The hat problem on cycles on at least nine vertices

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Abstract

The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The solution of the hat problem on a graph is known for trees and for the cycle C_4. We solve the problem on cycles on at least nine vertices.

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1 Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [24], Die Zeit [6], and abcNews [23]. It is also a one of subjects of the webpage [4].

The hat problem with $2^k - 1$ players was solved in [14], and for 2^k players in [11]. The problem with n players was investigated in [7]. The hat problem and Hamming codes were the subject of [8]. The generalized hat problem with n people and q colors was investigated in [22].

There are many known variations of the hat problem (for a comprehensive list, see [21]). For example in the papers [1, 10, 18] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [16] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [17] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science (for a survey on this topic, see [21]), for example: information technology [5], linear programming [16], genetic programming [9], economics [1, 18], biology [17], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15]. Therefore, it is hoped that the hat problem on a graph is worth exploring as a natural generalization, and may also have many applications.

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [19]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [20] the problem was solved on the cycle C_4. It has been proven that for both trees and the cycle C_4 the maximum chance of success is one by two. Thus in such graph an optimal strategy is for example such in which one vertex always guesses it is blue, while the remaining vertices always pass. It means that the structure of such graph does not improve the maximum chance of success in the hat problem on a graph comparing to the one-vertex graph.

We solve the hat problem on cycles on at least nine vertices.

2 Preliminaries

For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. Let $v \in V(G)$. The degree of vertex v, that is, the number of its neighbors, we denote by $d_G(v)$. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively).

Let $f\colon X \to Y$ be a function, and let $y \in Y$. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to blue, and 2 corresponds to red.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$; of course $|C(G)| = 2^{|V(G)|}$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$; of course $|St_i(G)| = 2^{d_G(v_i)}$.

We say that a case c for the graph G corresponds to a situation s_i of vertex v_i if $c(v_j) = s_i(v_j)$, for every v_j adjacent to v_i. This implies that a case corresponds to a situation of v_i if every vertex adjacent to v_i in that case has the same color as in that situation. Of course, to every situation of the vertex v_i correspond exactly $2^{|V(G)| - d_G(v_i)}$ cases.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, which for a given situation gives the color v_i guesses it is, or 0 if v_i passes. Thus, a guessing instruction is a rule determining the behavior of a vertex in every situation. We say that v_i never guesses its color if v_i passes in every situation, that is, $g_i \equiv 0$.

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). Let $S \in \mathcal{F}(G)$ and let $v_i \in V(G)$. By $L(S, v_i)$ we denote the set of cases for the graph G such that in the strategy S the vertex v_i guesses its color wrong. By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \dots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S) \colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

Let $t \in \{1, 2, \ldots, n\}$, and let $m_1, m_2, \ldots, m_t \in \{1, 2, \ldots, n\}$ be such that $m_j \neq m_k$ for every $j \neq k$. Let $c_{m_1}, c_{m_2}, \ldots, c_{m_t} \in \{1, 2\}$. The set of cases c for the graph G such that $c(v_{m_j}) = c_{m_j}$ we denote by $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

Now we give an example of notation for the hat problem on the graph K_3. Of course, there are $2^3 = 8$ possible cases. The vertices we denote by v_1, v_2, and v_3. Assume for example that in a case c the vertices v_1 and v_3 have the first color, and the vertex v_2 has the second color. Thus $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$. Now let us consider situations of some vertex, say v_1. The vertex v_1 can see that v_2 has the second color and v_3 has the first color. Of course, the vertex v_1 cannot see its own color. Thus $s_1(v_1) = 0$, $s_1(v_2) = 2$, and $s_1(v_3) = 1$. We say that a case corresponds to that situation if each one of the neighbors of v_1 has the same color as in that situation. It is easy to see that the case in which v_1 and v_2 have the second color and v_3 has the first color corresponds to that situation. These are the only two cases corresponding to that situation as $2^{|V(K_3)| - d_{K_3}(v_1)} = 2^{3-2} = 2$. Now let us consider a guessing instruction of some vertex, say v_2. Assume for example that the vertex v_2 guesses it has the first color when v_1 and v_3 have the second color; it guesses it has the second color when v_1 and v_3 have the first color; otherwise it passes. We have $g_2(202) = 1$, $g_2(101) = 2$, and $g_2(102) = g_2(201) = 0$. If a case c is such that $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$, then the guess of v_2 is correct as $g_2(101) = 2 = c(v_2)$.

The following theorems are from [19]. The first of them is a lower bound on the chance of success of an optimal strategy.

Theorem 1 *Let G be a graph. If S is an optimal strategy for G, then $p(S) \geq 1/2$.*

Now we give a sufficient condition for deleting a vertex of a graph without changing its hat number.

Theorem 2 *Let G be a graph and let v be a vertex of G. If there exists a strategy $S \in \mathcal{F}^0(G)$ such that v never guesses its color, then $h(G) = h(G - v)$.*

The following theorem is the solution of the hat problem on paths.

Theorem 3 *For every path P_n we have $h(P_n) = 1/2$.*

3 Results

In the next few pages we solve the hat problem on cycles on at least nine vertices.

We assume that $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Let S be a strategy for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. Let $\alpha_i(S), \beta_i(S), \gamma_i(S) \in \{1, 2\}$ (we write $\alpha_i, \beta_i, \gamma_i$) be such that the guess of v_i is wrong when $c(v_{i-1}) = \alpha_i$, $c(v_i) = \beta_i$, and $c(v_{i+1}) = \gamma_i$ ($i \in \{2, 3, \ldots, n-1\}$), the guess of v_1 is wrong when $c(v_n) = \alpha_1$, $c(v_1) = \beta_1$, and $c(v_2) = \gamma_1$, and the guess of v_n is wrong when $c(v_{n-1}) = \alpha_n$, $c(v_n) = \beta_n$, and $c(v_1) = \gamma_n$. For example, if the vertex v_2 guesses it has the second color when v_1 has the first color and v_3 has the second color, then it follows that the vertex v_2 guesses its color wrong when $c(v_1) = c(v_2) = 1$ and $c(v_3) = 2$. Therefore $\alpha(v_2) = \beta(v_2) = 1$ and $\gamma(v_2) = 2$.

Let us consider strategies such that every vertex guesses its color (rather than passing) in exactly one situation. In the following lemma we give such strategy for which the number of cases in which some vertex guesses its color wrong is as small as possible.

Lemma 4 *Let us consider the family of all strategies for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. The number of cases in which some vertex guesses its color wrong is minimal for a strategy S such that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$, and $\gamma_n = \beta_1 = \alpha_2$.*

Proof. First, we prove that we may assume that $\alpha_n = \gamma_{n-2}$. Consider the possibility $\alpha_n \neq \gamma_{n-2}$. Thus $\beta_{n-1} = \alpha_n$ or $\beta_{n-1} = \gamma_{n-2}$, otherwise $\alpha_n = \gamma_{n-2}$, a contradiction. Without loss of generality we assume that $\beta_{n-1} = \gamma_{n-2}$. Since $\alpha_n \neq \gamma_{n-2}$, we have $\gamma_{n-2} = \beta_{n-1} \neq \alpha_n$. Let a strategy S' differ from S only in that $\alpha_n(S') \neq \alpha_n(S) = \alpha_n$. Thus $\alpha_n(S') = \beta_{n-1} = \gamma_{n-2}$. Let B (B', respectively) denote the set of cases in which in the strategy S (S', respectively) the vertex v_n guesses its color wrong, and at the same time another vertex also guesses its color wrong. Thus $B = L(S, v_n) \cap \bigcup_{i=1}^{n-1} L(S, v_i)$ and $B' = L(S', v_n) \cap \bigcup_{i=1}^{n-1} L(S', v_i)$. We want to minimize the number of cases in which some vertex guesses its color wrong. Therefore we want the number of cases in which v_n guesses its color wrong, and at the same time another vertex also guesses its color wrong to be as great as possible. Since the strategies S and S' differ only in the behavior of the vertex v_n, and for each set A_i ($i \in \{1, 2, \ldots, n-3\}$) the color of the vertex v_{n-1} is not determined, we have $|L(S, v_n)| \cap \bigcup_{i=1}^{n-3} L(S, v_i)| = |L(S', v_n)| \cap \bigcup_{i=1}^{n-3} L(S', v_i)|$. We also get $|L(S, v_n) \cap L(S, v_{n-2})| = |C(C_n, v_{n-1}^{\alpha_n}, v_n^{\beta_n}, v_1^{\gamma_n}) \cap C(C_n, v_{n-3}^{\alpha_{n-2}}, v_{n-2}^{\beta_{n-2}}, v_{n-1}^{\gamma_{n-2}})| = 0$ as

$\alpha_n \neq \gamma_{n-2}$. Since $\alpha_n \neq \beta_{n-1}$, we have $|L(S, v_n) \cap L(S, v_{n-1})| = |C(C_n, v_{n-1}^{\alpha_n}, v_n^{\beta_n}, v_1^{\gamma_n}) \cap C(C_n, v_{n-2}^{\alpha_{n-1}}, v_{n-1}^{\beta_{n-1}}, v_n^{\gamma_{n-1}})| = 0$. This implies that $|B'| \geq |B|$, and therefore we may assume that $\alpha_n = \gamma_{n-2}$. Let us make this assumption.

Now we prove that we may assume that $\beta_{n-1} = \gamma_{n-2}$. Consider the possibility $\beta_{n-1} \neq \gamma_{n-2}$. Let a strategy S'' differ from S only in that $\beta_{n-1}(S'') \neq \beta_{n-1}(S) = \beta_{n-1}$, thus $\beta_{n-1}(S'') = \gamma_{n-2} = \alpha_n$. Let us define sets D and D'' analogically as the sets B and B'. Similarly we get $|D''| \geq |D|$. Therefore we may assume that $\beta_{n-1} = \gamma_{n-2}$.

Because of the possibility of cyclic renumbering of vertices of the cycle, we may assume that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$, and $\gamma_n = \beta_1 = \alpha_2$. ■

If $n \geq 3$ is an integer, then let

$$A_n = \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1, \text{ for an } i \in \{2, 3, \ldots, n-1\}\},$$

that is, A_n is the set of cases for C_n such that there are three vertices of the first color the indices of which are consecutive integers. Let the sequence $\{a_n\}_{n=1}^{\infty}$ be such that $a_n = |A_n|$ ($n \geq 3$), and also $a_1 = a_2 = 0$.

In the following lemma we give a recursive formula for a_n (with $n \geq 4$).

Lemma 5 *For every integer $n \geq 4$ we have $a_n = 2^{n-3} + a_{n-3} + a_{n-2} + a_{n-1}$.*

Proof. To find the number a_n, we have to count the cases for C_n such that $c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1$, for some $i \in \{2, 3, \ldots, n-1\}$. Let c be any case for C_n. We consider the following four possibilities: (1) $\min\{i\colon c(v_i) = 2\} = 1$; (2) $\min\{i\colon c(v_i) = 2\} = 2$; (3) $\min\{i\colon c(v_i) = 2\} = 3$; (4) $c(v_1) = c(v_2) = c(v_3) = 1$.

(1) There are a_{n-1} such cases, because there are $n-1$ vertices which can form a triple of vertices of the first color the indices of which are consecutive integers.

(2) There are a_{n-2} such cases, because there are $n-2$ vertices which can form a triple of vertices of the first color the indices of which are consecutive integers, as v_2 has the second color, and v_1 cannot belong to any triple of vertices of the first color the indices of which are consecutive integers because of the interruption of v_2.

(3) There are a_{n-3} such cases, due to reasons similar to those in (2).

(4) There are 2^{n-3} such cases, because v_1, v_2, and v_3 form a triple of vertices of the first color the indices of which are consecutive integers, and there are 2^{n-3} possibilities of coloring the remaining $n-3$ vertices.

From (1)–(4) it follows that $a_n = 2^{n-3} + a_{n-3} + a_{n-2} + a_{n-1}$. ■

If n is an integer such that $n \geq 3$, then let

$$B_n = \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1 \text{ (for an } i \in \{2,3,\ldots,n-1\}) \\ \text{or } c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1\},$$

that is, B_n is the set of cases for C_n such that there are three consecutive vertices of the first color. Let the sequence $\{b_n\}_{n=3}^{\infty}$ be such that $b_n = |B_n|$.

Now we give a relation between the number b_n (with $n \geq 6$), and the elements of the sequence $\{a_n\}_{n=1}^{\infty}$.

Lemma 6 *If $n \geq 6$ is an integer, then $b_n = 5 \cdot 2^{n-6} + a_n - 2a_{n-5} - a_{n-6}$.*

Proof. Let us consider the partition of the set B_n (the set of cases for C_n such that there are three consecutive vertices of the first color) into the following two sets. In the first set there are the cases for C_n such that there are three vertices of the first color the indices of which are consecutive integers. In the second set there are the cases for C_n such that there are three consecutive vertices of the first color, but there are not any three vertices of the first color the indices of which are consecutive integers. Thus

$$\begin{aligned} B_n &= \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1 \text{ (for an } i \in \{2,3,\ldots,n-1\}) \\ &\qquad \text{or } c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1\} \\ &= \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1, \text{ for an } i \in \{2,3,\ldots,n-1\}\} \\ &\quad \cup \{c \in C(C_n)\colon c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1, \\ &\qquad \text{and at the same time there is no } i \in \{2,3,\ldots,n-1\} \text{ such that} \\ &\qquad c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1\} \\ &= A_n \cup (B_n \setminus A_n). \end{aligned}$$

We have

$$b_n = |B_n| = |A_n \cup (B_n \setminus A_n)| = |A_n| + |B_n \setminus A_n| = a_n + |B_n \setminus A_n|.$$

Now let us find a formula for $|B_n \setminus A_n|$. Let c be any case for C_n belonging to the set $B_n \setminus A_n$. We consider the following three possibilities: (1) $c(v_{n-1}) = c(v_n) = c(v_1) = c(v_2) = 1$ (so also $c(v_{n-2}) = c(v_3) = 2$, as this case does not belong to the set A_n); (2) $c(v_{n-1}) = 2$ and $c(v_n) = c(v_1) = c(v_2) = 1$ (so also $c(v_3) = 2$, as this case does not belong to the set A_n); (3) $c(v_{n-1}) = c(v_n) = c(v_1) = 1$ and $c(v_2) = 2$ (so also $c(v_{n-2}) = 2$, as this case does not belong to the set A_n), see Figure 1.

(1) There are $2^{n-6} - a_{n-6}$ such cases, because there are 2^{n-6} possibilities of coloring the remaining $n-6$ vertices, and we do not count the a_{n-6} cases such that there are three vertices of the first color the indices of which are consecutive integers.

(2) There are $2^{n-5} - a_{n-5}$ such cases, due to reasons analogical to that in (1).

(3) There are $2^{n-5} - a_{n-5}$ such cases, also due to reasons analogical to that in (1).

It follows from (1), (2), and (3) that

$$|B_n \setminus A_n| = 2^{n-6} - a_{n-6} + 2(2^{n-5} - a_{n-5}) = 5 \cdot 2^{n-6} - 2a_{n-5} - a_{n-6}.$$

Since $b_n = a_n + |B_n \setminus A_n|$, we get $b_n = 5 \cdot 2^{n-6} + a_n - 2a_{n-5} - a_{n-6}$. ■

- ◉ a vertex of the first color
- ⋇ a vertex of the second color
- • a vertex of unknown color

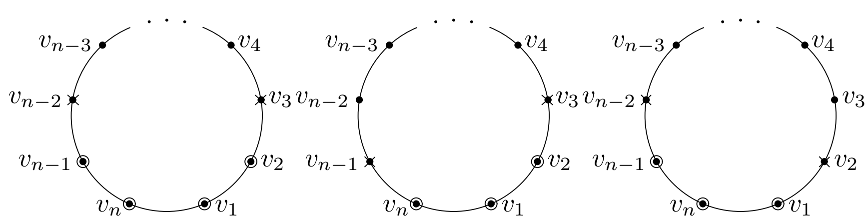

Figure 1: Illustrations to the proof of Lemma 6: possibilities (1), (2), and (3), respectively

Now we give a lower bound on the number b_n (with $n \geq 9$).

Lemma 7 *For every integer $n \geq 9$ we have $b_n > 2^{n-1}$.*

Proof. First, we find the eleven initial elements of the sequence $\{a_n\}_{n=1}^{\infty}$. We calculate them recursively. If we try to solve the recurrence which determines the elements of the sequence $\{a_n\}_{n=1}^{\infty}$, then in the generating function we get the expression x^3+x^2+x+1 corresponding to the so-called tribonacci sequence for which the iterative formula is not known. Solving the recurrence of the sequence $\{a_n\}_{n=1}^{\infty}$ using tribonacci numbers, we can only get a formula which is also recursive.

Using Lemma 5, the definition of the sequence $\{a_n\}_{n=1}^{\infty}$, and the fact that $a_3 = 1$ (as the case in which every vertex has the first color is the only one such case), we get

$$\begin{array}{lclllllllllllllll}
a_1 & = & 0, \\
a_2 & = & 0, \\
a_3 & = & 1, \\
a_4 & = & 2 & + & a_1 & + & a_2 & + & a_3 & = & 2 & + & 0 & + & 0 & + & 1 & = & 3, \\
a_5 & = & 2^2 & + & a_2 & + & a_3 & + & a_4 & = & 4 & + & 0 & + & 1 & + & 3 & = & 8, \\
a_6 & = & 2^3 & + & a_3 & + & a_4 & + & a_5 & = & 8 & + & 1 & + & 3 & + & 8 & = & 20, \\
a_7 & = & 2^4 & + & a_4 & + & a_5 & + & a_6 & = & 16 & + & 3 & + & 8 & + & 20 & = & 47,
\end{array}$$

$$
\begin{array}{lllllllllllllll}
a_8 & = & 2^5 & + & a_5 & + & a_6 & + & a_7 & = & 32 & + 8 & + 20 & + 47 & = 107, \\
a_9 & = & 2^6 & + & a_6 & + & a_7 & + & a_8 & = & 64 & + 20 & + 47 & + 107 & = 238, \\
a_{10} & = & 2^7 & + & a_7 & + & a_8 & + & a_9 & = & 128 & + 47 & + 107 & + 238 & = 520, \\
a_{11} & = & 2^8 & + & a_8 & + & a_9 & + & a_{10} & = & 256 & + 107 & + 238 & + 520 & = 1121.
\end{array}
$$

By Lemma 6 we get

$$
\begin{array}{lllllll}
b_9 & = & 5 \cdot 2^3 & + \; a_9 & - \; 2a_4 & - \; a_3 \\
& = & 40 & + \; 238 & - \; 2 \cdot 3 & - \; 1 \\
& = & 271 \\
& > & 256 & = 2^8.
\end{array}
$$

Now assume that $n \geq 10$. Since $a_n = |A_n|$, $b_n = |B_n|$, and $A_n \subseteq B_n$ (see the definition of the set B_n), we get $a_n \leq b_n$. This implies that it suffices to prove that $a_n > 2^{n-1}$. We prove this by induction. We have $a_{10} = 520 > 512 = 2^9$ and $a_{11} = 1121 > 1024 = 2^{10}$. Assume that $n \geq 10$ is an integer, and we have $a_n > 2^{n-1}$ and $a_{n+1} > 2^n$. We prove that $a_{n+2} > 2^{n+1}$. By Lemma 5 and the inductive hypothesis we get

$$
\begin{array}{lllll}
a_{n+2} & = & 2^{n-1} & + \; a_{n-1} & + \; a_n & + \; a_{n+1} \\
& > & 2^{n-1} & + \; 0 & + \; 2^{n-1} & + \; 2^n \\
& = & 2^{n+1}.
\end{array}
$$

■

Now we solve the hat problem on cycles on at least nine vertices.

Theorem 8 *For every integer $n \geq 9$ we have $h(C_n) = 1/2$.*

Proof. Let S be an optimal strategy for C_n. If some vertex, say v_i, never guesses its color, then by Theorem 2 we have $h(C_n) = h(C_n - v_i)$. Since $C_n - v_i = P_{n-1}$ and $h(P_{n-1}) = 1/2$ (by Theorem 3), we get $h(C_n) = 1/2$. Now assume that every vertex guesses its color (rather than passing) in at least one situation. We are interested in the possibility when the number of cases for which the team loses is as small as possible. We assume that every vertex guesses its color (rather than passing) in exactly one situation, and we prove that these guesses suffice to cause the loss of the team in more than half of all cases. Let us consider the strategy $S' \in \mathcal{F}(C_n)$ such that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$, and $\gamma_n = \beta_1 = \alpha_2$. Without loss of generality we assume that $\gamma_{i-1} = \beta_i = \alpha_{i+1} = 1$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1 = 1$, and $\gamma_n = \beta_1 = \alpha_2 = 1$. Some vertex guesses its color wrong in the cases such that there are three consecutive vertices of the first color. Using the definition of the sequence $\{b_n\}_{n=3}^{\infty}$, there are b_n such cases. From Lemma 4 we know that the number of cases in which some vertex guesses its color wrong in the strategy S' is minimal among all strategies for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. This implies that in the strategy S in at least b_n cases some vertex guesses its color wrong.

Therefore the team loses for at least b_n cases, that is, $|L(S)| \geq b_n$. Since $b_n > 2^{n-1}$ (by Lemma 7), we have $|L(S)| > 2^{n-1}$. Now we get

$$p(S) = \frac{|W(S)|}{|C(C_n)|} = \frac{|C(C_n)| - |L(S)|}{|C(C_n)|} < \frac{2^n - 2^{n-1}}{2^n} = \frac{1}{2},$$

a contradiction to Corollary 1. ■

Of course, $h(C_3) = 3/4$. A natural issue is to determine the hat numbers of cycles of length between four and eight. This will make the hat problem on cycles solved. One can also investigate the problem on another classes of graphs. This may be helpful for solving generally the hat problem on an arbitrary graph.

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