite graph with partite sets $V(G)$ and T. It follows that $\gamma_{\mathrm{con}}(H) \geq 2$. Since for any two vertices $x, y \in V(G)$, every (x, y)-geodesic in H contains at least one subdivision vertex, we conclude that $D_H \cap T \neq \emptyset$. By

Corollary 1, $D := D_H - T$ is a dominating set of G. Now, let $a, b \in D$ be two arbitrary vertices. If P is an (a,b)-geodesic in G, then clearly P corresponds to an (a,b)-geodesic P_H in H. Since D_H is convex in H, we deduce that $V(P) \subseteq D$ and so D is convex. Thus D is a convex dominating set of G of size smaller than of $\gamma_{\rm con}(H)$. This yields $\mathrm{sd}_{\gamma_{\rm con}}(G) \leq m$ and the proof is completed. □

A consequence of Theorem 2 is that $\mathrm{sd}_{\gamma_{\rm con}}(G)$ is defined for every connected graph G of order $n \geq 3$.
Given $S, T \subseteq V(G)$, we write $[S,T]$ for the set of edges having one end–point in S and the other in T. An edge cut is an edge set of the form $[S, \overline{S}]$, where S is a nonempty proper subset of $V(G)$ and $\overline{S}$ denotes $V(G) - S$.

Theorem 3. For any connected triangle-free graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{\rm con}}(G) \leq \kappa'(G)$.

Proof. Assume $E_T = [S, \overline{S}]$ is an edge cut of G of size $\kappa'(G)$, G_1 and G_2 are the components of $G - E_T$, and H is the graph obtained from G by subdividing the edges of E_T. Let T be the set of all subdivision vertices and let D_H be a convex dominating set of H and $D_i = D_H \cap V(G_i)$ for $i = 1, 2$. If $D_H \cap T = \emptyset$, then $D_i \neq \emptyset$ for $i = 1, 2$, and $D_H = D_1 \cup D_2$. Now for vertices $x_1 \in D_1$ and $x_2 \in D_2$, any (x_1, x_2)-geodesic path intersect T implying that $D_H \cap T \neq \emptyset$ which leads to a contradiction. Therefore $D_H \cap T \neq \emptyset$. By Corollary 1, $D := D_H - T$ is a dominating set of G. Now we show that D is convex in G. Assume, to the contrary, that D is not a convex set in G. Let $a, b \in D$ be two vertices of G such that there exists an (a,b)-geodesic P in G containing vertices of $V(G) - D$. We suppose that a and b have been chosen so that $d(a,b)$ is minimal with this property. Then

$$V(P) \cap D = \{a, b\}. \tag{2}$$

Let $P = a, a_1, a_2, \ldots, a_k$, where $a_k = b$. Clearly, $k \geq 2$. Now P corresponds to an (a,b)-path P_H in H. None of the edges of P, except possibly aa_1 and $a_{k-1}b$, are in E_T, since otherwise P would contain vertices of D in its interior, contradicting (2). Since D_H is convex in H, we conclude that P_H is not an (a,b)-geodesic in H. Hence P_H is longer than P, so at least one of the edges of P, without loss of generality aa_1, is in E_T. Assume that $a \in V(G_1)$ and $a_1 \in V(G_2)$. Let u be the subdivision vertex of aa_1. Now a_1 is dominated in H by some vertex $b_1 \in D_H$ (possibly $b_1 = b$). As in the proof of Theorem 1, we have $b_1 \neq u$. Since b_1, a_1, u, a is a path joining two vertices in D_H that contains vertices not in D_H, it follows by the convexity of D_H that there exists a (b_1, a)-path Q in H of length at most two. Since E_T is an edge-cut of G, we deduce that the (b_1, a)-path Q in H has length two. Let $Q = b_1 y a$. If $b_1 \in D$, then $b_1 \in V(G_2)$ and y is the subdivision vertex of the edge $b_1 a$ and this implies that aa_1b_1 is a triangle in G, a contradiction. If $b_1 \in T$, then $y \in V(G_1)$ and so aa_1y is a triangle in G, a contradiction again. Thus D is a convex set in G and hence D is a convex dominating set of G of size smaller than of $\gamma_{\rm con}(H)$. This yields $\mathrm{sd}_{\gamma_{\rm con}}(G) \leq \kappa'(G)$ and the proof is completed. □

A closer look at the proof of Theorem 3 shows that if $E_T = [S, \overline{S}]$ is an edge-cut of size one, then b_1, a_1, u, a is the unique (b_1, a)-geodesic in H which is impossible. Hence we obtain the next result.

Corollary 2. For any connected graph G of order at least 3 with a cut edge, $\mathrm{sd}_{\gamma_{\rm con}}(G) = 1$.

The next results are immediate consequences of Theorem 3.

Corollary 3. For any connected triangle-free graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{\rm con}}(G) \leq \delta(G)$.

Corollary 4. For any connected triangle-free graph G with a cut vertex v,

$$\mathrm{sd}_{\gamma_{\rm con}}(G) \leq \lfloor \deg(v)/2 \rfloor.$$

Theorem 4. If G is a connected graph of order n with $g(G) \geq 5$, then $\mathrm{sd}_{\gamma_{\rm con}}(G) = 1$. In particular, for every edge $e \in E(G)$, $\gamma_{\rm con}(G_e) > \gamma_{\rm con}(G)$.

Proof. Let $e = u_1u_2$ be an arbitrary edge of G. If e is a cut edge, then clearly $\gamma_{\rm con}(G_e) > \gamma_{\rm con}(G)$. Let $C = (u_1u_2 \ldots u_k)$ be a cycle containing e. Assume G_e is obtained from G by subdividing the edge e with subdivision vertex w and let D be a $\gamma_{\rm con}(G_e)$–set. First let $\{u_1, u_2\} \subseteq D$. Then we have $w \in D$. Since $g(G) \geq 5$, we conclude from Theorem 1 that $D - \{w\}$ is a convex dominating set of G of size smaller than of $\gamma_{\rm con}(G_e)$ as desired. Now, let $\{u_1, u_2\} \nsubseteq D$. Assume, without loss of generality, that $u_2 \notin D$. To dominate w, we must have $u_1 \in D$. If $w \in D$, then as above $D - \{w\}$ is a convex dominating set of G of size smaller than of $\gamma_{\rm con}(G_e)$, as desired. Suppose that $w \notin D$. To dominate u_2, we must $D \cap N_G(u_2) \neq \emptyset$. Suppose $v \in D \cap N_G(u_2)$. Since $g(G) \geq 5$, we deduce that $d_{G_e}(u_1, v) = 3$. Since D is a convex dominating set for G_e, we must have $u_1, w, u_2, v \in D$, a contradiction. It follows that $\gamma_{\rm con}(G_e) > \gamma_{\rm con}(G)$ and hence $\mathrm{sd}_{\gamma_{\rm con}}(G) = 1$. This completes the proof. □

Corollary 5. For any connected graph G of order $n \geq 6$ with $g(G) = 4$,

$$\mathrm{sd}_{\gamma_{\rm con}}(G) \leq \lfloor n/2 \rfloor.$$

Proof. Let $C = (v_1v_2v_3v_4)$ be a cycle of G and let without loss of generality that $\deg(v_1) = \min\{\deg(v_i) \mid 1 \leq i \leq 4\}$. Since $g(G) = 4$, $N(v_1) \cap N(v_2) = \emptyset$. It follows that $\delta(G) \leq \deg(v_1) \leq \frac{\deg(v_1)+\deg(v_2)}{2} \leq \frac{n}{2}$ and the result follows from Corollary 3. □

It could be of ample interest if one could find the bound for $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G)$ posed in the following open problems:

Problem 1. Let G be a connected graph of girth four. Is there a constant c such that $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq c$.

Problem 2. Let G be a connected graph of girth three. Is there a constant c such that $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq c$.
Let $\alpha'(G)$ be the maximum number of edges in a matching in G.

Proposition 4. Let G be a connected triangle-free graph of order $n \geq 3$. If $\alpha'(G) < \frac{n-1}{2}$, then $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq \alpha'(G)$.

Proof. Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of G and let X be the independent set of M-unsaturated vertices. Since $\alpha'(G) < \frac{n-1}{2}$, we have $|X| \geq 2$. Assume y and z are vertices of X such that $\deg(y) \leq \deg(z)$. If $yu_i \in E(G)$, then since the matching M is maximum, $zv_i \notin E(G)$. Therefore, for all $i \in \{1, 2, \ldots, \alpha'\}$ there are at most two edges between the sets $\{u_i, v_i\}$ and $\{y, z\}$. So $2\deg(y) \leq \deg(y) + \deg(z) \leq 2\alpha'$ and the result follows by Corollary 3. □

Proposition 5. Let G be a connected graph of order $n \geq 3$. If $\alpha'(G) > \gamma_{\mathrm{con}}(G)$, then $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq \alpha'(G)$.

Proof. Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of G and let G' be obtained by subdividing every edge of M. Each convex dominating set of G' has order at least $|M|$. Hence $\gamma_{\mathrm{con}}(G') > \gamma_{\mathrm{con}}(G)$ and thus $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq \alpha'(G)$. □

3. Graphs with small convex domination subdivision number

In this section, we consider graphs with small convex domination subdivision number.

Proposition 6. Let G be a connected graph of order $n \geq 3$. If G satisfies one of the following properties:

(i) $\gamma_{\mathrm{con}}(G) = 1$;

(ii) $\gamma_{\mathrm{con}}(G) = 2$ and G contains a $\gamma_{\mathrm{con}}(G)$–set $\{a, b\}$ such that $N(a) \cap N(b) = \emptyset$;

then $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) = 1$.

Proof. (i) Since $n \geq 3$, the graph G_e obtained by subdividing any edge e of G has no universal vertex. Hence $\gamma_{\mathrm{con}}(G_e) > 1 = \gamma_{\mathrm{con}}(G)$ and so $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) = 1$.
(ii) Let G' be the graph obtained from G by subdividing the edge ab with subdivision vertex x. Obviously every convex dominating set of G' contains at least one of a, b, say a, and either two vertices in $N(a) \cup N(b)$, or x and b. Hence $\gamma_{\mathrm{con}}(G') \geq 3 > \gamma_{\mathrm{con}}(G)$. □

Proposition 7. For any connected graph G of order $n \geq 3$ with $\gamma_{\mathrm{con}}(G) = 2$,

$$\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq 2.$$

Proof. Since $\gamma_{\mathrm{con}}(G) = 2$, $\Delta(G) \leq n - 2$. Let $S = \{u, v\}$ be a $\gamma_{\mathrm{con}}(G)$–set. Assume u' is a private neighbor of u with respect to S and v' is a private neighbor of v with respect to S. Let G' be the graph obtained from G by subdividing the edges uu', vv' with subdivision vertices x and y, respectively, and let D be a $\gamma_{\mathrm{con}}(G')$-set. We show that $|D| \geq 3$ which implies $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq 2$. Suppose to the contrary that $|D| \leq 2$. To dominate x, y, we must have $|D \cap \{u, u'\}| \geq 1$ and $|D \cap \{v, v'\}| \geq 1$. Since $|D| \leq 2$, we have $|D \cap \{u, u'\}| = 1$ and $|D \cap \{v, v'\}| = 1$. Since $G[D]$ is connected and since $uv' \notin E(G)$ and $vu' \notin E(G)$, we deduce that either $D = \{u, v\}$ or $D = \{u', v'\}$. In each case, D is not a dominating set of G' which is a contradiction. Hence $\gamma_{\mathrm{con}}(G') = |D| \geq 3 > \gamma_{\mathrm{con}}(G)$ and the proof is complete. □

Proposition 8. Let $k \geq 2$ be an integer. For the complete k-partite graph $G = K_{p_1, p_2, \ldots p_k}$ with $2 \leq p_1 \leq p_2 \leq \cdots \leq p_k$,

$$\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) = \begin{cases} 1 & \text{if } k = 2 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. It is clear that any two adjacent vertices form a minimum convex dominating set of G which implies $\gamma_{\mathrm{con}}(G) = 2$. If $k = 2$, the result follows from Proposition 6 (ii). Let $k \geq 3$ and let $V_1, V_2, \ldots, V_k$ be the partite sets of G. By Proposition 7, $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \leq 2$. For any edge $e = ab$, where $a \in V_i, b \in V_j$ $(i \neq j)$, the set $\{a, v\}$ for each $v \in V_k$ $(k \notin \{i, j\})$ forms a minimum convex dominating set of G. It follows that $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) \geq 2$. Thus $\mathrm{sd}_{\gamma_{\mathrm{con}}}(G) = 2$ and the proof is complete. □

Proposition 8 shows that the bound in Proposition 7 is sharp.

Proposition 9. Let G be a connected graph of order $n \geq 3$ with $\gamma_{\text{con}}(G) = 3$ or 4. If G has a triangle, then $\text{sd}_{\gamma_{\text{con}}}(G) \leq 3$.

Proof. Assume uvw is a triangle in G and let H be the graph obtained from G by subdividing the edges uv, uw, vw by subdivision vertices x, y, z, respectively. Let D_H be a $\gamma_{\text{con}}(H)$-set. To dominate the subdivision vertices, we must have $|D_H \cap \{u, v, w\}| \geq 2$. Assume without loss of generality that $u, v \in D$. Since D_H is convex, we must have $x \in D_H$. Hence $\{u, v, x\} \subseteq D_H$. We show that $|D_H| \geq 5$ which implies $\text{sd}_{\gamma_{\text{con}}}(G) \leq 3$. Suppose to the contrary that $|D_H| \leq 4$. To dominate w, we must have $D_H \cap N_H[w] \neq \emptyset$. Assume $a \in D_H \cap N_H[w]$. Then $\{u, v, x, a\} \subseteq D_H$. If $\gamma_{\text{con}}(G) = 3$, then we deuce that $\text{sd}_{\gamma_{\text{con}}}(G) \leq 3$ as desired. Let $\gamma_{\text{con}}(G) = 4$. If $a = y$ (the case $a = z$ is similar), then we deduce from $d_H(a, v) = 3$ that $w, z \in D_H$ which is a contradiction. Assume that $a \notin \{y, z\}$. If $a = w$, then we must have $y, z \in D_H$ which leads to a contradiction again. Hence $a \neq w$. It follows from $|D_H| = 4$ that $au, av \in E(G)$. Hence uva is a triangle in G. It follows that $D := D_H - \{x\} = \{u, v, a\}$ is a convex dominating set of G contradicting $\gamma_{\text{con}}(G) = 4$. Thus $|D_H| \geq 5$ and so $\text{sd}_{\gamma_{\text{con}}}(G) \leq 3$. This completes the proof. □

Next we show that the bound in Proposition 9 is sharp when $\gamma_{\text{con}}(G) = 4$. The following graph was introduced by Haynes et al. in [12].
Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subset X : |Y| = k\}$. Thus, $\mathcal{Y}$ consists of all k–subsets of X, and so $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let G_k be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, G_k is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces a clique in G_k, while the set $\mathcal{Y}$ is an independent set and each vertex of $\mathcal{Y}$ has degree k in G_k. Therefore $\delta(G_k) = k$. Dettlaff et al. [6] proved that $\gamma_{\text{wcon}}(G_k) = 2(k-1)$.

Proposition 10. For any integer $k \geq 3$, $\gamma_{\text{con}}(G_k) = 2(k-1)$.

Proof. It is easy to see that any subset of X of cardinality $2(k-1)$ is a convex dominating set of G, and so $\gamma_{\text{con}}(G_k) \leq 2(k-1)$. It follows from Proposition 2 that $\gamma_{\text{con}}(G_k) = \gamma_{\text{wcon}}(G_k) = 2(k-1)$ and the proof is complete. □

Proposition 11. For any integer $k \geq 3$, $\text{sd}_{\gamma_{\text{con}}}(G_k) \geq 3$.

Proof. Assume e_1, e_2 are two arbitrary edges of G_k and let G'_k be the graph obtained from G_k by subdividing the edges e_1, e_2. We show that $\gamma_{\rm con}(G'_k) \le \gamma_{\rm con}(G_k) = 2(k-1)$. Assume $e_i = u_iv_i$ for $i = 1, 2$. Since every edge of G is incident with at least one vertex of X, we may assume that $u_i \in X$ for $i = 1, 2$. If $v_i \in \mathcal{Y}$ for $i = 1, 2$, then let w_i be a neighbor of v_i in $X - \{u_1, u_2\}$. If $v_1 \in \mathcal{Y}$ and $v_2 \in X$ (the case $v_2 \in \mathcal{Y}$ and $v_1 \in X$ is similar), then let $w_2 = v_2$ and w_1 be a neighbor of v_1 in $X - \{u_1, u_2\}$. If $v_i \in X$ for $i = 1, 2$, then let w_i be any vertex of $X - \{u_i, v_i \mid i = 1, 2\}$. Assume that $D = \{u_1, u_2, w_1, w_2\}$. Then $|D| \le 4$. Now extend D to a set D' of size $2(k-1)$ by adding $2(k-1) - |D|$ vertices of $X - \{u_i, v_i \mid i = 1, 2\}$. Clearly D' is a convex dominating set of G'_k, and so $\gamma_{\rm con}(G'_k) \le 2(k-1) = \gamma_{\rm con}(G_k)$. This implies that $\mathrm{sd}_{\gamma_{\rm con}}(G_k) \ge 3$ and the proof is complete. $\square$

In the case $k = 3$, Propositions 10 and 11 demonstrate that the bound of Proposition 9 is sharp when $\gamma_{\rm con}(G) = 4$.

Proposition 12. For every connected triangle-free graph G with $\gamma_{\rm con}(G) = 3$, $\mathrm{sd}_{\gamma_{\rm con}}(G) \le 2$.

Proof. Let G be triangle-free and let $D = \{u_1, u_2, u\}$ be a $\gamma_{\rm con}(G)$-set. Since $G[D]$ is connected and since G is triangle-free, $G[D]$ is a path. Suppose $G[D] = u_1uu_2$. It follows from convexity of D that

$$N_G(u_1) \cap N_G(u_2) = \{u\}. \tag{3}$$

If u_i has no private neighbor with respect to D for some i, then clearly $D - \{u_i\}$ is a convex dominating set of G which is a contradiction. Hence, assume u_i has a private neighbor, say v_i, with respect to D, for $i = 1, 2$. It follows that

$$u \notin N_G(v_1) \cup N_G(v_2). \tag{4}$$

Let G' be the graph obtained from G by subdividing the edges u_1v_1, u_2v_2 with vertices x_1, x_2, respectively, and let D' be a $\gamma_{\rm con}(G')$-set. We show that $|D'| \ge 4$. Suppose to the contrary that $|D'| \le 3$. To dominate x_i, we must have $D' \cap \{u_i, v_i\} \ne \emptyset$ for $i = 1, 2$. If $\{u_i, v_i\} \subseteq D'$ for some i, then $x_i \in D'$ implying that $|D'| \ge 4$, a contradiction. Let $|\{u_i, v_i\} \cap D'| = 1$ for each i. If $u_1, u_2 \in D'$, then clearly $u \in D'$ and so $D' = \{u, u_1, u_2\}$. But then v_1 is not dominated by D' since v_1 is a private neighbor of u_1 with respect to D in G, a contradiction. If $u_1, v_2 \in D'$ (the case $u_2, v_1 \in D'$ is similar), then u_1 and v_2 must have a common neighbor, say w, such that $D' = \{u_1, v_2, w\}$. Now to dominate u_2, we must have $wu_2 \in E(G)$ which is a contradiction because G is triangle-free. Let $v_1, v_2 \in D'$ and let $D' = \{v_1, v_2, w\}$. By (4), we have $w \neq u$. On the other hand, $w \neq u_i$ for some i, say $i = 1$. Since D' is a dominating set, we must have $wu, wu_1 \in E(G)$ which leads to a contradiction because G is triangle-free. This completes the proof. $\square$

Theorem 5. For every connected graph G with $\gamma_{\text{con}}(G) = 4$, $\text{sd}_{\gamma_{\text{con}}}(G) \leq 3$.

Proof. If G has a triangle, then the result follows by Proposition 9. Henceforth, let G be triangle-free. Let $D = \{u_1, u_2, u_3, u_4\}$ be a $\gamma_{\text{con}}(G)$-set such that the size of $G[D]$ is as large as possible. Since the induced subgraph $G[D]$ is connected, we consider three cases.

Case 1. $G[D] = C_4 = (u_1, u_2, u_3, u_4)$.
Since D is a convex set, we deduce that $N_G(u_1) \cap N_G(u_3) = \{u_2, u_4\}$ and $N_G(u_2) \cap N_G(u_4) = \{u_1, u_3\}$. Let G' be the graph obtained from G by subdividing the edges u_1u_2, u_2u_3, u_3u_4 with subdivision vertices x_1, x_2, x_3, respectively. Suppose D_1 is a $\gamma_{\text{con}}(G')$-set. To dominate x_1, we must have $u_1 \in D_1$ or $u_2 \in D_1$, to dominate x_2, $u_2 \in D_1$ or $u_3 \in D_1$, and to dominate x_3, $u_3 \in D_1$ or $u_4 \in D_1$. Consider the following subcases.

Subcase 1.1. $u_1, u_3 \in D_1$ (the case $u_2, u_4 \in D_1$ is similar).
Since $N_G(u_1) \cap N_G(u_3) = \{u_2, u_4\}$ and $P = u_1u_4x_3u_3$ is a path of length 3 in G', we deduce that $d_{G'}(u_1, u_3) = 3$. This implies that $u_1, u_4, x_3, u_3 \in D_1$. Now to dominate u_2, we must have $N_{G'}(u_2) \cap D_1 \neq \emptyset$. Since G is triangle-free, we deduce that $|D_1| \geq 5$ as desired.

Subcase 1.2. $u_2, u_3 \in D_1$.
Since D_1 is a convex set, we have $x_2 \in D_1$. If $x_1, x_3 \in D_1$, then $|D_1| \geq 5$ as desired. Let without loss of generality that $x_1 \notin D_1$. This implies that $u_1 \notin D_1$. To dominate u_1, we must have $N_{G'}(u_1) \cap D_1 \neq \emptyset$. Let $w \in N_{G'}(u_1) \cap D_1$. Since G is triangle-free and since $N_G(u_1) \cap N_G(u_3) = \{u_2, u_4\}$, we have $d_{G'}(w, \{u_2, x_2, u_3\}) \geq 2$. It follows from the convexity of D_1 that $|D_1| \geq 5$ and we are done.

Case 2. $G[D] = P_4 = u_1u_2u_3u_4$.
Then $u_1u_4 \notin E(G)$. It follows from the convexity of D that $d_G(u_1, u_4) = 3$, $N_G(u_1) \cap N_G(u_3) = \{u_2\}$ and $N_G(u_2) \cap N_G(u_4) = \{u_3\}$. Suppose G' is the graph obtained from G by subdividing the edges u_1u_2, u_2u_3, u_3u_4 with subdivision vertices x_1, x_2, x_3, respectively. Assume D_2 is a $\gamma_{\text{con}}(G')$-set. It now will be shown that $|D_2| \geq 5$. To dominate x_2, we must have $D_2 \cap \{u_2, u_3\} \neq \emptyset$. Assume without loss of generality that $u_2 \in D_2$. Now to dominate x_3, we must have $D_2 \cap \{u_3, u_4\} \neq \emptyset$. Consider two subcases.

Subcase 2.1. $u_3 \in D_2$.
Since D_2 is a convex set, $x_2 \in D_2$. If $x_1, x_3 \in D_2$, then $|D_2| \geq 5$ and we

are done. Assume without loss of generality that $x_1 \notin D_2$. Now to dominate u_1, we must have $D_2 \cap N_{G'}(u_1) \neq \emptyset$. Let $w \in D_2 \cap N_{G'}(u_1)$. Therefore $\{u_2, x_2, u_3, w\} \subseteq D_2$. Since D_2 is a convex set and since G is triangle-free, u_4 is not dominated by the set $\{u_2, x_2, u_3, w\}$ implying that $|D_2| \geq 5$ as desired.

Subcase 2.2. $u_4 \in D_2$.
Since G is triangle-free and $N_G(u_2) \cap N_G(u_4) = \{u_3\}$, we have $d_{G'}(u_2, u_4) \geq 3$. If $d_{G'}(u_2, u_4) \geq 4$, then it follows from convexity of D_2 that $|D_2| \geq 5$ and we are done. Let $d_{G'}(u_2, u_4) = 3$ and let $Q = u_2w_1w_2u_4$ is a path with length 3 in G'. Then $\{u_2, w_1, w_2, u_4\} \subseteq D_2$. Since $u_1u_4 \notin E(G)$, $d_G(u_1, u_4) = 3$ and G is triangle-free, we deduce that u_1 is not dominated by $\{u_2, w_1, w_2, u_4\}$ implying that $|D_2| \geq 5$ as desired.

Case 3. $G[D] = K_{1,3}$.
Assume u is the center of $G[D] = K_{1,3}$ and u_1, u_2, u_3 are leaves adjacent to u. If u_i has no private neighbor with respect to D for some i, then clearly $D - \{u_i\}$ is a convex dominating set of G which is a contradiction. Henceforth, assume u_i has a private neighbor with respect to D, say v_i, for each i. Let G' be the graph obtained from G by subdividing the edges u_1v_1, u_2v_2, u_3v_3 with vertices x_1, x_2, x_3, respectively, and let D_3 be a $\gamma_{\mathrm{con}}(G')$-set. We show that $|D_3| \geq 5$. Assume, to the contrary, that $|D_3| \leq 4$. To dominate x_i, we must have $D_3 \cap \{u_i, v_i\} \neq \emptyset$ for each i. If $\{u_i, v_i\} \subseteq D_3$ for some i, then $x_i \in D_3$ implying that $|D_3| \geq 5$, a contradiction. Let $|\{u_i, v_i\} \cap D_3| = 1$ for each i. Now we consider the following subcases.

Subcase 3.1. $u_i, u_j \in D_3$.
Assume without loss of generality that $u_1, u_2 \in D_3$. Since $d(u_1, u_2) = 2$, we must have $u \in D_3$ because D_3 is a convex set. If $u_3 \in D_3$, then $D_3 = D = \{u, u_1, u_2, u_3\}$ and v_1 is not dominated by D_3 since v_1 is a private neighbor of u_1 with respect to D, a contradiction. Let $v_3 \in D_3$. Then $D_3 = \{u, u_1, u_2, v_3\}$. Since v_3 is a private neighbor of u_3 with respect to D, we deduce that $d_{G'}(v_3, \{u, u_1, u_2\}) \geq d_G(v_3, \{u, u_1, u_2\}) \geq 2$. Hence, v_3 is an isolated vertex in $G'[D_3]$ which contradicts the connectedness of $G'[D_3]$.

Subcase 3.2. $u_i, v_j, v_k \in D_3$ where $\{j, k\} = \{1, 2, 3\} - \{i\}$.
Assume without loss of generality that $i = 1$. Since v_2 is a private neighbor of u_2 with respect to D, $d_{G'}(u_1, v_2) \geq d_G(u_1, v_2) \geq 2$. First let $d_{G'}(u_1, v_2) = 2$. Assume $w \in N(u_1) \cap N(v_2)$. Then $D_3 = \{u_1, w, v_1, v_2\}$ and w must dominate u_2 which leads to a contradiction because G is triangle-free. Now let $d_{G'}(u_1, v_2) \geq 3$. Similarly, we may assume $d_{G'}(u_1, v_3) \geq 3$. It follows from the convexity of D_3 that $|D_3| \geq 5$, a contradiction again.

Subcase 3.3. $v_1, v_2, v_3 \in D_3$.
Let $D_3 = \{v_1, v_2, v_3, w\}$. Then w must be adjacent to u_i for each i. Since $G'[D_3]$ is connected, we may assume that $wv_1 \in E(G)$. This leads to a contradiction because G is triangle-free and the proof is complete. □

We conclude this paper with an open problem.
A connected graph G is called *convex domination subdivision critical* if subdividing every edge of G increases the convex domination number of G.

Problem 3. Characterize the convex domination subdivision critical graphs.

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