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Research Article

Maciej Starostka* and Nils Waterstraat

The E-Cohomological Conley Index, Cup-Lengths and the Arnold Conjecture on T^{2n}

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Abstract: We show that the E-cohomological Conley index, that was introduced by the first author recently, has a natural module structure. This yields a new cup-length and a lower bound for the number of critical points of functionals on Hilbert spaces. When applied to the setting of the Arnold conjecture, this paves the way to a short proof on tori, where it was first shown by C. Conley and E. Zehnder in 1983.

Keywords: E-Cohomology, Critical Points, Arnold Conjecture

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1 Introduction

Motivated by questions of celestial mechanics from the beginning of the 20th century, Arnold conjectured in the sixties that every Hamiltonian diffeomorphism on a compact symplectic manifold (M, ω) has at least as many fixed points as a function on M has critical points. Let us recall that a diffeomorphism $\psi : M \to M$ is called Hamiltonian if there exists a smooth map $H : \mathbb{R} \times M \to \mathbb{R}$, $H(t+1, x) = H(t, x)$, such that $\psi = \eta^1$, where the family $\{\eta^t\}_{t\in\mathbb{R}}$ satisfies

$$\begin{cases} \dfrac{d}{dt}\eta^t = X_H(\eta^t), \\ \eta^0 = \mathrm{id}, \end{cases} \tag{1.1}$$

and X_H stands for the time-dependent vector field given by

$$dH(\,\cdot\,) = \omega(X_H, \cdot\,).$$

Consequently, p is a fixed point of ψ if and only if it is the initial condition of a 1-periodic solution of (1.1), and so Arnold's famous conjecture can be reformulated dynamically as follows.

Arnold Conjecture. The Hamiltonian system

$$\dot{x}(t) = X_H(x(t)) \tag{1.2}$$

has at least as many 1-periodic orbits as a function on M has critical points.

The aim of this paper is to point out a new approach to the Arnold conjecture which proves it on tori, where it was first shown by C. Conley and A. Zehnder in [3]. Let us point out that several approaches to the Arnold

***Corresponding author: Maciej Starostka,** Ruhr-Universität Bochum, Bochum, Germany; and Gdańsk University of Technology, Gdańsk, Polen, e-mail: maciejstarostka@gmail.com
Nils Waterstraat, Institut für Mathematik, Naturwissenschaftliche Fakultät II, Martin-Luther-Universität Halle-Wittenberg, 06099 Halle (Saale), Germany, e-mail: nils.waterstraat@mathematik.uni-halle.de

Conjecture have appeared since then. We refer to [10, pp. 215–216], but want to mention in particular that Chaperon proved it in [2] for tori by a concise geometric argument (cf. also [9]). Our proof is short as well, and it will be future work to investigate if our methods also apply to cases where the conjecture is still open. For example, the Arnold conjecture has not been proved for $T^{2n} \times \mathbb{C}P^m$ where a similar analytical setting can be introduced (see [6]). To the best of our knowledge, the previous methods only work to some extent in this case (see, however, [12] for partial results), and therefore it is worthwhile to develop new approaches.

However, let us point out that, apart from these important applications, our methods are of independent interest and can be outlined as follows. In [15] the first author introduced the E-cohomological Conley index for isolated invariant sets of flows in Hilbert spaces. Roughly speaking, it is a generalization of the classical Conley index for flows on locally compact spaces by using E-cohomology, which is a generalized cohomology theory for subsets of Hilbert spaces that was constructed by Abbondandolo in [1] (cf. also [7]). The first aim of this paper is to introduce a module structure for the E-cohomological Conley index, which allows us to define a relative cup-length for triples of closed and bounded subsets of Hilbert spaces. Secondly, we consider this numerical invariant for isolating neighborhoods of $\mathcal{LS}$-flows in Hilbert spaces (cf. [8, 16]), and show that it is a lower bound for the number of critical points of gradient flows as in classical Ljusternik–Schnirelman theory. Here we substantially use properties of the E-cohomological Conley index that were recently obtained by the first author in a joint work with Izydorek, Rot, Styborski and Vandervorst in [11]. Finally, we apply our Ljusternik–Schnirelman-type theorem to the functionals in the setting of the Arnold conjecture on T^{2n}. This yields an estimate from below for the number of contractible 1-periodic solutions of (1.2), and the obtained bound is indeed the one that Arnold conjectured.

This paper is organized with the intention of guiding the reader through our proof of the Arnold conjecture in as straightforward a manner as possible. Therefore, in the second section, we only introduce the material that is necessary to understand the basics of our approach and postpone more technical proofs to Section 4. Our discussion of the Arnold conjecture can be found in between, in the third section.

2 The *E*-Cohomological Conley Index and Cup-Lengths

2.1 Module Structure for *E*-Cohomology

We begin this section by recalling E-cohomology from [1], where we slightly modify the definition as in [11]. Let E be a separable real Hilbert space and E^+, E^- closed subspaces such that $E = E^+ \oplus E^-$. In what follows we denote by H^* Alexander–Spanier cohomology with compact supports, for which we refer to [14] and the nice survey in [1, Section 1]. Moreover, we let $\mathcal{V}$ be the set of all finite-dimensional subspaces of E^-, which is partially ordered by inclusion and directed.

If $U, V, W \in \mathcal{V}$ are such that $W = V \oplus U$ and $\dim(U) = 1$, then we can decompose W into two subspaces by setting

$$W^+ = \{w \in W : \langle w, u\rangle \geq 0\},$$
$$W^- = \{w \in W : \langle w, u\rangle \leq 0\},$$

where $u \neq 0$ is a fixed element in U. Note that the choice of u corresponds to an orientation of the one-dimensional space U, and changing this orientation swaps W^+ and W^-.

We set for a closed and bounded subset X of E,

$$X_W = X \cap (E^+ \times W), \quad X^+_W = X \cap (E^+ \times W^+), \quad X^-_W = X \cap (E^+ \times W^-)$$

and note that $X_W = X^+_W \cup X^-_W$ as well as $X_V := X \cap (E^+ \times V) = X^+_W \cap X^-_W$. (See Figure 1.) If now $A \subset X$ is closed, then we obtain a relative Meyer–Vietoris sequence

$$\cdots \longrightarrow H^k(X^+_W, A^+_W) \oplus H^k(X^-_W, A^-_W) \longrightarrow H^k(X_V, A_V) \xrightarrow{\Delta^k_{V,W}} H^{k+1}(X_W, A_W)$$
$$\longrightarrow H^{k+1}(X^+_W, A^+_W) \oplus H^{k+1}(X^-_W, A^-_W) \longrightarrow \cdots .$$

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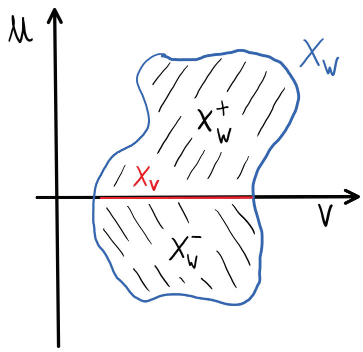

Figure 1: Decomposition of X by W.

In the more general case that $W = V \oplus U$ and $\dim(U) = n \geq 2$, we decompose U into n one-dimensional subspaces $U = U_1 \oplus \cdots \oplus U_n$ and set $W_i = V \oplus U_1 \oplus \cdots \oplus U_i$ for $1 \leq i \leq n$ as well as $W_0 = V$. Then the previous construction yields n Mayer–Vietoris homomorphisms

$$\Delta^{k+i-1}_{W_{i-1},W_i} : H^{k+i-1}(X_{W_{i-1}}, A_{W_{i-1}}) \to H^{k+i}(X_{W_i}, A_{W_i})$$

and their composition is a homomorphism $H^k(X_V, A_V) \to H^{k+n}(X_W, A_W)$. Hence we have constructed for any $q \in \mathbb{Z}$ and $V, W \in \mathcal{V}$, $V \subset W$, a homomorphism

$$\Delta^q_{V,W}(X) : H^{q+\dim(V)}(X_V, A_V) \to H^{q+\dim(W)}(X_W, A_W).$$

As noted in [1, Proposition 2.2], these maps do not depend on the choice of the one-dimensional subspaces U_i and their orientations. In summary, $\{H^{q+\dim(V)}(X_V, A_V), \Delta^q_{VW}(X, A)\}$ is a direct system of abelian groups over the directed set $\mathcal{V}$.

Definition 2.1. Let $A \subset X$ be closed and bounded subsets of E. The E-cohomology group of index $q \in \mathbb{Z}$ of (X, A) is the direct limit

$$H^q_E(X, A) = \varinjlim_{V\in\mathcal{V}} \{H^{q+\dim(V)}(X_V, A_V), \Delta^q_{V,W}(X, A)\},$$

and we set as usual $H^q_E(X) := H^q_E(X, \emptyset)$.

The inclusions $\iota_{V,W} : X_V \to X_W$ for $V, W \in \mathcal{V}$ yield an inverse system $\{H^p(X_V), \iota^*_{V,W}\}$ over $\mathcal{V}$. We define for $p \in \mathbb{Z}$ the group $H^p_0(X)$ as the inverse limit

$$H^p_0(X) := \varprojlim_{V\in\mathcal{V}} \{H^p(X_V), \iota^*_{V,W}\}.$$

In what follows, we denote elements of $H^p_0(X)$ by $[\alpha_V]_0$ if $\alpha_V \in H^p(X_V)$, and correspondingly elements of $H^q_E(X, A)$ by $[\alpha_V]_E$ if $\alpha_V \in H^{q+\dim(V)}(X_V, A_V)$.

Let us point out that $H^*_0(X)$ is a ring if we define the product of $[\alpha_V]_0 \in H^p_0(X)$ and $[\beta_V]_0 \in H^q_0(X)$ by

$$[\alpha_V]_0 \cup [\beta_V]_0 = [\alpha_V \cup \beta_V]_0 \in H^{p+q}_0(X).$$

It is readily seen from the naturality of the cup product that this is a sensible definition.

Proposition 2.2. *The group $H^*_E(X, A)$ is a right module over $H^*_0(X)$, where the module multiplication is induced by the cup product.*

Proof. We define for $[\alpha_V]_0 \in H^r_0(X)$ and $[\beta_V]_E \in H^q_E(X, A)$,

$$[\beta_V]_E \cup [\alpha_V]_0 := [\beta_V \cup \alpha_V]_E \in H^{q+r}_E(X, A).$$

This product is well defined, as if $\beta_W = \Delta^q_{V,W}\beta_V$ and $\alpha_V = \iota^*_{V,W}\alpha_W$, then

$$\Delta^{q+r}_{V,W}(\beta_V \cup \alpha_V) = \Delta^{q+r}_{V,W}(\beta_V \cup \iota^*_{V,W}\alpha_W) = (\Delta^q_{V,W}\beta_V) \cup \alpha_W = \beta_W \cup \alpha_W,$$

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where we have used that the coboundary operators of the Mayer–Vietoris sequence commute with products in multiplicative cohomology theories (cf. [4, Proposition 17.2.1]). □

Let now $\Omega \subset E$ be closed and bounded and such that $X \subset \Omega$. The inclusions $j_V : X_V \to \Omega_V$ induce homomorphisms $j_V^* : H^p(\Omega_V) \to H^p(X_V)$ for $V \in \mathcal{V}$, and it is readily seen that they actually yield a ring homomorphism

$$j^* : H_0^*(\Omega) \to H_0^*(X).$$

Consequently, we obtain the following corollary from Proposition 2.2.

Corollary 2.3. *For every $X \subset \Omega \subset E$, $H_E^*(X, A)$ is a right $H_0^*(\Omega)$-module.*

Henceforth we denote the module product of $\alpha \in H_0^r(\Omega)$ and $\beta \in H_E^p(X, A)$ by

$$\beta \cup \alpha \in H_E^{p+r}(X, A).$$

We conclude this section with the following crucial definition of a *relative cup-length*.

Definition 2.4. Let $A \subset X \subset \Omega$ be closed and bounded subsets of E.

- If $H_E^*(X, A) = 0$, we set
$$\mathrm{CL}(\Omega; X, A) = 0.$$
- If $H_E^*(X, A) \neq 0$ but $\beta \cup \alpha = 0$ for every $\beta \in H_E^*(X, A)$ and $\alpha \in H_0^{>0}(\Omega)$, then we set
$$\mathrm{CL}(\Omega; X, A) = 1.$$
- If there are $k \geq 2$, $\beta_0 \in H_E^*(X, A)$ and $\alpha_1, \alpha_2, \dots, \alpha_{k-1} \in H_0^{>0}(\Omega)$ such that
$$\beta_0 \cup \alpha_1 \cup \cdots \cup \alpha_{k-1} \neq 0,$$
then
$$\mathrm{CL}(\Omega; X, A) \geq k.$$

In order to keep the definition short we have not defined when actually $\mathrm{CL}(\Omega; X, A) = k$ for $k \geq 2$ as the stated estimate in the final part of Definition 2.4 is good enough for our purposes (see Theorem 2.10).

2.2 The *E*-Cohomological Conley Index and Critical Points

The first aim of this subsection is to introduce the E-cohomological Conley index and to define a module structure for it. Let E be a real separable Hilbert space and $L : E \to E$ an invertible selfadjoint operator for which there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of finite-dimensional subspaces of E such that $L(E_n) = E_n$, $E_n \subset E_{n+1}$ and $\overline{\bigcup_{n\in\mathbb{N}} E_n} = E$. Let $U \subset E$ be open. Following [8], we call a vector field $F : U \subset E \to E$, $F(u) = Lu + K(u)$ an $\mathcal{LS}$-vector field if $K : U \subset E \to E$ is a locally Lipschitz compact operator. Note that every $\mathcal{LS}$-vector field generates a local flow η^t satisfying

$$\frac{d}{dt}\eta^t = -F \circ \eta^t, \quad \eta^0 = \mathrm{id}, \tag{2.1}$$

which we call an $\mathcal{LS}$-flow.

Let us now assume that η is a global $\mathcal{LS}$-flow on U, and let us denote by

$$\mathrm{Inv}(\Omega, \eta) = \{x \in \Omega : \eta^t(x) \in \Omega,\ t \in \mathbb{R}\}$$

the maximal η-invariant subset of $\Omega \subset U$.

Definition 2.5. A closed and bounded set $\Omega \subset U$ is called an *isolating neighborhood* of η if $\mathrm{Inv}(\Omega, \eta) \subset \mathrm{int}(\Omega)$, where $\mathrm{int}(\Omega)$ denotes the interior of Ω.

Let now Ω be an isolating neighborhood of η and $S := \mathrm{Inv}(\Omega, \eta)$.

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Definition 2.6. We call a closed and bounded pair (X, A) of subsets of Ω an *index pair* for S if
- A is positively invariant with respect to X, i.e. given $x \in A$ and $t > 0$ with $\eta^{[0,t]}(x) \subset X$, then $\eta^{[0,t]}(x) \subset A$,
- $S = \operatorname{Inv}(\overline{X \setminus A}, \eta) \subset \operatorname{int}(\overline{X \setminus A})$,
- if $y \in X$, $t > 0$ and $\eta^t(y) \notin X$, then there exists $t' < t$ such that $\eta^{[0,t']}(y) \subset X$ and $\eta(t', y) \in A$.

It was shown in [11, Lemma 2.7] that every isolated invariant set S as above has an index pair.

Note that the space E splits as $E = E^+ \oplus E^-$, where $E^\pm$ are the spectral subspaces with respect to the positive and negative part of the spectrum of L. Henceforth, we denote by H^*_E the E-cohomology with respect to this splitting. The following crucial result was proved in [11, Proposition 2.8].

Proposition 2.7. *If (X, A) and (X', A') are index pairs for S, the groups $H^*_E(X, A)$ and $H^*_E(X', A')$ are isomorphic.*

Hence the next definition is sensible (cf. [11, Definition 2.9]).

Definition 2.8. The *E-cohomological Conley index* of S is defined by

$$\operatorname{ch}_E(S) = H^*_E(X, A),$$

where (X, A) is an index pair for S.

If we want to emphasize the isolating neighborhood Ω instead of the isolated invariant set S, we will also write $\operatorname{ch}_E(\Omega)$ to denote the E-cohomological Conley index.

When taking the module structure from Section 2 into account, it is readily seen by arguing as in [11, Proposition 2.8] that $H^*_E(X, A)$ and $H^*_E(X', A')$ are actually isomorphic as $H^*_0(\Omega)$-modules. Hence we obtain as a consequence of Proposition 2.7 the following important result.

Corollary 2.9. *The cup-length $\operatorname{CL}(\Omega; X, A)$ does not depend on the choice of the index pair (X, A) such that $X \subset \Omega$.*

Consequently, we can define

$$\operatorname{CL}(\Omega, S) := \operatorname{CL}(\Omega; X, A),$$

where (X, A) is any index pair for S such that $X \subset \Omega$. As S is uniquely determined by Ω and the flow η, we will sometimes denote this cup-length by $\operatorname{CL}(\Omega, \eta)$ if we want to emphasize η.

Let us now assume that η is the gradient flow with respect to a differentiable functional $f : U \to \mathbb{R}$, i.e. the map $F : U \subset E \to E$ is of the form $F = \nabla f$. As before, we assume that η is global. Let Ω be an isolating neighborhood of η and $S = \operatorname{Inv}(\Omega, \eta)$. We denote by $\operatorname{Crit}(f, \Omega)$ the set of critical values of $f|_\Omega$ and can now state the main theorem of this paper.

Theorem 2.10. *If f has only finitely many critical points in Ω, then the number of critical values of $f|_\Omega$ is bounded below by the cup-length of Ω with respect to S, i.e.*

$$\# \operatorname{Crit}(f, \Omega) \geq \operatorname{CL}(\Omega, S). \tag{2.2}$$

Note that by Theorem 2.10, the right-hand side in (2.2) is obviously also a lower bound for the number of critical points of f in Ω. We will prove Theorem 2.10 in Section 4.

3 The Arnold Conjecture on the Torus T^{2n}

Let T^{2n} denote the standard Torus of dimension $2n$ and let ω_0 be its standard symplectic structure. Let $H \in C^2(S^1 \times T^{2n}, \mathbb{R})$ be a 1-periodic Hamiltonian and X_H the induced vector field on T^{2n} given by

$$dH(\,\cdot\,) = \omega(X_H, \cdot\,).$$

We consider the Hamiltonian equation

$$\dot{x}(t) = X_H(x(t)), \tag{3.1}$$

and the aim of this section is to prove the following deep theorem.

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Theorem 3.1 (Arnold Conjecture on T^{2n}). *For every C^2-Hamiltonian on T^{2n} there exist at least $2n+1$ contractible solutions of* (3.1).

The above theorem was first proved by Conley and Zehnder in [3] (cf. also [10]). We now recall the analytical setting from their proof in the next section, and then use Theorem 2.10 for a short proof of Theorem 3.1.

3.1 The Analytical Setting

Before proving Theorem 3.1, let us first recall the analytical setting from [10] (see also [13]). In what follows, we let J be the symplectic standard matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

which is related to the symplectic form on T^{2n} by

$$\omega_0(x,y) = \langle x, Jy\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean scalar product.

We now start with the case of $\mathbb{R}^{2n}$ and consider the space of smooth loops $C^\infty(S^1,\mathbb{R}^{2n})$ in $\mathbb{R}^{2n}$. If we set $e_k(t) := e^{tk2\pi J}$, $k\in\mathbb{Z}$, then any $x\in C^\infty(S^1,\mathbb{R}^{2n})$ is represented by its Fourier series

$$x(t) = \sum_{k\in\mathbb{Z}} e_k(t)x_k, \tag{3.2}$$

where $x_k\in\mathbb{R}^{2n}$, $k\in\mathbb{Z}$. The Sobolev space $H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n})$ is the Hilbert space which is obtained as the completion of $C^\infty(S^1,\mathbb{R}^{2n})$ with respect to the scalar product

$$\langle x,y\rangle_{\frac{1}{2}} = \langle x_0,y_0\rangle + 2\pi\sum_{k\in\mathbb{Z}}|k|\langle x_k,y_k\rangle.$$

There is an orthogonal decomposition

$$H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n}) = Z_0\oplus Z^-\oplus Z^+$$

into a $2n$-dimensional subspace Z_0 and closed infinite-dimensional subspaces Z^+ and Z^- which correspond to $k=0$, $k>0$ and $k<0$ in the Fourier-series expansion (3.2), respectively. In what follows, we denote by P_0, P^+ and P^- the corresponding orthogonal projections.

Now let $H\in C^2(S^1\times\mathbb{R}^{2n},\mathbb{R})$ be a Hamiltonian such that $|H(x)|\le C\cdot|x|^2$ at infinity and such that the second spatial derivative H'' is globally bounded. We define a functional $\Phi_H : C^\infty(S^1,\mathbb{R}^{2n})\to\mathbb{R}$ by the formula

$$\Phi_H(x) = a(x) - b(x) := \frac{1}{2}\int_0^1 \langle -J\dot{x}(t), x(t)\rangle\,dt - \int_0^1 H(t,x(t))\,dt. \tag{3.3}$$

The importance of Φ_H comes from the fact that the critical points of Φ_H are periodic solutions of the Hamilton equation (3.1). It is easy to see that Φ_H extends to $H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n})$, and

$$\nabla\Phi_H = L + K, \tag{3.4}$$

where $L=\nabla a = P^+ - P^-$ is a selfadjoint Fredholm operator and $K=-\nabla b = -j^*\nabla H$ is a compact map because of the compactness of the adjoint $j^* : L^2\to H^{\frac{1}{2}}$ of the inclusion.

On a general manifold, it is a delicate problem to define spaces $H^{\frac{1}{2}}(S^1,M)$ as $H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n})$ contains non-continuous functions which consequently have no local meaning. However, for a torus one can overcome this problem by using the universal covering $\mathbb{R}^{2n}\to T^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Then smooth Hamiltonians on T^{2n} are in one-to-one correspondence with $\mathbb{Z}^{2n}$-invariant smooth Hamiltonians on $\mathbb{R}^{2n}$, where $\mathbb{Z}^{2n}$ acts on $\mathbb{R}^{2n}$ by translations. By a slight abuse of notation, we will denote by H both the Hamiltonian on the torus and the

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Hamiltonian lifted to $\mathbb{R}^{2n}$. Note that the lifted Hamiltonian on $\mathbb{R}^{2n}$ is $\mathbb{Z}^{2n}$-invariant and therefore its second spatial derivative is bounded and it obviously satisfies the growth condition mentioned above. Now the corresponding functional Φ_H in (3.3) is $\mathbb{Z}^{2n}$-invariant as well, and therefore it descends to a functional on the quotient space

$$\mathcal{M} := Z_0/\mathbb{Z}^{2n} \times Z^+ \times Z^- = T^{2n} \times Z^+ \times Z^-.$$

3.2 Proof of Theorem 3.1

We suppose as in the previous subsection that $H \in C^2(S^1 \times T^{2n}, \mathbb{R})$ is a given Hamiltonian. Let us note at first that $F = \nabla\Phi_H$ in (3.4) is an $\mathcal{LS}$-vector field, even though the operator L is not invertible. Indeed, if we write $F = \hat{L} + \hat{K} := (L + P_0) + (K - P_0)$, where P_0 is the orthogonal projection onto the finite-dimensional kernel of L as introduced above, then F is the sum of an invertible selfadjoint operator and a compact map.

As $\mathcal{M}$ is a Hilbert manifold, we cannot directly apply the E-cohomological Conley index which we only have defined for flows on open subsets of a Hilbert space. However, if we use a tubular neighborhood, the definition can easily be extended to Hilbert manifolds of the type $M \times E$, where M is a closed manifold and E is a Hilbert space. In the case of $\mathcal{M}$, the construction is as follows. We embed $\mathcal{M}$ into $\hat{E} = \mathbb{R}^{4n} \times Z^+ \times Z^-$ in such a way that every S^1 in $T^{2n} = S^1 \times \cdots \times S^1$ is mapped to the unit circle in $\mathbb{R}^2$. We consider the open set

$$U := D_0^{2n} \times Z^+ \times Z^- \subset \hat{E}$$

of $\hat{E}$, where $D_0 = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}$ is a punctured disc of radius 2 in $\mathbb{R}^2$, and we let $\pi : \mathcal{N} \to \mathcal{M}$ be the standard projection to T^{2n} on D_0^{2n} and the identity on Z^+ and Z^-. The map Φ_H can be extended to U by

$$\Psi_H(x) = \Phi_H(\pi(x)) + \sum_{i=1}^{2n}(1 - r_i(x))^2,$$

where $r_i(x)$ denotes the polar coordinate in $\mathbb{R}^2$ of the projection of $x \in U$ to the i-th component of $(\mathbb{R}^2)^{2n}$. Note that the extension is done in such a way that Ψ_H and Φ_H have the same critical points. We denote by $\tilde{K}$ the compact operator which is the sum of $\hat{K}$ and $\nabla(\sum_{i=1}^{2n}(1 - r_i(x))^2)$.

Note that $\nabla\Psi_H = \hat{L} + \tilde{K}$ is an $\mathcal{LS}$-vector field, and the negative and positive spectral subspaces of the selfadjoint isomorphism $\hat{L}$ are given by

$$E^+ = \mathbb{R}^{4n} \oplus Z^+, \quad E^- = Z^-.$$

Now Theorem 3.1 can be obtained as follows. Since K is bounded, there is $R > 0$ such that $R > \|K(x)\|$ for all $x \in U$. We set

$$X = C^{2n} \times B(Z^+, R) \times B(Z^-, R),$$

where $B(Z^\pm, R)$ are the closed balls of radius R in $Z^\pm$ and $C \subset D_0$ is a closed annulus containing S^1. Note that the boundary ∂X is given by the non-disjoint union

$$\partial X = (\partial C^{2n} \times B(Z^+, R) \times B(Z^-, R)) \cup (C^{2n} \times \partial B(Z^+, R) \times B(Z^-, R)) \cup (C^{2n} \times B(Z^+, R) \times \partial B(Z^-, R)).$$

Let now X_1, X_2, X_3 denote the three parts of ∂X in the above order and let η be the flow induced by $-\nabla\Psi_H$ as in (2.1). Firstly, if $x \in X_1$ but neither in X_2 nor in X_3, then there is some $t > 0$ such that $\eta^{(0,t]}(x) \subset X \setminus \partial X$. Secondly, if $x \in X_2$, then $\|P^+x\| = R$ and we have

$$\left\langle \hat{L}x + \tilde{K}(x), \frac{P^+x}{R} \right\rangle = \left\langle LP^+x, \frac{P^+x}{R} \right\rangle + \left\langle K(x), \frac{P^+x}{R} \right\rangle > R - \|K(x)\| > 0.$$

Consequently, the vector field $\hat{L}x + \tilde{K}(x)$ is pointing outwards the sphere $\partial B(Z^+, R)$. Hence, if $x \in X_1 \cup X_2$ but $x \notin X_3$, then η, which is the flow induced by $-\nabla\Psi_H$, moves x into the interior of X. Finally, if $x \in X_3$, we analogously see that

$$\left\langle \hat{L}x + \tilde{K}(x), \frac{P^-x}{R} \right\rangle < 0$$

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as $\|P^- x\| = R$. Hence those x leave X under η. It is now readily seen that

$$(X, A) = (C^{2n} \times B(Z^+, R) \times B(Z^-, R), C^{2n} \times B(Z^+, R) \times \partial B(Z^-, R))$$

is an index pair for $S = \operatorname{Inv}(X, \eta)$ in the sense of Definition 2.6, where we have set $A := X_3$ for simplicity of notation.

To find the E-cohomology of (X, A), let $V \subset Z^-$ be of finite dimension. Then

$$(X_V, A_V) = (C^{2n} \times B(Z^+, R) \times B(V, R), C^{2n} \times B(Z^+, R) \times \partial B(V, R)),$$

where $B(V, R)$ denotes the ball of radius R in V. Hence we get for $k \in \mathbb{Z}$

$$H^k(X_V, A_V) = H^k(X_V/A_V) = H^k(S(V, R) \wedge T^{2n}) = H^{k-\dim(V)}(T^{2n}),$$

where $S(V, R)$ denotes the sphere of radius R in V. Moreover, if $W \supset V$ is another finite-dimensional subspace, then the Mayer–Vietoris homomorphism $\Delta^k_{V,W}$ mapping

$$H^{k+\dim(V)}(X_V, A_V) = H^{k+\dim(V)}(S(V) \wedge T^{2n})$$

to

$$H^{k+\dim(W)}(X_W, A_W) = H^{k+\dim(W)}(S(W) \wedge T^{2n})$$

is by definition just the suspension isomorphism. Hence we obtain

$$H^*_E(X, A) = H^*(T^{2n}).$$

Finally, to find the cup-length, we note at first that for the isolating neighborhood X, and any finite-dimensional subspace $V \subset Z^-$,

$$H^*(X_V) = H^*(C^{2n} \times B(Z^+; R) \times B(V; R)) = H^*(T^{2n}).$$

Hence $\operatorname{CL}(X, \eta)$ is just the ordinary cup-length of the torus T^{2n}, which is $2n + 1$. By Theorem 2.10, this is a lower bound for the number of critical points of Φ_H in X, and so we have proved the Arnold conjecture on T^{2n}.

4 Proof of Theorem 2.10

We will need the following two properties of the cup-length CL that we introduced in Definition 2.4. As the proofs are purely algebraic, we leave it to the reader to check that they follow by obvious modifications from [5, Lemmas 2.2 and 2.3].

Lemma 4.1. *If $B \subset A \subset X \subset Y$ are closed and bounded subsets of E, then*

$$\operatorname{CL}(Y; X, B) \le \operatorname{CL}(Y; X, A) + \operatorname{CL}(Y; A, B).$$

Lemma 4.2. *If $A \subset X \subset Y_1 \subset Y_2$ are closed and bounded subsets of E, then*

$$\operatorname{CL}(Y_2; X, A) \le \operatorname{CL}(Y_1; X, A).$$

Now let us consider an isolating neighborhood Ω for the flow η generated by the gradient of the function $f : U \to \mathbb{R}$ in Theorem 2.10. As we suppose that there are only finitely many critical points of f in Ω, the set of critical values $\operatorname{Crit}(f, \Omega)$ is finite as well, say, $c_1 < \cdots < c_k$. Let $M_i \subset \Omega$ denote the set of stationary points with values c_i, and set for $1 \le i \le j \le k$,

$$M_{ij} = \{x \in \Omega : \omega(x) \cup \alpha(x) \subset M_i \cup M_{i+1} \cup \cdots \cup M_j\},$$

where $\alpha(x)$ and $\omega(x)$ denote as above the α and ω limits of $x \in E$ under the flow η. Note that M_{1k} consists of all the critical points of f inside Ω and all the orbits connecting them. Consequently,

$$M_{1k} = \operatorname{Inv}(\Omega, \eta).$$

Now let (X, A) be an index pair for M_{1k}.

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Lemma 4.3 (Morse Filtration). *There exist sets*

$$X_0 = A \subset X_1 \subset \cdots \subset X_k = X$$

such that (X_j, X_{i-1}) *is an index pair for* M_{ij}.

Proof. We let $b_i \in (c_i, c_{i+1})$, $i = 1, \dots, k-1$, be regular values of f, set $b_k = \infty$, and define $X_0 = A$ as well as $X_i := X \cap f^{-1}(-\infty, b_i]$, $i = 1, \dots, k$. Then it is readily seen that (X_j, X_{i-1}) is an index pair for M_{ij} as M_{ij} consists of all critical points x such that $f(x) \in \{c_i, \dots, c_j\}$ and all the orbits connecting them. □

If we now apply Lemma 4.1 k times, we get

$$\mathrm{CL}(\Omega; X, A) \le \sum_{i=1}^{k} \mathrm{CL}(\Omega; X_i, X_{i-1}). \tag{4.1}$$

On the other hand, (X_i, X_{i-1}) is an index pair for M_{ii}, which is a set consisting of a finite number of stationary points. Therefore we can choose an isolating neighborhood Ω_i for M_{ii}, where Ω_i is a disjoint union of discs. If now (X'_i, X'_{i-1}) is an index pair for M_{ii} such that $X'_i \subset \Omega_i$, then by Corollary 2.9 and Lemma 4.2

$$\mathrm{CL}(\Omega; X_i, X_{i-1}) = \mathrm{CL}(\Omega; X'_i, X'_{i-1}) \le \mathrm{CL}(\Omega_i; X'_i, X'_{i-1}) \le 1,$$

where the last inequality follows from the fact that the groups $H_0^{q>0}(\Omega_i)$ are trivial. Hence, by (4.1),

$$\mathrm{CL}(\Omega; X, A) \le k$$

and Theorem 2.10 is shown, as k is the number of critical values of f in Ω.

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