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The Hopf type theorem for equivariant gradient local maps

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Abstract. We construct a degree-type otopy invariant for equivariant gradient local maps in the case of a real finite-dimensional orthogonal representation of a compact Lie group. We prove that the invariant establishes a bijection between the set of equivariant gradient otopy classes and the direct sum of countably many copies of $\mathbb{Z}$.

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Introduction

In the 1920s H. Hopf discovered the homotopy classification of maps from n-dimensional closed oriented manifold into the n-sphere. Namely, he proved that the topological degree of the map determines its homotopy class. More precisely, there is a bijection from the set of homotopy classes of such maps to the integers. A broad and comprehensive description of the degree theory and its applications can be found in [21].

In 1985, to obtain new bifurcation results, Dancer [14] introduced a new degree-type invariant for S^1-equivariant gradient maps, which provides more information than the usual equivariant degree. Interestingly, 5 years later Parusiński [26] showed that in the absence of group action such an extra homotopy invariant for gradient maps does not exist. The idea of the more subtle invariant in the equivariant gradient case was developed later by Rybicki and his collaborators. Many applications of this construction can be found in [20, 25, 27].

In this paper, we present the Hopf type classification of the set of otopy classes of equivariant gradient local maps $\mathcal{F}_G^\nabla[V]$ in the case of a real finite-dimensional orthogonal representation V of a compact Lie group G. More precisely, first we show a decomposition of $\mathcal{F}_G^\nabla[V]$ into factors indexed by orbit types appearing in V and then we give a description of each factor as the direct sum of a countable number of $\mathbb{Z}$. It should be emphasized that

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our result explains the phenomenon discovered by Dancer. Namely, in the decomposition of $\mathcal{F}_G^\nabla[V]$ additional factors appear, which do not occur in the decomposition of the set of otopy classes of equivariant local maps $\mathcal{F}_G[V]$. Thus we obtain essentially stronger otopy invariant in the gradient case. Recall that otopy is a generalization of the concept of homotopy introduced by Becker and Gottlieb [12,13] and independently by Dancer et al. [15].

Our study is the natural continuation of the previous work. Earlier we investigated the classification of otopy classes in cases of gradient local maps in $\mathbb{R}^n$ [8,9], gradient local fields on manifolds [11] and equivariant local maps on a representation of a compact Lie group [5,6].

It is worth pointing out that the ideas presented here were inspired by [1–4,19,22–24]. In particular, our paper develops and clarifies the material contained in [7,18]. Moreover, we emphasize that the techniques used here are not new. Namely, the construction of our classifying invariant is analogous to that presented in [15]. Two key ideas appearing in the proof of our Main Theorem, i.e. a perturbation of an equivariant gradient map to (H)-normal one and a splitting of the set of equivariant otopy classes with respect to orbit types come from [15,17]. However, [15] contains homotopy classification, whereas our result gives otopy classification. Furthermore, as opposed to [15] our approach does not require to appeal to the Parusiński method (see [26]), which allowed us to simplify the presentation of the topic.

The arrangement of the paper is as follows. Section 1 contains some preliminaries. In Sects. 2 and 3 we present constructions of the functions Φ and Ψ, which are essential to define the invariant Θ. Section 4 provides the description of the formula for Θ. In Sects. 5 and 6, we introduce the notions of (H)-normal and orbit-normal maps. Our two main results are stated in Sect. 7, where also the proof of the second is presented. In turn, the first result is proved in Sect. 8. In Sect. 9, we show the Parusiński type theorem, which establishes the relation between the sets of equivariant and equivariant gradient otopy classes. Finally, Sect. 10 contains some remarks concerning the parametrized case.

1. Preliminaries

The notation $A \Subset B$ means that A is a compact subset of B. For a topological space X, we denote by $\tau(X)$ the topology on X. For any topological spaces X and Y, let $\mathcal{M}(X,Y)$ be the set of all continuous maps $f\colon D_f \to Y$ such that D_f is an open subset of X. Let $\mathcal{R}$ be a family of subsets of Y. We define

$$\operatorname{Loc}(X,Y,\mathcal{R}) := \{\, f \in \mathcal{M}(X,Y) \mid f^{-1}(R) \Subset D_f \quad \text{for all } R \in \mathcal{R}\,\}.$$

We introduce a topology in $\operatorname{Loc}(X,Y,\mathcal{R})$ generated by the subbasis consisting of all sets of the form

- $H(C,U) := \{\, f \in \operatorname{Loc}(X,Y,\mathcal{R}) \mid C \subset D_f,\ f(C) \subset U\,\}$ for $C \Subset X$ and $U \in \tau(Y)$,
- $M(V,R) := \{\, f \in \operatorname{Loc}(X,Y,\mathcal{R}) \mid f^{-1}(R) \subset V\,\}$ for $V \in \tau(X)$ and $R \in \mathcal{R}$.

Elements of $\mathrm{Loc}(X,Y,\mathcal{R})$ are called *local maps*. The natural base point of $\mathrm{Loc}(X,Y,\mathcal{R})$ is the empty map. Let $\sqcup$ denote the union of two disjoint local maps. Moreover, in the case when $\mathcal{R}=\{\{y\}\}$ we will write $\mathrm{Loc}(X,Y,y)$ omitting double curly brackets.

Assume that V is a real finite-dimensional orthogonal representation of a compact Lie group G. Let X be an arbitrary G-space. We say that $f\colon X\to V$ is *equivariant*, if $f(gx)=gf(x)$ for all $x\in X$ and $g\in G$. We will denote by $\mathcal{F}_G(X)$ the space $\{f\in \mathrm{Loc}(X,V,0)\mid f \text{ is equivariant}\}$ with the induced topology. Assume that Ω is an open invariant subset of V. Elements of $\mathcal{F}_G(\Omega)$ are called *equivariant local maps*.

Let $I=[0,1]$. We assume that the action of G on I is trivial. Any element of $\mathcal{F}_G(I\times\Omega)$ is called an *otopy*. Each otopy corresponds to a path in $\mathcal{F}_G(\Omega)$ and vice versa. Given an otopy $h\colon \Lambda\subset I\times\Omega\to V$ we can define for each $t\in I$:

- sets $\Lambda_t=\{x\in\Omega\mid (t,x)\in\Lambda\}$,
- maps $h_t\colon \Lambda_t\to V$ with $h_t(x)=h(t,x)$.

In this situation we say that h_0 and h_1 are *otopic*. Otopy gives an equivalence relation on $\mathcal{F}_G(\Omega)$. The set of otopy classes will be denoted by $\mathcal{F}_G[\Omega]$.

Let $\mathcal{F}_G^{\nabla}(\Omega)$ denote the subspace of $\mathcal{F}_G(\Omega)$ (with the relative topology) consisting of those maps f for which there is an invariant C^1-function $\varphi\colon D_f\to\mathbb{R}$ such that $f=\nabla\varphi$. We call such maps *gradient*. Similarly, we write $\mathcal{F}_G^{\nabla}(I\times\Omega)$ for the subspace of $\mathcal{F}_G(I\times\Omega)$ consisting of such otopies h that $h_t\in\mathcal{F}_G^{\nabla}(\Omega)$ for each $t\in I$. These otopies are called *gradient*. Let us denote by $\mathcal{F}_G^{\nabla}[\Omega]$ the set of the equivalence classes of the *gradient otopy* relation.

If H is a closed subgroup of G then

- (H) stands for the conjugacy class of H,
- NH is the normalizer of H in G,
- WH is the Weyl group of H, i.e. $WH=NH/H$.

Recall that $G_x=\{g\in G\mid gx=x\}$. We define the following subsets of V:

$$V^H=\{x\in V\mid H\subset G_x\},$$
$$\Omega_H=\{x\in\Omega\mid H=G_x\}.$$

Set $\mathrm{Iso}(\Omega):=\{(H)\mid H \text{ is a closed subgroup of } G \text{ and } \Omega_H\neq\emptyset\}$. The set $\mathrm{Iso}(\Omega)$ is partially ordered. Namely, $(H)\leq(K)$ if H is conjugate to a subgroup of K.

We will make use of the following well-known facts:

- WH is a compact Lie group,
- V^H is a linear subspace of V and an orthogonal representation of WH,
- Ω_H is open in V^H,
- the action of WH on Ω_H is free,
- the set $\mathrm{Iso}(\Omega)$ is finite.

Assume that M is a smooth (i.e. C^1) connected manifold without boundary. Let $\mathcal{F}(M)\subset\mathrm{Loc}\,(M,TM,\{M\})$ denote the space of local vector fields equipped with the induced topology.

Suppose, in addition, that M is Riemannian. Then a local vector field v is called *gradient* if there is a smooth function $\varphi\colon D_v \to \mathbb{R}$ such that $v = \nabla\varphi$. In that case $\mathcal{F}(M)$ contains the subspace $\mathcal{F}^\nabla(M)$ consisting of gradient local vector fields. Any element of Map $(I, \mathcal{F}(M))$ is called an *otopy* and any element of Map $\left(I, \mathcal{F}^\nabla(M)\right)$ is called a *gradient otopy*. If two local vector fields are connected by a (gradient) otopy, we call them *(gradient) otopic*. Of course, (gradient) otopy gives an equivalence relation on $\mathcal{F}(M)$ $(\mathcal{F}^\nabla(M))$. The sets of the respective equivalence classes will be denoted by $\mathcal{F}[M]$ and $\mathcal{F}^\nabla[M]$.

2. Definition of Φ

Assume that V is a real finite-dimensional orthogonal representation of a compact Lie group G and Ω is an open invariant subset of V. The main goal of this section is to define a function

$$\Phi\colon \mathcal{F}_G^\nabla[\Omega] \to \prod_{(H)} \mathcal{F}_{WH}^\nabla\left[\Omega_H\right],$$

where the product is taken over $\mathrm{Iso}(\Omega)$. Before we get into the details, let us sketch the general idea of the construction on which our definition is based. Let $f \in \mathcal{F}_G^\nabla(\Omega)$. Natural approach suggests to take as a value of $\Phi([f])$ classes of restrictions

$$f_H := f\restriction_{D_f \cap \Omega_H}$$

for every orbit type (H) in $\mathrm{Iso}(\Omega)$. Unfortunately, so defined f_H may not be an element of $\mathcal{F}_{WH}^\nabla(\Omega_H)$, because the set of zeroes of f_H does not need to be compact. However, it is compact if (H) is maximal in $\mathrm{Iso}(\Omega)$, because in that case $\Omega_{(H)}$ is closed in Ω. Since it is possible to arrange orbit types in $\mathrm{Iso}(\Omega)$ so that $(H_i) \le (H_j)$ implies $j \le i$ we can define $\Phi_i([f])$ inductively with respect to that linear order. Namely, in the first step we set $\Phi_1([f]) = [f_{H_1}]$. Then, since the set of zeroes of f contained in $\Omega\backslash\Omega_{(H_1)}$ does not need to be compact, we perturb f to f' in such a way that $D_{f'} \subset D_f$, f' is otopic to f and the set of zeroes of f' in $\Omega\backslash\Omega_{(H_1)}$ becomes compact. Note that now (H_2) is maximal in $\Omega\backslash\Omega_{(H_1)}$. Hence we put $\Phi_2([f]) = \left[f'_{H_2}\right]$ and proceed as before for all subsequent orbit types.

The formal definition of Φ will be divided into four steps.

2.1. Definition of otopy perturbation from f to $f_{U,\epsilon}$

Assume that (H) is maximal in $\mathrm{Iso}(\Omega)$. Let $f = \nabla\varphi \in \mathcal{F}_G^\nabla(\Omega)$. We will define a map $f_{U,\epsilon}$, which is otopic to f and whose zeros contained in $\Omega\backslash\Omega_{(H)}$ are compact. To do that we choose an open invariant subset U of $D_f \cap \Omega_{(H)}$ such that

$$f^{-1}(0) \cap \Omega_{(H)} \subset U \subset \mathrm{cl}\, U \Subset \Omega_{(H)}.$$

For $\epsilon > 0$ we introduce the following notation:

$$U^\epsilon = \{x + v \mid x \in U, v \in N_x, |v| < \epsilon\},$$
$$B^\epsilon = \{x + v \mid x \in \mathrm{bd}\, U, v \in N_x, |v| \le \epsilon\},$$

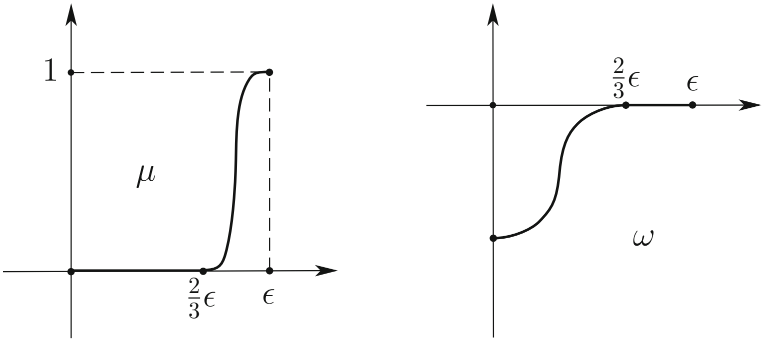

FIGURE 1. Graphs of μ and ω

where N_x denotes $\left(T_x\Omega_{(H)}\right)^{\perp}$. Throughout the paper whenever symbols U^ϵ and B^ϵ appear we assume tacitly that the set $\mathrm{cl}\,(U^\epsilon)$ is contained in some tubular neighbourhood of $\Omega_{(H)}$ in V. Observe that this condition is satisfied for ϵ sufficiently small. Moreover, to define $f_{U,\epsilon}$ we will need two more assumptions:

$$U^\epsilon \subset D_f \quad \text{and} \quad f^{-1}(0) \cap B^\epsilon = \emptyset, \tag{2.1}$$

which are also satisfied for ϵ small enough.

Let us introduce an auxiliary smooth function $\mu\colon [0,\epsilon] \to \mathbb{R}$ as in Fig. 1. Now define for each $t \in [0,1]$ the function $\mu_t\colon [0,\epsilon] \to \mathbb{R}$ by $\mu_t(s) = t\mu(s) + 1 - t$. Using the family of functions μ_t we will define for $t \in [0,1]$ a family of maps $r_t\colon U^\epsilon \to U^\epsilon$ by the formula

$$r_t(x+v) = x + \mu_t(|v|)v,$$

where $x \in U, v \in N_x, |v| < \epsilon$. Observe that $r_0 = \mathrm{Id}$ and $r_1(x+v) = x$ for $|v| \leq 2\epsilon/3$.

We will also need another auxiliary function $\omega\colon [0,\epsilon] \to \mathbb{R}$ (see Fig. 1) given by

$$\omega(s) = \begin{cases} \frac{1}{2}s^2 - \frac{1}{9}\epsilon^2 & \text{for } s \in [0, \epsilon/3]\,, \\ -\frac{1}{2}\left(s - \frac{2}{3}\epsilon\right)^2 & \text{for } s \in [\epsilon/3, 2\epsilon/3]\,, \\ 0 & \text{for } s \in [2\epsilon/3, \epsilon]\,. \end{cases}$$

Now we define for $t \in [0,1]$ a family of potentials

$$\varphi_t\colon D_f \backslash B^\epsilon \to \mathbb{R}$$

by the formula

$$\varphi_t(z) = \begin{cases} \varphi(r_{2t}(z)) & \text{if } z \in U^\epsilon \text{ and } t \in [0,1/2], \\ \varphi(r_1(z)) + (2t-1)\omega(|v|) & \text{if } z = x+v \in U^\epsilon \text{and } t \in [1/2,1], \\ \varphi(z) & \text{if } z \in D_f \backslash (U^\epsilon \cup B^\epsilon) \text{ and } t \in [0,1]. \end{cases} \tag{2.2}$$

We are now ready to define $f_{U,\epsilon}$ by the formula (see Fig. 2)

$$f_{U,\epsilon} = \nabla\varphi_1.$$

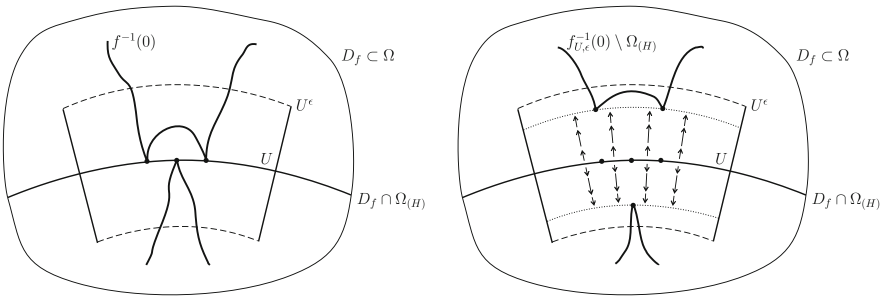

FIGURE 2. Perturbation of f

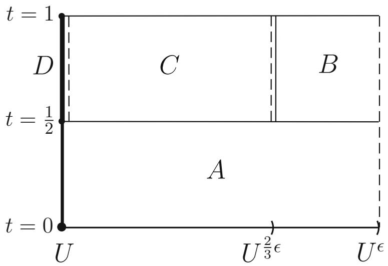

FIGURE 3. Partition of $I \times U^\epsilon$

Proposition 2.1. *The homotopy $h(t,z) = \nabla\varphi_t(z)$ is an otopy from $f \restriction_{D_f \setminus B^\epsilon}$ to $f_{U,\epsilon}$.*

Proof. It is sufficient to show that the set $h^{-1}(0)$ is compact. Consider the partition of $I \times U^\epsilon$ into four subsets (see Fig. 3):

$$\begin{aligned} A &= [0,1/2] \times U^\epsilon, \\ B &= [1/2,1] \times \left(U^\epsilon \backslash U^{2\epsilon/3}\right), \\ C &= [1/2,1] \times \left(U^{2\epsilon/3} \backslash U\right), \\ D &= [1/2,1] \times U. \end{aligned}$$

Since for $t \in [0,1/2]$ and $z \in U^\epsilon$ we have

$$\nabla\varphi_t(z) = Dr_{2t}^T(z) \cdot \nabla\varphi_t\left(r_{2t}(z)\right)$$

and for $t \in [1/2,1]$ and $z \in U^\epsilon$ the summand $\omega(|v|)$ occurs in formula (2.2), we obtain the following description of the set of zeroes of h in $I \times U^\epsilon$:

- $h(t,z) = 0$ iff $f\left(r_{2t}(z)\right) = 0$ in A,
- $h(t,z) = 0$ iff $f\left(r_1(z)\right) = 0$ in B,
- $h(t,z) \neq 0$ in C,
- $h(t,z) = f(z)$ in D.

From this we get our claim. $\square$

Let us introduce the following notation:

$$\begin{aligned} f^n &= f^n_{U,\epsilon} = f_{U,\epsilon} \restriction_{U^{\epsilon/3}}, \\ f^c &= f^c_{U,\epsilon} = f_{U,\epsilon} \restriction_{D_{f_{U,\epsilon}} \setminus \Omega_{(H)}}, \\ f^a &= f^c \restriction_{D_{f^c} \setminus \operatorname{cl}\left(U^{\epsilon/3}\right)} . \end{aligned}$$

Remark 2.2. The following observations will be useful in the study of the function $\widehat{\Phi}$, which will be defined in the next subsection:

- f^n is (H)-normal in the sense of Definition 5.1,
- $[f^c] = [f^a]$ in $\mathcal{F}^{\nabla}_{G}\left[\Omega \setminus \Omega_{(H)}\right]$,
- $[f] = [f_{U,\epsilon}] = [f^n \sqcup f^a]$ in $\mathcal{F}^{\nabla}_{G}[\Omega]$.

2.2. Definition of $\widehat{\Phi}$

Let us define the function

$$\widehat{\Phi} \colon \mathcal{F}^{\nabla}_{G}[\Omega] \to \mathcal{F}^{\nabla}_{WH}\left[\Omega_H\right] \times \mathcal{F}^{\nabla}_{G}\left[\Omega \setminus \Omega_{(H)}\right]$$

by

$$\widehat{\Phi}([f]) = (\widehat{\Phi}'([f]), \widehat{\Phi}''([f])) = \left(\left[f_H\right], \left[f^c_{U,\epsilon}\right]\right),$$

where $f_H = f \restriction_{D_f \cap \Omega_H}$. The fact that $\widehat{\Phi}$ is well-defined will be proved in Sect. 8.

2.3. Definition of $\widehat{\Phi}_i$

As we have mentioned the orbit types are enumerated $(H_1), (H_2), \ldots, (H_m)$ according to the reverse partial order in $\operatorname{Iso}(\Omega)$. Now consider a sequence of open subsets of Ω

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_m,$$

where $\Omega_1 = \Omega$ and $\Omega_{i+1} = \Omega_i \setminus \Omega_{(H_i)}$. Let

$$\widehat{\Phi}_i = (\widehat{\Phi}'_i, \widehat{\Phi}''_i) \colon \mathcal{F}^{\nabla}_{G}[\Omega_i] \to \mathcal{F}^{\nabla}_{WH_i}\left[\Omega_{H_i}\right] \times \mathcal{F}^{\nabla}_{G}\left[\Omega_{i+1}\right]$$

be defined as $\widehat{\Phi}$ in the previous subsection with Ω replaced by Ω_i and H by H_i.

2.4. Definition of Φ

Let us start with the inductive definition of

$$\Xi_i \colon \mathcal{F}^{\nabla}_{G}[\Omega] \to \mathcal{F}^{\nabla}_{G}[\Omega_{i+1}].$$

Set $\Xi_0 = \mathrm{Id}$ and $\Xi_i = \widehat{\Phi}''_i \circ \Xi_{i-1}$. Let

$$\Phi_i \colon \mathcal{F}^{\nabla}_{G}[\Omega] \to \mathcal{F}^{\nabla}_{WH_i}\left[\Omega_{H_i}\right]$$

be defined by $\Phi_i = \widehat{\Phi}'_i \circ \Xi_{i-1}$. Finally, let

$$\Phi \colon \mathcal{F}^{\nabla}_{G}[\Omega] \to \prod_{i=1}^{m} \mathcal{F}^{\nabla}_{WH_i}\left[\Omega_{H_i}\right]$$

be given by $\Phi = (\Phi_1, \ldots, \Phi_m)$.

Remark 2.3. It is worth pointing out that the obtained Φ does not depend on the choice of linear extension of the partial order in $\mathrm{Iso}(\Omega)$, since $\Omega_{(H)}$ is closed in Ω for every maximal orbit type (H) and therefore the inductive step can be performed simultaneously on all maximal types (H) in Ω.

Remark 2.4. Alternatively, for given $f \in \mathcal{F}_G^\nabla(\Omega)$ we can define two finite sequences of maps $f_i \in \mathcal{F}_G^\nabla(\Omega_i)$ and $f_i' \in \mathcal{F}_{WH_i}^\nabla(\Omega_{H_i})$ given by

$$f_1 = f, \quad f_{i+1} = (f_i)^c_{U_i,\epsilon_i}, \quad f_i' = f_i \restriction_{D_{f_i} \cap \Omega_{H_i}}.$$

Observe that $D_{f_{i+1}} \subset D_{f_i}$ and $\Phi_i([f]) = [f_i']$. In this way we obtain the equivalent definition of Φ.

3. Definition of Ψ

The main result of our paper (Main Theorem in Sec. 7) describes the properties of the function $\Theta\colon \mathcal{F}_G^\nabla[\Omega] \to \prod_{i=1}^m (\sum_j \mathbb{Z})$, which provides a degree-type invariant. In the previous section we have constructed the decomposition $\Phi\colon \mathcal{F}_G^\nabla[\Omega] \to \prod_{i=1}^m \mathcal{F}_{WH_i}^\nabla[\Omega_{H_i}]$. The function Θ will be defined as a composition of the function Φ and a family of bijections between the factors $\mathcal{F}_{WH_i}^\nabla[\Omega_{H_i}]$ and the direct sum of countably many copies of $\mathbb{Z}$. Below we present the construction of such a bijection.

In this section, we assume that V is a real finite-dimensional orthogonal representation of a compact Lie group G ($\dim V > 0$), Ω is an open invariant subset of V, G acts freely on Ω and $M := \Omega/G$. It is well known that M is a Riemannian manifold of positive dimension equipped with the so-called quotient Riemannian metric (see for instance [16, Prop. 2.28]).

If U is an open invariant subset of Ω and $\varphi\colon U \to \mathbb{R}$ is an invariant function, then $\widetilde{\varphi}$ stands for the quotient function $\widetilde{\varphi}\colon U/G \to \mathbb{R}$. Let the function $\Psi\colon \mathcal{F}_G^\nabla[\Omega] \to \mathcal{F}^\nabla[M]$ be given by $\Psi([\nabla\varphi]) = [\nabla\widetilde{\varphi}]$. The following result was proved in [11, Cor. 5.2].

Theorem 3.1. Ψ *is a well-defined bijection.*

Remark 3.2. Let $\{M_j\}$ denote the set of components of M. In [11] we proved that the intersection number I establishes a bijection $\mathcal{F}^\nabla[M_j] \approx \mathbb{Z}$. Consequently, the restrictions of $f \in \mathcal{F}^\nabla(M)$ to the components of M establish a natural bijection $\mathrm{I}\colon \mathcal{F}^\nabla[M] \to \sum_j \mathbb{Z}$. Note that a direct sum (not product) appears in the last formula, since for any gradient local vector field f the preimage of the zero section meets only a finite number of components of M and, in consequence, almost all restrictions of f are otopic to the empty map.

Corollary 3.3. *The composition*

$$\mathrm{I} \circ \Psi\colon \mathcal{F}_G^\nabla[\Omega] \to \mathcal{F}^\nabla[\Omega/G] \to \sum_j \mathbb{Z},$$

where the direct sum is taken over the set of connected components of Ω/G, *is a bijection. Moreover, if* $f, g \in \mathcal{F}_G^\nabla(\Omega)$ *such that* $D_f \cap D_g = \emptyset$ *then*

$$\mathrm{I} \circ \Psi\,([f \sqcup g]) = \mathrm{I} \circ \Psi\,([f]) + \mathrm{I} \circ \Psi\,([g])\,.$$

4. Definition of Θ

We are now ready to define the invariant Θ. Assume that V is a real finite-dimensional orthogonal representation of a compact Lie group G and Ω is an open invariant subset of V. Let $\Psi_i\colon \mathcal{F}^{\nabla}_{WH_i}[\Omega_{H_i}] \to \mathcal{F}^{\nabla}[\Omega_{H_i}/WH_i]$ denote the function Ψ defined in the previous section with G replaced by WH_i and Ω replaced by Ω_{H_i} (WH_i acts freely on Ω_{H_i}). Recall that $(\Omega_{H_i}/WH_i)_j$ denotes the jth component of Ω_{H_i}/WH_i. Let

$$\pi_{ij}\colon \mathcal{F}^{\nabla}[\Omega_{H_i}/WH_i] \to \mathcal{F}^{\nabla}[(\Omega_{H_i}/WH_i)_j]$$

denote the function induced by the restriction of a gradient local vector field to jth component of Ω_{H_i}/WH_i. Finally, set

$$\Theta_{ij} = \mathrm{I} \circ \pi_{ij} \circ \Psi_i \circ \Phi_i$$

and

$$\Theta = \{\Theta_{ij}\}\colon \mathcal{F}^{\nabla}_G[\Omega] \to \prod_{i=1}^{m}\Big(\sum_j \mathbb{Z}\Big).$$

Correctness of the above definition requires that $\dim \Omega_{H_i}/WH_i > 0$ for $i = 1, \ldots, m$, which is necessary for calculation of the intersection number. This is the case when $\dim V^{H_1} > 0$. The opposite case, in which $0 \in \Omega$ and $\dim V^G = 0$, will be discussed in Remark 7.2 after the proof of Main Result.

5. (H)-normal maps

The maps discussed in this section are essential for the formulation of Theorem 7.1. These maps are important, because in some sense they are "generic" with respect to Φ. Namely, if f is (H_i)-normal then $\Phi_i([f]) = [f_{H_i}]$, where $f_{H_i} = f\restriction_{D_f \cap \Omega_{H_i}}$, and $\Phi_i([f]) = [\emptyset]$ for $i \neq j$. Let U be an open bounded invariant subset of $\Omega_{(H)}$. Recall that $U^\epsilon = \{x + v \mid x \in U, v \in N_x, |v| < \epsilon\}$ and $B^\epsilon = \{x + v \mid x \in \operatorname{bd} U, v \in N_x, |v| \le \epsilon\}$.

Definition 5.1. A map $f \in \mathcal{F}^{\nabla}_G(\Omega)$ is called (H)-*normal* (on U^ϵ) if

- $f^{-1}(0) \subset U$,
- $U^\epsilon \subset D_f$,
- $f\restriction_{U^\epsilon} = \nabla\varphi$, where $\varphi\colon U^\epsilon \to \mathbb{R}$ is an invariant C^1-function satisfying $\varphi(x+v) = \varphi(x) + \frac{1}{2}|v|^2$ for $x \in U$, $v \in N_x\Omega_{(H)}$, $|v| < \epsilon$.

The following result describes a basic property of (H)-normal maps and their behaviour under perturbation.

Proposition 5.2. *Assume that (H) is maximal in* $\mathrm{Iso}(\Omega)$ *and f is (H)-normal on U^ϵ. Then $\left[f^c_{U,\epsilon}\right] = [\emptyset]$ in $\mathcal{F}^{\nabla}_G\left[\Omega\backslash\Omega_{(H)}\right]$.*

Proof. By definition, $f^{-1}(0) \subset U \subset \Omega_{(H)}$, $f = \nabla\varphi$, where $\varphi(x+v) = \varphi(x) + \frac{1}{2}|v|^2$ for $x + v \in U^\epsilon \subset D_f$, and $f^c_{U,\epsilon} = \nabla\varphi_1\restriction_{D_{\varphi_1}\backslash\Omega_{(H)}}$, where

$$\varphi_1(z) = \begin{cases} \varphi(x) + \frac{1}{2}\mu^2(|v|)\,|v|^2 + \omega(|v|) & \text{if } z = x + v \in U^\epsilon, \\ \varphi(z) & \text{if } z \in D_f\backslash(U^\epsilon \cup B^\epsilon). \end{cases}$$

Define the family of potentials $h_t \colon D_{\varphi_1} \backslash \Omega_{(H)} \to \mathbb{R}$

$$h_t(z) = \begin{cases} \varphi(x) + \frac{1}{2}\mu^2(|v|)\,|v|^2 + \omega(|v|) + t\omega\left(\frac{2}{3}\,|v|\right) & \text{if } z = x + v \in U^\epsilon \backslash U, \\ \varphi(z) & \text{if } z \in D_f \backslash (U^\epsilon \cup B^\epsilon \cup \Omega_{(H)}). \end{cases}$$

Since ∇h_t is a path from $f^c_{U,\epsilon}$ to ∇h_1 in $\mathcal{F}^\nabla_G\left(\Omega\backslash\Omega_{(H)}\right)$ and $(\nabla h_1)^{-1}(0) = \emptyset$, we have $\left[f^c_{U,\epsilon}\right] = [\emptyset]$ in $\mathcal{F}^\nabla_G\left[\Omega\backslash\Omega_{(H)}\right]$. □

6. Orbit-normal maps

Here we introduce an important subclass of (H)-normal maps, which will appear in the formulation of Main Theorem as base functions for Θ. Let $\mathcal{O}$ denote a G-orbit in Ω. Assume that $\mathcal{O}^\epsilon := \{x+v \mid x \in \mathcal{O}, v \in (T_x\mathcal{O})^\perp, |v| < \epsilon\}$ is contained with its closure in some tubular neighbourhood of $\mathcal{O}$ in Ω.

Definition 6.1. A map $f \in \mathcal{F}^\nabla_G(\Omega)$ is called *orbit-normal* around $\mathcal{O}$ if

- $f^{-1}(0) = \mathcal{O}$,
- $\mathcal{O}^\epsilon \subset D_f$,
- $f(x+v) = v$ for $x + v \in \mathcal{O}^\epsilon$.

The three following properties of orbit-normal maps will be needed in the proof of Main Theorem. The first one explains the relation between the notions of orbit-normal and (H)-normal maps.

Proposition 6.2. *If (H) is an orbit type of an orbit $\mathcal{O}$ then every orbit-normal map around $\mathcal{O}$ is also (H)-normal.*

Proof. Assume that $f \in \mathcal{F}^\nabla_G(\Omega)$ is orbit-normal around $\mathcal{O}$. Let for $x \in \mathcal{O}$ $(G_x = H)$

$$\begin{aligned} N_1^x &:= (T_x(\mathcal{O}))^\perp \cap T_x\Omega_H, \\ N_2^x &:= (T_x\Omega_{(H)})^\perp. \end{aligned}$$

By definition, $N_1^x \perp N_2^x$. We will show that $(T_x\mathcal{O})^\perp = N_1^x \oplus N_2^x$. It is well-known that

$$T_x\Omega_{(H)} = T_x\mathcal{O} \oplus ((T_x\mathcal{O})^\perp \cap V^H).$$

Hence

$$\begin{aligned} (T_x(\mathcal{O}))^\perp \cap T_x\Omega_{(H)} &= (T_x(\mathcal{O}))^\perp \cap (T_x\mathcal{O} \oplus ((T_x\mathcal{O})^\perp \cap V^H)) \\ &= (T_x\mathcal{O})^\perp \cap V^H) = (T_x(\mathcal{O}))^\perp \cap T_x\Omega_H = N_1^x \end{aligned}$$

and, in consequence,

$$(T_x\mathcal{O})^\perp = ((T_x\mathcal{O})^\perp \cap T_x\Omega_{(H)}) \oplus (T_x\Omega_{(H)})^\perp = N_1^x \oplus N_2^x.$$

Set $U = \{x + v_1 \mid x \in \mathcal{O}, v_1 \in N_1^x, |v_1| < \frac{\sqrt{2}}{2}\epsilon\}$. Since

$$U^{\frac{\sqrt{2}}{2}\epsilon} = \{x+v_1+v_2 \mid x \in \mathcal{O}, v_1 \in N_1^x, |v_1| < \tfrac{\sqrt{2}}{2}\epsilon, v_2 \in N_2^x, |v_2| < \tfrac{\sqrt{2}}{2}\epsilon\} \subset \mathcal{O}^\epsilon,$$

we have $f(x+v_1+v_2) = v_1 + v_2 = f(x+v_1) + v_2$ for $x + v_1 + v_2 \in U^{\frac{\sqrt{2}}{2}\epsilon}$, which proves that f is (H)-normal. □

It turns out that the property of being orbit-normal is inherited by the restriction to Ω_H.

Proposition 6.3. *Assume that G-orbit $\mathcal{O}$ has orbit type (H). If $f \in \mathcal{F}_G^{\nabla}(\Omega)$ is orbit-normal around $\mathcal{O}$ then $f \restriction_{D_f \cap \Omega_H} \in \mathcal{F}_{WH}^{\nabla}(\Omega_H)$ is orbit-normal around WH-orbit $\mathcal{O} \cap \Omega_H$.*

Proof. The assertion follows from the observation that

$$\{x + v \mid x \in \mathcal{O} \cap \Omega_H, v \in V^H, |v| < \epsilon\} = \mathcal{O}^\epsilon \cap \Omega_H$$

is a tubular neighbourhood of the WH-orbit $\mathcal{O} \cap \Omega_H$, in which $f(x+v) = v$. □

The next result describes what happens when we divide out the free action in an orbit-normal map.

Proposition 6.4. *If G acts freely on Ω and $f = \nabla\varphi \in \mathcal{F}_G^{\nabla}(\Omega)$ is orbit-normal around $\mathcal{O}$ then $\mathcal{O}/G \in \Omega/G$ is a source for $\nabla\widetilde{\varphi} \in \mathcal{F}^{\nabla}(\Omega/G)$.*

Proof. Let $p = \mathcal{O}/G \in \Omega/G = M$. Choose $x \in \mathcal{O}$. We can identify some neighbourhood U of p in M with the set $\{v \in (T_x\mathcal{O})^\perp \mid |v| < \epsilon\}$ and for $v \in U$ the tangent space T_vM with $(T_x\mathcal{O})^\perp$. Since $\nabla\varphi(x+v) = v$, for $v \in U$ we have $\nabla\widetilde{\varphi}(v) = v \in T_vM$. Hence p is a source for $\nabla\widetilde{\varphi}$. □

7. Main results

We can now formulate main results of our paper. Theorem 7.1 will be proved in the next section.

Theorem 7.1. *The function*

$$\Phi \colon \mathcal{F}_G^{\nabla}[\Omega] \to \prod_{i=1}^{m} \mathcal{F}_{WH_i}^{\nabla}[\Omega_{H_i}],$$

where the product is taken over $\operatorname{Iso}(\Omega)$*, is a bijection. Moreover, if f is (H_j)-normal then*

$$\Phi_i([f]) = \begin{cases} [\emptyset] & \text{if} \quad i \neq j, \\ [f_{H_j}] & \text{if} \quad i = j, \end{cases}$$

where $f_{H_j} = f \restriction_{D_f \cap \Omega_{H_j}}$.

Main Theorem. *Assume that $0 \notin \Omega$ or $\dim V^G > 0$. Then the function*

$$\Theta \colon \mathcal{F}_G^{\nabla}[\Omega] \to \prod_{i=1}^{m} \Big(\sum_j \mathbb{Z}\Big),$$

where the product is taken over $\operatorname{Iso}(\Omega)$ *and the respective direct sums are indexed by either finite or countably infinite sets of connected components of the quotients Ω_{H_i}/WH_i, is a bijection. Moreover*

1. *$\Theta([f \sqcup g]) = \Theta([f]) + \Theta([g])$ for $f, g \in \mathcal{F}_G^{\nabla}(\Omega)$ such that $D_f \cap D_g = \emptyset$,*
2. *$\Theta([\emptyset]) = 0$,*
3. *if $\Theta([f]) \neq 0$ then there is $x \in D_f$ such that $f(x) = 0$,*

4. *if f is orbit-normal around $\mathcal{O} \subset \Omega_{(H_k)}$ and $p_k\left(\mathcal{O} \cap \Omega_{H_k}\right) \in (\Omega_{H_k}/WH_k)_l$, where $p_k \colon \Omega_{H_k} \to \Omega_{H_k}/WH_k$ denotes the quotient map and $(\Omega_{H_k}/WH_k)_l$ the respective component, then*

$$\Theta_{ij}([f]) = \begin{cases} 1 & \text{if} \quad i = k \quad \text{and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we show that Θ is a bijection. Define

$$\Psi = \prod_{i=1}^{m} \Psi_i \colon \prod_{i=1}^{m} \Psi_i \mathcal{F}_{WH_i}^{\nabla}\left[\Omega_{H_i}\right] \to \prod_{i=1}^{m} \mathcal{F}^{\nabla}\left[\Omega_{H_i}/WH_i\right]$$

and

$$\pi = \sum_{i,j} \pi_{ij} \colon \prod_{i=1}^{m} \mathcal{F}^{\nabla}\left[\Omega_{H_i}/WH_i\right] \to \prod_{i=1}^{m} \sum_{j} \mathcal{F}^{\nabla}\left[(\Omega_{H_i}/WH_i)_j\right],$$

where for each i the index j runs over the set of connected components of Ω_{H_i}/WH_i. Observe that Ψ and π are bijections (the first from Theorem 3.1 and the second by definition). Let $\mathrm{I}_{ij} \colon \mathcal{F}^{\nabla}\left[(\Omega_{H_i}/WH_i)_j\right] \to \mathbb{Z}$ denote the intersection number restricted to the respective component. By Remark 3.2, I_{ij} is a bijection and, in consequence, so is

$$\mathrm{I} = \sum_{i,j} \mathrm{I}_{ij} \colon \prod_{i=1}^{m} \sum_{j} \mathcal{F}^{\nabla}\left[(\Omega_{H_i}/WH_i)_j\right] \to \prod_{i=1}^{m} \sum_{j} \mathbb{Z}.$$

Since, by Theorem 7.1, Φ is a bijection, we obtain that the composition $\Theta = \mathrm{I} \circ \pi \circ \Psi \circ \Phi$ is also a bijection.

Next we prove the additivity property (1). Following the notation from Remark 2.4 we obtain the sequences $f_i, g_i, (f \sqcup g)_i \in \mathcal{F}_G^{\nabla}(\Omega_i)$ and $f_i', g_i', (f \sqcup g)_i' = f_i' \sqcup g_i' \in \mathcal{F}_{WH_i}^{\nabla}(\Omega_{H_i})$. Since $\Phi_i([f \sqcup g]) = [f_i' \sqcup g_i']$, by Corollary 3.3 we have

$$\begin{aligned} \Theta_{ij}([f \sqcup g]) &= \mathrm{I} \circ \pi_{ij} \circ \Psi_i\left([f_i' \sqcup g_i']\right) \\ &= \mathrm{I} \circ \pi_{ij} \circ \Psi_i\left([f_i']\right) + \mathrm{I} \circ \pi_{ij} \circ \Psi_i\left([g_i']\right) \\ &= \Theta_{ij}([f]) + \Theta_{ij}([g]). \end{aligned}$$

The property (2) follows from (1) as well as from the direct construction.

To prove (3) observe that if $f^{-1}(0) = \emptyset$ then f is otopic to the empty map and hence $\Theta_{ij}([f]) = \Theta_{ij}([\emptyset]) = 0$.

Finally, we show the normalization property (4). By Proposition 6.2, f is (H_k)-normal. First consider the case $i = k$. Note that $\Phi_k([f]) = [f_{H_k}]$ and $\Psi_k([f_{H_k}]) = \Psi_k([\nabla\varphi]) = [\nabla\widetilde{\varphi}_k]$. By Propositions 6.3 and 6.4, the only zero of $\nabla\widetilde{\varphi}_k$ is a source and, by assumption, it is contained in $(\Omega_{H_k}/WH_k)_l$. Consequently,

$$\Theta_{kj}([f]) = \mathrm{I} \circ \pi_{kj} \circ \Psi_k\left([f_{H_k}]\right) = \mathrm{I} \circ \pi_{kj}\left([\nabla\widetilde{\varphi}_k]\right) = \begin{cases} 1 & \text{for } k = l, \\ 0 & \text{for } k \neq l. \end{cases}$$

In turn, for $i \neq k$, $\Phi_i([f]) = [\emptyset]$ and, in consequence, $\Theta_{ij}([f]) = 0$. $\square$

Remark 7.2. Now consider the case $0 \in \Omega$ and $\dim V^G = 0$. In that situation $H_1 = G$, $V^{H_1} = \Omega_{H_1} = \{0\}$, and WH_1 is trivial. Since

$$\Psi_1 \circ \Phi_1 \colon \mathcal{F}_G^{\nabla}[\Omega] \to \mathcal{F}^{\nabla}[\Omega_{H_1}/WH_1] = \mathcal{F}^{\nabla}[\{0\}] = \{\emptyset, 0\},$$

we can express Θ the in following form

$$\Theta \colon \mathcal{F}_G^{\nabla}[\Omega] \to \{0,1\} \times \prod_{i=2}^{m} \sum_{j} \mathbb{Z},$$

where $\Theta_{11} \colon \mathcal{F}_G^{\nabla}[\Omega] \to \{0,1\}$ and

$$\Theta_{11}([f]) = \begin{cases} 1 & \text{for } 0 \in D_f, \\ 0 & \text{for } 0 \notin D_f. \end{cases}$$

Main Theorem holds as well in the above case. Regarding the additivity property let us mention that all $\mathbb{Z}$ have complete additive structure, but the addition 1+1 in the set $\{0,1\}$ makes no sense. Nevertheless, when $D_f \cap D_g = \emptyset$ then either $\Theta_{11}([f])$ or $\Theta_{11}([g])$ is equal to 0 and therefore condition (1) makes no problem.

8. Proof of Theorem 7.1

This section contains the proof of Theorem 7.1 preceded by a series of lemmas and notations. Let us assume that (H) is maximal in $\operatorname{Iso}(\Omega)$. Recall that below $\sim$ denotes the relation of gradient otopy.

Lemma 8.1. *$\widehat{\Phi}$ is well-defined.*

Proof. Observe that the definition of $\widehat{\Phi}$ does not depend on the choice of U and ϵ if only they satisfy Condition (2.1). Since for the fixed U the definition of $\widehat{\Phi}$ does not depend on the choice of ϵ, it remains to check that it does not depend on the choice of U if ϵ is fixed. Let φ_1 and φ_1' be potentials from the definition of $\widehat{\Phi}$ corresponding to U and V. We can assume that $U \subset V$ because otherwise we can pass from U and V through $U \cap V$. Our assertion follows from the observation that

$$\nabla\varphi_1' \restriction_{D_{\varphi_1'} \setminus \Omega_{(H)}} \sim \nabla\varphi_1' \restriction_A \sim \nabla\varphi_1 \restriction_A \sim \nabla\varphi_1 \restriction_{D_{\varphi_1} \setminus \Omega_{(H)}},$$

where $A = D_f \setminus \left(\Omega_{(H)} \cup B_U^{\epsilon} \cup B_V^{\epsilon}\right)$.

Now we will show that if $[f] = [g]$ in $\mathcal{F}_G^{\nabla}[\Omega]$ then

1. $\widehat{\Phi}'([f]) = [f_H] = [g_H] = \widehat{\Phi}'([g])$ in $\mathcal{F}_{WH}^{\nabla}[\Omega_H]$,
2. $\widehat{\Phi}''([f]) = \widehat{\Phi}''([g])$ in $\mathcal{F}_G^{\nabla}\left[\Omega \setminus \Omega_{(H)}\right]$.

By assumption, there is an otopy $h \colon \Lambda \subset I \times \Omega \to V$ such that $f = h_0$ and $g = h_1$. The proof of (1) is straightforward. Namely, let $A = \Lambda \cap (I \times \Omega_H)$ and $k = h \restriction_A$. Then $k \colon A \subset I \times \Omega_H \to V^H$ is an otopy such that $k_0 = f_H$ and $k_1 = g_H$, and (1) is proved. To show (2) we will perturb the otopy h treating every section $h_t(\cdot) = h(t, \cdot)$ analogously to the perturbation $f_{U,\epsilon}$ of the map f. Let W be an open and invariant subset of $I \times \Omega_{(H)}$ such that

$$h^{-1}(0) \cap \left(I \times \Omega_{(H)}\right) \subset W \subset \operatorname{cl} W \Subset \Lambda \cap \left(I \times \Omega_{(H)}\right).$$

Let $B = \operatorname{bd} W$. For $\epsilon > 0$ let us define the sets

$$W^\epsilon = \{(t, x+v) \mid (t,x) \in W, v \in N_x, |v| < \epsilon\},$$
$$B^\epsilon = \{(t, x+v) \mid (t,x) \in B, v \in N_x, |v| \le \epsilon\}.$$

Observe that for ϵ sufficiently small we have $W^\epsilon \subset \Lambda$, $\operatorname{cl}(W^\epsilon)$ is contained in some tubular neighbourhood of $I \times \Omega_{(H)}$ and $h^{-1}(0) \cap B^\epsilon = \emptyset$. Recall that for $X \subset I \times \Omega$ we denote by X_t the set $\{x \in \Omega \mid (t,x) \in A\}$. Define the map

$$h_{W,\epsilon} \colon \Lambda \backslash B^\epsilon \to V$$

by the formula

$$h_{W,\epsilon}(z) = \begin{cases} (h_t)_{W_t,\epsilon}(x) & \text{if } z = (t,x) \in W^\epsilon, \\ h(z) & \text{if } z \in \Lambda \backslash (W^\epsilon \cup B^\epsilon). \end{cases}$$

In the above formula we use notation of perturbation introduced in Sect. 2.1. Set

$$h^c_{W,\epsilon} = h_{W,\epsilon} \restriction_{\Lambda \backslash \left(B^\epsilon \cup \left(I \times \Omega_{(H)}\right)\right)} .$$

Since $h_{W,\epsilon}$ is an otopy in $\mathcal{F}^\nabla_G(\Omega)$ and $\left(h^c_{W,\epsilon}\right)^{-1}(0)$ is compact, $h^c_{W,\epsilon}$ is an otopy in $\mathcal{F}^\nabla_G\left(\Omega \backslash \Omega_{(H)}\right)$ connecting $f^c_{W_0,\epsilon} \restriction_{D_f \backslash B^\epsilon_0}$ and $g^c_{W_1,\epsilon} \restriction_{D_g \backslash B^\epsilon_1}$. Consequently,

$$\left[f^c_{W_0,\epsilon}\right] = \left[f^c_{W_0,\epsilon} \restriction_{D_f \backslash B^\epsilon_0}\right] = \left[g^c_{W_1,\epsilon} \restriction_{D_g \backslash B^\epsilon_1}\right] = \left[g^c_{W_1,\epsilon}\right] \quad \text{in } \mathcal{F}^\nabla_G\left[\Omega \backslash \Omega_{(H)}\right],$$

which gives $\widehat{\Phi}''([f]) = \widehat{\Phi}''([g])$ and (2) is proved. □

The following two constructions will be needed in the proof of Lemma 8.4.

Assume $k = \nabla\varphi \in \mathcal{F}^\nabla_{WH}(\Omega_H)$. Let $\varphi_G \colon GD_k \to \mathbb{R}$ be given by $\varphi_G(gx) = \varphi(x)$. Let U be an open bounded invariant subset of GD_k such that $k^{-1}(0) \subset U$. Define the function $\widetilde{\varphi} \colon U^\epsilon \to \mathbb{R}$ by $\widetilde{\varphi}(x+v) = \varphi_G(x) + \frac{1}{2}|v|^2$ for $x + v \in U^\epsilon$. Set

$$k^{U,\epsilon} = \nabla\widetilde{\varphi}.$$

For $l \in \mathcal{F}^\nabla_G\left(\Omega \backslash \Omega_{(H)}\right)$ and $Y \subset \Omega$ closed invariant such that $l^{-1}(0) \cap Y = \emptyset$, we define

$$l^Y = l \restriction_{D_l \backslash Y} .$$

In the following proposition we use the notation introduced in Sect. 2.1.

Proposition 8.2. *The maps $k^{U,\epsilon} \in \mathcal{F}^\nabla_G(\Omega)$ and $l^Y \in \mathcal{F}^\nabla_G\left(\Omega \backslash \Omega_{(H)}\right)$ have the following properties:*

1. *$k^{U,\epsilon}$ is (H)-normal,*
2. *in $\mathcal{F}^\nabla_G\left[\Omega \backslash \Omega_{(H)}\right]$ we have*
   - (a) $\left[\left(k^{U,\epsilon}\right)^c_{U,\epsilon}\right] = [\emptyset]$,
   - (b) $\left[\left(k^{U,\epsilon} \sqcup l^{\operatorname{cl}(U^\epsilon)}\right)^c_{U,\epsilon}\right] = \left[l^{\operatorname{cl}(U^\epsilon)}\right]$,
   - (c) $\left[l^{\operatorname{cl}(U^\epsilon)}\right] = [l]$,
3. $\left(f^c_{U,\epsilon}\right)^{\operatorname{cl}\left(U^{\epsilon/3}\right)} = f^a_{U,\epsilon}$,
4. $\left(f_H\right)^{U,\epsilon/3} = f_{U,\epsilon} \restriction_{U^{\epsilon/3}} = f^n_{U,\epsilon}$.

Proof. Property (2a) follows from (1) and Proposition 5.2. All other properties are obvious. □

Let us define the function

$$\widetilde{\Phi}\colon \mathcal{F}^{\nabla}_{WH}\left[\Omega_H\right] \times \mathcal{F}^{\nabla}_{G}\left[\Omega\backslash\Omega_{(H)}\right] \to \mathcal{F}^{\nabla}_{G}[\Omega]$$

by the formula

$$\widetilde{\Phi}\left([k],[l]\right) = \left[k^{U,\epsilon} \sqcup l^{\mathrm{cl}(U^{\epsilon})}\right].$$

It will turn out that $\widetilde{\Phi}$ is inverse to $\widehat{\Phi}$, which will imply that $\widehat{\Phi}$ is a bijection.

Lemma 8.3. $\widetilde{\Phi}$ *is well-defined.*

Proof. It is easy to see that the definition of $\widetilde{\Phi}$ does not depend on the choice of U and ϵ. We show that $\widetilde{\Phi}$ is also independent of the choice of the representative in the otopy class. Let $k\colon D_k \subset I\times\Omega_H \to V^H$ and $l\colon D_l \subset I\times\left(\Omega\backslash\Omega_{(H)}\right) \to V$ be otopies. Let $W \subset I\times\Omega_{(H)}$ be an open invariant subset such that

$$k^{-1}(0) \subset W \subset \mathrm{cl}\, W \Subset GD_k.$$

Choose $\epsilon > 0$ such that $l^{-1}(0) \cap \mathrm{cl}\,(W^{\epsilon}) = \emptyset$. Recall that for $X \subset I\times\Omega$ we denote by X_t the set $\{x\in\Omega \mid (t,x)\in X\}$. Since

$$h_t := (k_t)^{W_t,\epsilon} \sqcup (l_t)^{(\mathrm{cl}(W^{\epsilon}))_t}$$

is an otopy and

$$(k_i)^{W_i,\epsilon} \sqcup (l_i)^{\mathrm{cl}(W_i^{\epsilon})} \sim (k_i)^{W_i,\epsilon} \sqcup (l_i)^{(\mathrm{cl}(W^{\epsilon}))_i} \qquad \text{for } i=0,1$$

we obtain $\widetilde{\Phi}\left([k_0],[l_0]\right) = \widetilde{\Phi}\left([k_1],[l_1]\right)$, which is the desired conclusion. □

Lemma 8.4. $\widetilde{\Phi}$ *is inverse to* $\widehat{\Phi}$ *and therefore* $\widehat{\Phi}$ *is a bijection.*

Proof. The calculations below are based on Proposition 8.2. Observe that

$$\begin{aligned}\widehat{\Phi}\circ\widetilde{\Phi}\left([k],[l]\right) &= \widehat{\Phi}\left(\left[k^{U,\epsilon}\sqcup l^{\mathrm{cl}(U^{\epsilon})}\right]\right)\\ &= \left(\left[k\restriction_U\right], \left[\left(k^{U,\epsilon}\sqcup l^{\mathrm{cl}(U^{\epsilon})}\right)^{c}_{U,\epsilon}\right]\right)\\ &= \left([k], \left[l^{\mathrm{cl}(U^{\epsilon})}\right]\right) = \left([k],[l]\right).\end{aligned}$$

In turn, if in $\widetilde{\Phi}$ we take $\epsilon/3$ instead of ϵ we obtain

$$\begin{aligned}\widetilde{\Phi}\circ\widehat{\Phi}\left([f]\right) &= \widetilde{\Phi}\left(\left[f_H\right], \left[f^{c}_{U,\epsilon}\right]\right) = \left[\left(f_H\right)^{U,\epsilon/3}\sqcup\left(f^{c}_{U,\epsilon}\right)^{\mathrm{cl}\left(U^{\epsilon/3}\right)}\right]\\ &= \left[f^{n}_{U,\epsilon}\sqcup f^{a}_{U,\epsilon}\right] = [f],\end{aligned}$$

which completes the proof. □

The next lemma follows directly from the construction of $\widehat{\Phi}$ and the definition of the sequence f_i (Remark 2.4).

Lemma 8.5. *Let $f \in \mathcal{F}_G^{\nabla}(\Omega)$. For each i the bijection $\widehat{\Phi}_i$ has the following properties:*

1. $\widehat{\Phi}_i([\emptyset]) = ([\emptyset],[\emptyset])$,
2. $\widehat{\Phi}_i''([f_i]) = [f_{i+1}]$,
3. $\widehat{\Phi}_i'([f_i]) = \Phi_i([f])$.

Now we can move on to the final purpose of this section.

Proof of Theorem 7.1. First we show that $(\Phi_1,\Phi_2,\dots,\Phi_m)$ is a bijection. Since

1. $(\Phi_1,\Xi_1) = \widehat{\Phi}_1$ is a bijection,
2. the bijectivity of $(\Phi_1,\dots,\Phi_i,\Xi_i)$ and $\widehat{\Phi}_{i+1}$ imply the bijectivity of
$$(\Phi_1,\dots,\Phi_i,\Phi_{i+1},\Xi_{i+1}) = (\Phi_1,\dots,\Phi_i,\widehat{\Phi}_{i+1}\circ\Xi_i),$$
3. the set of values of Ξ_m is equal to the singleton $\mathcal{F}_G^{\nabla}[\emptyset]$,

by induction on i we obtain that $(\Phi_1,\Phi_2,\dots,\Phi_m)$ is a bijection.

Now we will prove the second part of our statement. Assume that f is (H_j)-normal. Note that $f^{-1}(0)\subset\Omega_{(H_j)}$. First observe that since $f_i = f\restriction_{D_f\cap\Omega_i}$ for $i\le j$ we have

$$\Phi_i([f]) = [f\restriction_{D_f\cap\Omega_{H_i}}] = \begin{cases}[\emptyset] & \text{if} \quad i<j,\\ [f_{H_j}] & \text{if} \quad i=j.\end{cases}$$

Note that (H_j) is maximal in $\mathrm{Iso}(\Omega_j)$. By definition, $f_j = f\restriction_{D_f\cap\Omega_j}$ and $f_{j+1} = (f_j)^{c}_{U,\epsilon}$. Hence, by Proposition 5.2, $[f_{j+1}] = [\emptyset]$ in $\mathcal{F}_G^{\nabla}[\Omega_{j+1}]$. By Lemma 8.5, $[f_i] = [\emptyset]$ in $\mathcal{F}_G^{\nabla}[\Omega_i]$ for $i>j$, and, in consequence,

$$\Phi_i([f]) = \widehat{\Phi}_i'([f_i]) = [\emptyset] \quad \text{for}\, i>j.$$

□

9. The Parusiński type theorem for equivariant local maps

The aim of this section is to compare the sets of equivariant and equivariant gradient otopy classes. Note that the inclusion $\mathcal{F}_G^{\nabla}(\Omega)\hookrightarrow\mathcal{F}_G(\Omega)$ induces the well-defined function $\mathcal{J}\colon\mathcal{F}_G^{\nabla}[\Omega]\to\mathcal{F}_G[\Omega]$. In [8] we proved that in the absence of group action the function $\mathcal{J}$ is a bijection. It turns out that in the equivariant case the following Parusiński type theorem (see [26]) holds. Define $\mathrm{Iso}_0(\Omega) := \{(H)\in\mathrm{Iso}(\Omega)\mid \dim WH = 0\}$.

Theorem 9.1. *$\mathcal{J}$ is a bijection if and only if* $\mathrm{Iso}_0(\Omega) = \mathrm{Iso}(\Omega)$.

The remainder of this section will be devoted to the proof of this result. Recall that in [5] we showed the existence of the bijection $\Upsilon\colon\mathcal{F}_G[\Omega]\to\prod_{i=1}^m\mathcal{F}_{WH_i}[\Omega_{H_i}]$ and in the present paper the existence of the bijection

$\Phi\colon \mathcal{F}_G^{\nabla}[\Omega] \to \prod_{i=1}^m \mathcal{F}_{WH_i}^{\nabla}[\Omega_{H_i}]$. It is natural to consider the following diagram

$$\begin{array}{ccc} \mathcal{F}_G^{\nabla}[\Omega] & \xrightarrow{\Phi} & \prod_{i=1}^m \mathcal{F}_{WH_i}^{\nabla}[\Omega_{H_i}] \\ \mathcal{J}\Big\downarrow & & \Big\downarrow \prod \mathcal{J}_i \\ \mathcal{F}_G[\Omega] & \xrightarrow{\Upsilon} & \prod_{i=1}^m \mathcal{F}_{WH_i}[\Omega_{H_i}], \end{array} \tag{9.1}$$

where $\mathcal{J}_i\colon \mathcal{F}_{WH_i}^{\nabla}[\Omega_{H_i}] \to \mathcal{F}_{WH_i}[\Omega_{H_i}]$ are also induced by the inclusions. The commutativity of diagram (9.1) follows from the inductive definitions of Φ and Υ and the following result.

Lemma 9.2. *Assume that (H) is maximal in* $\mathrm{Iso}(\Omega)$. *Let $\mathcal{J}'\colon \mathcal{F}_{WH}^{\nabla}[\Omega_H] \to \mathcal{F}_{WH}[\Omega_H]$ and $\mathcal{J}''\colon \mathcal{F}_G^{\nabla}\left[\Omega\backslash\Omega_{(H)}\right] \to \mathcal{F}_G\left[\Omega\backslash\Omega_{(H)}\right]$ be induced by the respective inclusions. Then the diagram*

$$\begin{array}{ccc} \mathcal{F}_G^{\nabla}[\Omega] & \xrightarrow{\widehat{\Phi}} & \mathcal{F}_{WH}^{\nabla}[\Omega_H] \times \mathcal{F}_G^{\nabla}\left[\Omega\backslash\Omega_{(H)}\right] \\ \mathcal{J}\Big\downarrow & & \Big\downarrow \mathcal{J}'\times\mathcal{J}'' \\ \mathcal{F}_G[\Omega] & \xrightarrow{\widehat{\Upsilon}} & \mathcal{F}_{WH}[\Omega_H] \times \mathcal{F}_G\left[\Omega\backslash\Omega_{(H)}\right] \end{array} \tag{9.2}$$

commutes.

Proof. Let $f = \nabla\varphi \in \mathcal{F}_G^{\nabla}(\Omega)$. Recall that $\widehat{\Phi}([f]) = ([f_H], [k])$, where

$$k(z) = \begin{cases} \nabla\left(\varphi \circ r_1\right)(z) + \nabla\omega(|v|) & \text{if } z = x + v \in U^{\epsilon}\backslash\Omega_{(H)}, \\ \nabla\varphi(z) & \text{if } z \in D_f\backslash\left(U^{\epsilon} \cup B^{\epsilon} \cup \Omega_{(H)}\right), \end{cases}$$

and $\widehat{\Upsilon}([f]) = \left([f_H], \left[\widetilde{k}\right]\right)$, where

$$\widetilde{k}(z) = \begin{cases} \nabla\varphi\left(r_1(z)\right) + \nabla\omega(|v|) & \text{if } z = x + v \in U^{\epsilon}\backslash\Omega_{(H)}, \\ \nabla\varphi(z) & \text{if } z \in D_f\backslash\left(U^{\epsilon} \cup B^{\epsilon} \cup \Omega_{(H)}\right). \end{cases}$$

First observe that $\mathcal{J}' \circ \widehat{\Phi}'([f]) = [f_H] = \widehat{\Upsilon}' \circ \mathcal{J}([f])$. To prove that $\mathcal{J}'' \circ \widehat{\Phi}'' = \widehat{\Upsilon}'' \circ \mathcal{J}$ it is enough to show that the straight-line homotopy $h_t = (1-t)k + t\widetilde{k}$ is an otopy in $\mathcal{F}_G\left(\Omega\backslash\Omega_{(H)}\right)$. To see that, note that

$$k(z) = \nabla\left(\varphi \circ r_1\right)(z) + \nabla\omega(|v|) = \left(Dr_1(z)\right)^T \cdot \nabla\varphi(r_1(z)) + \nabla\omega(|v|) \quad \text{on } U^{\epsilon}\backslash\Omega_{(H)}.$$

Using orthogonal coordinates x being principal directions corresponding to the principal curvatures of $\Omega_{(H)}$ and orthogonal coordinates v, we get that the matrix $\left(Dr_1(z)\right)^T = \left(D(x + \mu(|v|)v)\right)^T$ has the following form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right),$$

where A is diagonal with positive entries on the diagonal and $B = D(\mu(|v|)v)$ is nonsingular iff $2\epsilon/3 < |v| < \epsilon$. Therefore,

$$h_t(z) = 0 \equiv k(z) = 0 \equiv \widetilde{k}(z) = 0$$

for all $t \in I$ and, in consequence, h is an otopy. □

Since from Diagram (9.1) $\mathcal{J}$ is a bijection if and only if all $\mathcal{J}_i$ are bijections and the action of WH_i on Ω_{H_i} is free, it remains to study the function $\mathcal{J}\colon \mathcal{F}_G^\nabla[\Omega] \to \mathcal{F}_G[\Omega]$ assuming G acts freely on Ω. We will consider two cases. If $\dim G > 0$ then $\mathcal{F}_G[\Omega]$ is trivial (see [5, Thm 3.1])) and $\mathcal{F}_G^\nabla[\Omega]$ is nontrivial by Corollary 3.3, and so $\mathcal{J}$ is not a bijection. Now consider the case $\dim G = 0$. Recall that $M = \Omega/G$, $E = (\Omega \times V)/G$ and $\Gamma[M,E]$ denotes the set of otopy classes of local cross sections of the bundle $E \to M$. The following commutative diagram (9.3) relates the sets of different otopy classes. Since a, b, c, d are bijections by Theorem 3.1, [5, Thm 3.4], [11, Thm 5.1] and the natural identification of vector bundles E and TM, so is $\mathcal{J}$.

$$\begin{tikzcd} \mathcal{F}_G^\nabla[\Omega] \arrow[r, "a"] \arrow[dd, dashed, "\mathcal{J}"'] & \mathcal{F}^\nabla[M] \arrow[d, "c"] \\ & \mathcal{F}[M] \\ \mathcal{F}_G[\Omega] \arrow[r, "b"] & \Gamma[M,E] \arrow[u, "d"'] \end{tikzcd} \tag{9.3}$$

The commutativity of Diagram (9.1) and the above considerations concerning the single function $\mathcal{J}_i\colon \mathcal{F}_{WH_i}^\nabla[\Omega_{H_i}] \to \mathcal{F}_{WH_i}[\Omega_{H_i}]$ complete the proof of Theorem 9.1.

Remark 9.3. It may be worth noting the relation of our constructions with the additive subgroups $U(V)$ and $A(V)$ of the Euler ring $U(G)$ and the Burnside ring $A(G)$, taking into account only their additive structure. Namely, denoting by Θ_U the function Θ from Main Theorem, by Θ_A its nongradient version from [5], by π_U and π_A the summing on j and by $\mathcal{U}$ the forgetful functor, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{F}_G^\nabla[\Omega] \arrow[r, "\Theta_U"] \arrow[d, "\mathcal{J}"'] & \prod\limits_{\mathrm{Iso}(\Omega)} \left(\sum\limits_j \mathbb{Z}\right) \arrow[r, "\pi_U"] \arrow[d, "\mathcal{U}"] & U(V) = \prod\limits_{\mathrm{Iso}(\Omega)} \mathbb{Z} \arrow[d, "\mathcal{U}"] \\ \mathcal{F}_G[\Omega] \arrow[r, "\Theta_A"] & \prod\limits_{\mathrm{Iso}_0(\Omega)} \left(\sum\limits_j \mathbb{Z}\right) \arrow[r, "\pi_A"] & A(V) = \prod\limits_{\mathrm{Iso}_0(\Omega)} \mathbb{Z}. \end{tikzcd} \tag{9.4}$$

Moreover, observe that $\pi_U \circ \Theta_U = \deg_G^\nabla$ and $\pi_A \circ \Theta_A = \deg_G$ using the notation of the respective degrees from [7].

10. Parametrized equivariant gradient local maps

We close the paper with some remarks concerning the parametrized case. Recall that $\mathbb{R}^k$ denotes a trivial representation of G. Let Ω be an open invariant subset of $\mathbb{R}^k \oplus V$. Define $\mathcal{F}_G(\Omega) = \mathrm{Loc}_G(\Omega, V, 0)$. Let (y,x) denote the coordinates $\mathbb{R}^k \oplus V$. A map $f \in \mathcal{F}_G(\Omega)$ is called *gradient* if there is a function (not necessarily continuous) $\varphi\colon D_f \to \mathbb{R}$ such that φ is C^1 with respect to

x and $f(x,y) = \nabla_x\varphi(x,y)$. Define $\mathcal{F}_G^{\nabla}(\Omega) = \{f \in \mathcal{F}_G(\Omega) \mid f \text{ is gradient}\}$. Similarly as in the nonparametrized case, we define gradient otopies and the set of gradient otopy classes $\mathcal{F}_G^{\nabla}[\Omega]$.

It is easily seen that Theorem 7.1 also holds in that case, because both the construction of the function $\Phi\colon \mathcal{F}_G^{\nabla}[\Omega] \to \prod_{i=1}^m \mathcal{F}_{WH_i}^{\nabla}[\Omega_{H_i}]$ and the proof of Theorem 7.1 are essentially the same.

Now let us describe the single factor of the above decomposition. Assume that G acts freely on $\Omega \subset \mathbb{R}^k \oplus V$. Set $M = \Omega/G$. Define $\pi\colon M \to \mathbb{R}$ by $\pi([y,x]) = y$. Observe that $M_y = \pi^{-1}(y)$ is a submanifold of M (possibly empty) for every $y \in \mathbb{R}^k$. Let us consider a subbundle $E \subset TM$ defined by $E_p = T_pM_{\pi(p)}$ for $p \in M$. The set of all local continuous sections of the bundle E will be denoted by $\Gamma(M,E)$. A local section $s\colon D_s \subset M \to E$ is called *gradient* if there is a function $\varphi\colon D_s \to \mathbb{R}$ (not necessarily continuous) such that φ is C^1 on every $D_s \cap M_y$ and

$$s(p) := \nabla_\pi\varphi(p) = \nabla\left(\varphi\restriction_{D_s\cap M_{\pi(p)}}\right)(p).$$

Write $\mathcal{F}_\pi^{\nabla}(M) := \{s \in \Gamma(M,E) \mid s \text{ is gradient}\}$. Recall that if $\varphi\colon U \to \mathbb{R}$ is an invariant function then $\widetilde{\varphi}$ stands for the quotient function $\widetilde{\varphi}\colon U/G \to \mathbb{R}$. Finally, let $\Psi\colon \mathcal{F}_G^{\nabla}(\Omega) \to \mathcal{F}_\pi^{\nabla}(M)$ be defined by $\Psi(\nabla_x\varphi) = \nabla_\pi\widetilde{\varphi}$. The following result, which is an analogue of Theorem 3.1, allows us to replace single factors of our decomposition Φ by the sets of gradient otopy classes of parametrized maps on quotient manifolds (the free G-action has been divided out).

Proposition 10.1. *Ψ is a bijection and induces a bijection between the sets of gradient otopy classes $\mathcal{F}_G^{\nabla}[\Omega]$ and $\mathcal{F}_\pi^{\nabla}[M]$.*

Remark 10.2. It is worth pointing out that for now we do not have a satisfactory classification of the set $\mathcal{F}_\pi^{\nabla}[M]$ even in the simplest case of trivial action, where $M = \Omega = \mathbb{R}^k \oplus \mathbb{R}^n$ (see Question 1 in [10]).

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