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The outer-connected domination number of a graph

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Abstract

For a given graph $G = (V, E)$, a set $D \subseteq V(G)$ is said to be an *outer-connected dominating set* if D is dominating and the graph $G - D$ is connected. The *outer-connected domination number* of a graph G, denoted by $\widetilde{\gamma}_c(G)$, is the cardinality of a minimum outer-connected dominating set of G. We study several properties of outer-connected dominating sets and give some bounds on the outer-connected domination number of a graph. We also show that the decision problem for the outer-connected domination number of a graph G is NP-complete even for bipartite graphs.

1 Introduction

Graph theory terminology not presented here can be found in [1, 5].

Let $G = (V, E)$ be a simple graph. The *neighbourhood* of a vertex v, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G. If v is a vertex of G then the integer $\deg_G(v) = |N_G(v)|$ is said to be the *degree* of v in G. The *minimum* and *maximum degree* among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree one in a graph is called an *end-vertex*. A *support* is the unique neighbour of an end-vertex. Let $\Omega = \Omega(G)$ be the set of all end-vertices of G.

A set $D \subseteq V(G)$ is a *dominating set* in G if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V(G) - D$. The *domination number* of a graph G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G.

A set $D \subseteq V(G)$ is said to be an *outer-connected dominating set* of G if D is dominating and either $D = V(G)$ or $G - D$ is connected. The cardinality of a minimum outer-connected dominating set in G is called the *outer-connected domination number* of G and is denoted by $\widetilde{\gamma}_c(G)$. Observe that every graph G has an outer-connected dominating set, since the set of all vertices of G is an outer-connected dominating set in G.

2 Preliminary results

Let K_n, C_n and P_n denote the complete graph, the cycle and the path of order n, respectively. For positive integers $n_1, \dots, n_t$ let $K_{n_1,\dots,n_t}$ be the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \dots \cup S_t$, where $|S_i| = n_i$ for $1 \leq i \leq t$. By a star of order n we mean the bipartite graph $K_{1,n-1}$ for $n \geq 2$.

In our first observation we present the outer-connected domination number of complete graphs, cycles, paths and complete multipartite graphs.

Observation 1

(i) $\widetilde{\gamma}_c(K_n) = 1$ *for* $n \geq 1$*;*

(ii) $\widetilde{\gamma}_c(C_n) = n - 2$ *for* $n \geq 3$*;*

(iii) $\widetilde{\gamma}_c(P_n) = \begin{cases} n-1, & n = 2,3, \\ n-2, & n \geq 4; \end{cases}$

(iv) If $t \geq 2$ *and* $n_1 \leq n_2 \leq \dots \leq n_t$*, then*

$$\widetilde{\gamma}_c(K_{n_1,\dots,n_t}) = \begin{cases} n_2 & if \quad t = 2 \quad and \quad n_1 = 1, \\ 1 & if \quad t \geq 3 \quad and \quad n_1 = 1, \\ 2 & if \quad t \geq 2 \quad and \quad n_1 > 1. \end{cases}$$

It follows from the next theorem and from its proof that outer-connected dominating sets and outer-connected domination numbers of a disconnected graph are determined by outer-connected dominating sets and outer-connected domination numbers of its components.

Theorem 1 *If* $G_1, \dots, G_r$ *are the components of a graph* G*, then*

$$\widetilde{\gamma}_c(G) = |V(G)| - \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \dots, r\}.$$

Proof. Let $D_1, \dots, D_r$ be minimum outer-connected dominating sets of $G_1, \dots, G_r$ respectively. Then $V(G) - (V(G_1) - D_1), \dots, V(G) - (V(G_r) - D_r)$ are outer-connected dominating sets of G and therefore

$$\begin{aligned} \widetilde{\gamma}_c(G) &\leq \min\{|V(G) - (V(G_i) - D_i)| \colon i = 1, \dots, r\} \\ &= |V(G)| - \max\{|V(G_i) - D_i| \colon i = 1, \dots, r\} \\ &= |V(G)| - \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \dots, r\}. \end{aligned}$$

Now let D be a minimum outer-connected dominating set of G. Then $|D| = \widetilde{\gamma}_c(G)$ and in addition $G - D$ is connected. Hence $V(G) - D \subseteq V(G_l)$ for some $l \in \{1, \dots, r\}$ and from the minimality of D it follows that $D \cap V(G_l)$ is a minimum outer-connected dominating set of G_l. Thus $D \cap V(G_l) = \widetilde{\gamma}_c(G_l)$ and $|V(G)| - \widetilde{\gamma}_c(G) = |V(G) - D| = |V(G_l) - (D \cap V(G_l))| = |V(G_l)| - \widetilde{\gamma}_c(G_l) \leq \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \dots, r\}$ and therefore $\widetilde{\gamma}_c(G) \geq |V(G)| - \max\{|V(G_i)| - \widetilde{\gamma}_c(G_i) \colon i = 1, \dots, r\}$ which completes the proof. □

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3 Bounds

It is obvious that if G is a graph of order n, then $1 \leq \widetilde{\gamma}_c(G) \leq n$. In addition, $\widetilde{\gamma}_c(G) = 1$ if and only if $G = K_1 + H$, where H is a connected graph of order $n-1$, while $\widetilde{\gamma}_c(G) = n$ if and only if $G = \overline{K_n}$. Hence $\widetilde{\gamma}_c(G) \leq n-1$ if G has at least one edge. Moreover, $\widetilde{\gamma}_c(G) \leq n-2$ if and only if G has at least one edge which is not an end-edge. In general, $\widetilde{\gamma}_c(G) \leq n-k$ if and only if there exists a proper connected subgraph H of G such that $|V(H)| = k$ and every vertex of H has a neighbour which belongs to $V(G) - V(H)$.

A characterization of graphs G of order n for which $\widetilde{\gamma}_c(G) = n-1$ is given in the following observation.

Observation 2 *If G is a connected graph on $n \geq 2$ vertices, then $\widetilde{\gamma}_c(G) = n-1$ if and only if G is a star.*

Sampathkumar and Walikar [7] have proved that $\frac{n(G)}{\Delta(G)+1} \leq \gamma_c(G) \leq 2m(G) - n(G)$ for a connected graph G. Now we present similar inequalities for the outer-connected domination number.

Let $\mathcal{A}$ be the family of graphs defined as follows: a graph G belongs to $\mathcal{A}$ if and only if there exists an outer-connected dominating set A of G such that $|PN_G[v, A]| = \Delta(G) + 1$ for every vertex v belonging to A, where $PN_G[v, A]$ is the private neighbourhood of v with respect to A, i.e. $PN_G[v, A] = N_G[v] - N_G[A - \{v\}]$.

Theorem 2 *If G is a connected graph with $n(G) \geq 2$, then*

$$\frac{n(G)}{\Delta(G)+1} \leq \widetilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1.$$

In addition, $\widetilde{\gamma}_c(G) = \frac{n(G)}{\Delta(G)+1}$ if and only if G belongs to the family $\mathcal{A}$, while $\widetilde{\gamma}_c(G) = 2m(G) - n(G) + 1$ if and only if G is a star.

Proof. Since $\frac{n(G)}{\Delta(G)+1} \leq \gamma(G)$ (see [8]) and $\gamma(G) \leq \widetilde{\gamma}_c(G)$, we certainly have $\frac{n(G)}{\Delta(G)+1} \leq \widetilde{\gamma}_c(G)$. Moreover, since G is connected and has at least two vertices, we have $m(G) \geq n(G) - 1$ and $\widetilde{\gamma}_c(G) \leq n(G) - 1$. Consequently, $\widetilde{\gamma}_c(G) \leq 2m(G) - n(G) + 1$.

If G belongs to the family $\mathcal{A}$, then there exists an outer-connected dominating set A of G for which $\widetilde{\gamma}_c(G) \leq |A| = \frac{n(G)}{\Delta(G)+1} \leq \widetilde{\gamma}_c(G)$. Now assume that $\widetilde{\gamma}_c(G) = \frac{n(G)}{\Delta(G)+1}$ and let D be a minimum outer-connected dominating set in G. Then $|D| = \frac{n(G)}{\Delta(G)+1}$ and this forces each vertex of D to dominate exactly $\Delta(G)+1$ vertices and moreover $|PN_G[v, D]| = \Delta(G) + 1$. Consequently $G \in \mathcal{A}$.

If G is a star, then $\widetilde{\gamma}_c(G) = n(G) - 1 = 2m(G) - n(G) + 1$. If $\widetilde{\gamma}_c(G) = 2m(G) - n(G) + 1$, then, since G has at least one edge, $2m(G) - n(G) + 1 = \widetilde{\gamma}_c(G) \leq n(G) - 1$. Thus $m(G) \leq n(G) - 1$ and by the connectivity of G, $m(G) = n(G) - 1$. Consequently, $\widetilde{\gamma}_c(G) = n(G) - 1$ and, according to Observation 2, G is a star. □

Before stating the next theorem, we describe a family $\mathcal{S}$ of graphs, which are the extremal graphs of the theorem.

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Let $\mathcal{S}$ be the family of graphs, where a graph G belongs to $\mathcal{S}$ if and only if there exists an independent set I in G such that $G-I$ is a tree and every vertex of $G-I$ is adjacent to exactly one vertex of I.

Theorem 3 *If G is a graph, then*

$$\widetilde{\gamma}_c(G) \geq n(G) - \frac{m(G)+1}{2}. \tag{1}$$

In addition, $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$ if and only if G belongs to the family $\mathcal{S}$.

Proof. Let D be a minimum outer-connected dominating set in G and let $m_G(D)$ denote the number of edges joining D and $V(G)-D$ in G. Then

$$m(G-D) \geq n(G) - \widetilde{\gamma}_c(G) - 1, \tag{2}$$

and

$$m_G(D) \geq n(G) - \widetilde{\gamma}_c(G). \tag{3}$$

Hence

$$m(G) \geq m(G-D) + m_G(D) \geq 2n(G) - 2\widetilde{\gamma}_c(G) - 1 \tag{4}$$

and thus $\widetilde{\gamma}_c(G) \geq n(G) - \frac{m(G)+1}{2}$.

We now prove that $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$ if and only if G belongs to the family $\mathcal{S}$.

Assume first that G belongs to $\mathcal{S}$. Then there exists an independent set I in G such that $G-I$ is a tree and every vertex of $G-I$ is adjacent to exactly one vertex of I. The set I is an outer-connected dominating set in G. Thus $\widetilde{\gamma}_c(G) \leq |I| = n(G) - n(G-I)$ and because $n(G-I) = m_G(I) = m(G) - m(G-I) = m(G) - n(G-I) + 1$ (and therefore $n(G-I) = \frac{m(G)+1}{2}$),

$$\widetilde{\gamma}_c(G) \leq n(G) - n(G-I) = n(G) - \frac{m(G)+1}{2}.$$

Consequently, by (1), $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$.

Assume now that $\widetilde{\gamma}_c(G) = n(G) - \frac{m(G)+1}{2}$. Let D be a minimum outer-connected dominating set of G. Then by (4) we have

$$\begin{aligned} m(G) &\geq m(G-D) + m_G(D) \\ &\geq 2n(G) - 2\widetilde{\gamma}_c(G) - 1 \\ &= 2n(G) - 2\left(n(G) - \tfrac{m(G)+1}{2}\right) - 1 = m(G). \end{aligned} \tag{5}$$

Hence, and by (2) and (3), it follows that

$$m(G-D) = n(G) - \widetilde{\gamma}_c(G) - 1 \tag{6}$$

and

$$m_G(D) = n(G) - \widetilde{\gamma}_c(G). \tag{7}$$

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Since $G-D$ is connected (by the choice of D) and has $n(G-D)-1$ edges, $G-D$ is a tree. Moreover, since $m(G)=m(G-D)+m_G(D)+m(G[D])$ and $m(G)=m(G-D)+m_G(D)$ by (5), $m(G[D])=0$ and D is an independent set. Now, since D is dominating, it follows from (7) that each vertex of $G-D$ has exactly one neighbour in D. This completes the proof of the fact that G belongs to the family S. □

Probably, the statement of the next lemma is well-known, but since we have not seen such a result anywhere, we state it here with a short proof.

Lemma 4 *In a graph G with $\delta(G)\geq 2$ there is a cycle of length at least $\delta(G)+1$.*

Proof. Let $(v_0,v_1,\ldots,v_l)$ be a longest path in G. Then $N_G(v_0)\subseteq\{v_1,v_2,\ldots,v_l\}$ and therefore $v_k\in N_G(v_0)\cap\{v_1,v_2,\ldots,v_l\}$ for some $k\geq \deg_G(v_0)\geq\delta(G)$. Consequently $(v_0,v_1,\ldots,v_k,v_0)$ is a required cycle. □

Theorem 5 *If G is a connected graph of order n, then*

$$\widetilde{\gamma}_c(G)\leq n-\delta(G).$$

Proof. The result is obvious if $\delta(G)\leq 2$. Now assume that $\delta(G)\geq 3$. By Lemma 4 there exists a cycle of length at least $\delta(G)+1$ in G. Let $C=(v_0,v_1\ldots,v_l,v_0)$ be a shortest cycle in G of length at least $\delta(G)$. We claim that the set $D=V(G)-V(C)$ is an outer-connected dominating set of G. Certainly $G-D=G[V(C)]$ is connected. Suppose D is not dominating. Then $N_G(v)\cap D=\emptyset$ for some vertex $v\in V(C)$. We may assume, without loss of generality, that $v=v_0$ and $\deg_G(v_0)=r$. Then $N_G(v_0)=\{v_1,v_{i_1},v_{i_2},\ldots,v_{i_{r-2}},v_l\}$ where $1<i_1<i_2<\ldots<i_{r-2}<l$. Now $(v_0,v_{i_1},v_{i_1+1},\ldots,v_l,v_0)$ is a cycle of length at least $\delta(G)$ which is shorter than C, a contradiction. □

In the next observation we describe the main properties of minimum outer-connected dominating sets of a graph.

Observation 3 *Let G be a connected graph on at least 3 vertices. If D is a minimum outer-connected dominating set in G and Ω is the set of end-vertices of G, then*

(i) $\Omega\subseteq D$ *if G is not a star;*

(ii) $D=\Omega$ *or* $|\Omega\cap D|=|\Omega|-1$ *if G is a star;*

(iii) $\widetilde{\gamma}_c(G)\geq|\Omega|$;

(iv) $\widetilde{\gamma}_c(G)=|\Omega|$ *if and only if every vertex of G is either a support or an end-vertex.*

Proof. Since (ii) is obvious and (iii) easily follows from (i) and Observation 2, we only prove (i) and (iv).
(i) Assume G is not a star and suppose to the contrary that $\Omega-D\neq\emptyset$. Then, since $G-D$ is connected, $\widetilde{\gamma}_c(G)=|V(G)|-1$ and therefore, by Observation 2, G is a star, a contradiction.
(iv) The statement is trivial for stars. Thus assume G is not a star and $\widetilde{\gamma}_c(G)=|\Omega|$.

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Then, by (i), Ω is a minimum outer-connected dominating set of G and this implies that every vertex belonging to $V(G) - \Omega$ is a support. Conversely, if every vertex of the graph G is a support or an end-vertex, then Ω is an outer-connected dominating set of G and, by (i), it is a minimum outer-connected dominating set of G and therefore $\widetilde{\gamma}_c(G) = |\Omega|$. □

A subdivision of an edge uv is obtained by inserting a new vertex w and replacing the edge uv with the edges uw and wv. A spider is the tree obtained from a star by subdividing all of its edges. A wounded spider is a tree obtained from a spider by removing at least one end-vertex. Certainly, a star is also a wounded spider.

The next theorem provides a lower bound for the outer-connected domination number of a tree.

Theorem 6 *If T is a tree of order $n \geq 3$, then*

$$\widetilde{\gamma}_c(T) \geq \Delta(T).$$

Furthermore, $\widetilde{\gamma}_c(T) = \Delta(T)$ if and only if T is a wounded spider.

Proof. The result is obvious if T is a star. Now let T be a tree of order $n \geq 3$ and assume T is not a star. Since T has at least $\Delta(T)$ end-vertices and since all end-vertices belong to every outer-connected dominating set of T we certainly have $\widetilde{\gamma}_c(T) \geq \Delta(T)$.

Clearly, if T is a wounded spider, then $\widetilde{\gamma}_c(T) = \Delta(T)$. Now assume T is a tree for which $\widetilde{\gamma}_c(T) = \Delta(T)$. Then since $\Delta(T) \leq |\Omega(T)| \leq \widetilde{\gamma}_c(T)$ we have $\Delta(T) = |\Omega(T)|$ (and $\widetilde{\gamma}_c(T) = |\Omega(T)|$). From the equality $\Delta(T) = |\Omega(T)|$, it follows that there exists a unique vertex, say u, of maximum degree, and $\Omega(T)$ is a minimum outer-connected dominating set of T. In addition, every inner vertex of a path joining u to an end-vertex of T (if any) is of degree 2. This and the fact that $\Omega(T)$ is dominating implies that every such a path is of length at most two and at least one of them is of length one. This proves that T is a wounded spider. □

Let $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{R}''$ be families of trees on at least 3 vertices defined as follows: a tree T belongs to $\mathcal{R}$ if T is the corona of another tree, while a tree T belongs to $\mathcal{R}'$ or $\mathcal{R}''$, respectively, if T is obtained from a tree S belonging to $\mathcal{R}$ by adding a new vertex and joining it to an end-vertex of S or to an inner vertex of S, respectively.

Theorem 7 *If T is a tree of order $n \geq 3$, then*

$$\widetilde{\gamma}_c(T) \geq \left\lceil \frac{n}{2} \right\rceil$$

with equality $\widetilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$ if and only if T belongs to $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$.

Proof. Let $T = (V, E)$ be a tree and let D be a minimum outer-connected dominating set of T. Suppose, on the contrary, that $\widetilde{\gamma}_c(T) < \lceil \frac{n}{2} \rceil$. Then $\widetilde{\gamma}_c(T) < \frac{n}{2}$ and by the pigeon hole principle $|N_T(v) \cap (V - D)| \geq 2$ for some $v \in D$. But then any path joining two vertices of $N_T(v) \cap (V - D)$ in the connected graph $T - D$ together with v form a cycle in T, which is impossible.

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Now we prove that $\widetilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$ if and only if T belongs to $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$.

If $T \in \mathcal{R}$, then T is a corona and $\Omega(T)$ is a minimum outer-connected dominating set of T and $\widetilde{\gamma}_c(T) = |\Omega(T)| = \frac{n}{2} = \lceil \frac{n}{2} \rceil$.

Assume $T \in \mathcal{R}' \cup \mathcal{R}''$. Then there exists an end-vertex v such that $T - v$ is a corona. If $T \in \mathcal{R}'$ and u is a neighbour of v, then $\Omega(T) \cup \{u\}$ is a minimum outer-connected dominating set of T and $\widetilde{\gamma}_c(T) = |\Omega(T)| + 1 = \frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$. Finally, if $T \in \mathcal{R}''$, then $\Omega(T)$ is a minimum outer-connected dominating set of T and $\widetilde{\gamma}_c(T) = |\Omega(T)| = \lceil \frac{n}{2} \rceil$.

Let T be a tree of order at least 3 such that $\widetilde{\gamma}_c(T) = \lceil \frac{n}{2} \rceil$. If $n = 3$, then certainly $T = P_3 \in \mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$. Thus assume T has at least 4 vertices. Then $\lceil \frac{n}{2} \rceil < n - 1$ which implies that $\widetilde{\gamma}_c(T) < n - 1$, so T is not a star. Consequently, by Observation 3, $\Omega(T) \subseteq D$. If $D = \Omega(T)$, then $|\Omega(T)| = \lceil \frac{n}{2} \rceil$ and therefore every vertex belonging to $V - \Omega(T)$ is adjacent to exactly one vertex in $\Omega(T)$ or one of them is adjacent to two end-vertices and each of the other vertices is adjacent to exactly one end-vertex. This implies T belongs to $\mathcal{R}$ or $\mathcal{R}''$.

Finally assume $\Omega(T) \subsetneq D$. Then there exists a vertex $v \in D$ such that $\deg_T(v) \geq 2$. We shall prove that $\deg_T(v) = 2$ and v is the only such vertex. From the connectivity of $T - D$ it follows that $|N_T(v) \cap (V - D)| \leq 1$. We claim that $|N_T(v) \cap D| \leq 1$. Suppose, to the contrary, that two vertices x and y belong to $N_T(v) \cap D$. Since T is a tree we have $|N_T(\{x, y, v\}) \cap (V - D)| \leq 1$. This, and the fact that no two vertices in $V - D$ share common neighbour in D, imply that $|V - D| = |N_T(\{x, y, v\}) \cap (V - D)| + |N_T(D - \{x, y, v\}) \cap (V - D)| \leq 1 + |D| - 3 = |D| - 2$. Hence, $n - |D| \leq |D| - 2$ and $|D| \geq \frac{n}{2} + 1 > \lceil \frac{n}{2} \rceil$. Thus v has exactly one neighbour in D and exactly one neighbour in $V - D$. Suppose now that $|D - \Omega(T)| \geq 2$. Then there exist $u, v \in D$ such that $\deg_T(u) = \deg_T(v) = 2$. Denote by u_1 and v_1 the neighbour of u and v in D, respectively. Then $|V - D| = |N_T(\{u, u_1, v, v_1\}) \cap (V - D)| + |N_T(D - \{u, u_1, v, v_1\}) \cap (V - D)| \leq 2 + |D| - 4 = |D| - 2$. Hence, $n - |D| \leq |D| - 2$ and $\lceil \frac{n}{2} \rceil = |D| \geq \frac{n}{2} + 1 > \lceil \frac{n}{2} \rceil$, a contradiction. We obtain that v is the unique vertex of degree two in D and the vertex $x \in N_T(v) \cap D$ is an end-vertex of T. Thus, we conclude that T belongs to the family $\mathcal{R}'$. □

As an immediate consequence of this theorem and of Ore's theorem [6] we have the following corollary.

Corollary 1 *For a tree $T \neq K_1$ we have $\widetilde{\gamma}_c(T) = \gamma(T)$ if and only if T is a corona.*

4 Edge subdivision and vertex removing

Now we examine the effects on $\widetilde{\gamma}_c(G)$ when G is modified by an edge subdivision. We start with some notation. If uv is an edge of G then by $G \oplus w_{uv}$ we denote the graph obtained from G by the subdivision of uv.

Theorem 8 *For every integer k there exist a graph G and an edge uv of G such that $\widetilde{\gamma}_c(G \oplus w_{uv}) - \widetilde{\gamma}_c(G) = k$.*

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Proof. We consider three cases.

Case 1. If $k \leq -2$ then we construct graphs G and $G \oplus w_{uv}$ as follows. We begin with four spiders S_i with $|V(S_i)| = 2|k| - 1$ and denote its centers by x_i, $i = 1, 2, 3, 4$. Next we add four end-vertices y_i and four edges $x_i y_i$. Finally, to obtain the graph G, we add vertices u, v and edges $uv, ux_1, ux_2, vx_3, vx_4$, x_1x_3 (see Fig. 1). It is easy to observe that $D = N_G[x_2] \cup \Omega(G)$ is a minimum outer-connected dominating set of G and thus $\widetilde{\gamma}_c(G) = 5|k| + 1$.

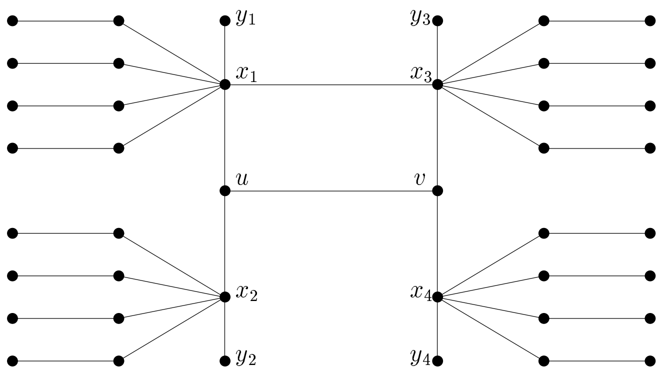

Figure 1: Graph G for $k = -5$

Let $G \oplus w_{uv}$ be a graph which results if the edge $e = uv$ is subdivided (see Fig. 1). Notice that $D = \{w\} \cup \Omega(G)$ is the minimum outer-connected dominating set of $G \oplus w_{uv}$ of cardinality $4|k|+1$. Thus $\widetilde{\gamma}_c(G \oplus w_{uv}) = 4|k|+1$ and $\widetilde{\gamma}_c(G \oplus w_{uv}) - \widetilde{\gamma}_c(G) = -|k| = k$.

Case 2. Define $A = \{u, v, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$. Then for $k = -1, 0, 1$ let H be the subgraph of G induced by A, $A - \{y_4\}$, $A - \{y_3, y_4\}$, respectively. The difference $\widetilde{\gamma}_c(H \oplus w_{uv}) - \widetilde{\gamma}_c(H) = k$ is easy to verify.

Case 3. If $k \geq 2$, then let G be the join $P_{3k} + K_1$, where $x_1, x_2, \ldots, x_{3k}$ are consecutive vertices of P_{3k} and x is the universal vertex of G. Then obviously $\{x\}$ is a minimum outer-connected dominating set of G, so $\widetilde{\gamma}_c(G) = 1$. It is also easy to see that $D = \{x, x_1, \ldots, x_k\}$ is a minimum outer-connected dominating set of $G \oplus w_{x_k x_{k+1}}$. Hence $\widetilde{\gamma}_c(G \oplus w_{x_k x_{k+1}}) = k + 1$ and the proof is complete. □

Now we investigate how removing a vertex influences an outer-connected domination number. We have the following two propositions.

Proposition 9 *For every connected graph G and a vertex $v \in V(G)$ such that $G - v$ is connected we have*

$$\widetilde{\gamma}_c(G) \leq \widetilde{\gamma}_c(G - v) + 1.$$

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Proof. If D is a minimum outer-connected dominating set of $G-v$, then clearly $D\cup\{v\}$ is a minimum outer-connected dominating set in G and therefore $\widetilde{\gamma}_c(G)\leq\widetilde{\gamma}_c(G-v)+1$. □

Proposition 10 *For every integer $k\geq -1$, there exists a graph G such that $\widetilde{\gamma}_c(G-v)-\widetilde{\gamma}_c(G)=k$.*

Proof. If $k\geq 1$, then let G be the graph which results if we add to a path P_{k+3} a vertex v and edges joining v to all vertices from the path. The vertex v is a universal non-cut vertex of G and thus we have $\widetilde{\gamma}_c(G)=1$. Next we remove v with all edges incident to v. Notice that $G-v$ is a path on at least four vertices, so by Observation 1, $\widetilde{\gamma}_c(G-v)=k+3-2$. Thus $\widetilde{\gamma}_c(G-v)-\widetilde{\gamma}_c(G)=k$. For $k=0$ and $k=-1$ let G be a path P_2 and P_3, respectively, and let v be an end-vertex of G. It is easy to verify that $\widetilde{\gamma}_c(G-v)-\widetilde{\gamma}_c(G)=k$. □

5 Comparing $\widetilde{\gamma}_c$ to other types of domination numbers

In this section we investigate relations between the outer-connected domination number and other types of domination numbers. We begin with some definitions.

A set $D\subseteq V(G)$ is a *connected dominating set* of G if it is dominating and the induced subgraph $G[D]$ is connected. The cardinality of a minimum connected dominating set of G is the *connected domination number* and is denoted by $\gamma_c(G)$.

We say that a set $D\subseteq V(G)$ is a *doubly connected dominating set* of G if it is dominating and the induced subgraphs $G[D]$ and $G[V(G)-D]$ are connected. The cardinality of a minimum doubly connected dominating set in G is a *doubly connected domination number* and is denoted by $\gamma_{cc}(G)$. Properties of the doubly connected domination number of a graph are studied in [2].

A set $D\subseteq V(G)$ is a *restrained dominating set* if every vertex in $V(G)-D$ is adjacent to a vertex in D and to another vertex in $V(G)-D$. By $\gamma_r(G)$ we denote the size of a smallest restrained dominating set of G. This type of domination was studied for example in [3].

Since for an arbitrary graph G every connected dominating set is a dominating set and every doubly connected dominating set is a connected dominating set, we have the following inequality chain

$$\gamma(G)\leq\widetilde{\gamma}_c(G)\leq\gamma_{cc}(G).$$

However, each of the differences $\widetilde{\gamma}_c(G)-\gamma(G)$ and $\gamma_{cc}(G)-\widetilde{\gamma}_c(G)$ may be arbitrarily large.

Proposition 11 *For any non-negative integers r and t, there exists a graph G such that $\widetilde{\gamma}_c(G)-\gamma(G)=r$ and $\gamma_{cc}(G)-\widetilde{\gamma}_c(G)=t$.*

Proof. Let G be the graph obtained from the star $K_{1,r+t+1}$ by subdividing t of its edges. It easy to verify that $\gamma(G)=1+t$, $\widetilde{\gamma}_c(G)=r+t+1$ and $\gamma_{cc}(G)=|V(G)|-1=r+2t+1$. □

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In the next proposition we prove that the numbers $\widetilde{\gamma}_c(G)$ and $\gamma_c(G)$ are incomparable.

Proposition 12 *For every positive integer r there exist graphs G_1 and G_2 such that $\widetilde{\gamma}_c(G_1) - \gamma_c(G_1) = r$ and $\gamma_c(G_2) - \widetilde{\gamma}_c(G_2) = r$.*

Proof. Let G_1 be a star of order $r+2$ and let G_2 be a graph pictured in Figure 2. It is straightforward to verify that $\widetilde{\gamma}_c(G_1) = r+1$, $\gamma_c(G_1) = 1$ and $\widetilde{\gamma}_c(G_2) = r+2$, $\gamma_c(G_2) = 2r+2$. Therefore, $\widetilde{\gamma}_c(G_1) - \gamma_c(G_1) = r$ and $\gamma_c(G_2) - \widetilde{\gamma}_c(G_2) = r$. □

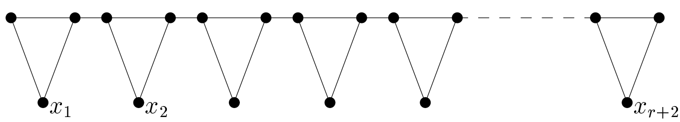

Figure 2: Graph G_2

Theorem 13 *For any connected graph G with $n(G) > 1$,*

(i) $\gamma_r(G) \leq \widetilde{\gamma}_c(G) + 1$;

(ii) $\gamma_r(G) = \widetilde{\gamma}_c(G) + 1$ *if and only if G is a star;*

(iii) *For any non-negative integer k there exists a graph G such that $\widetilde{\gamma}_c(G) - \gamma_r(G) = k$.*

Proof.

(i) If $\widetilde{\gamma}_c(G) \leq n(G) - 2$, then every outer-connected dominating set of G is a restrained dominating set of G and therefore $\gamma_r(G) \leq \widetilde{\gamma}_c(G) \leq \widetilde{\gamma}_c(G) + 1$. Otherwise $\widetilde{\gamma}_c(G) = n(G) - 1$ and, by Observation 2, G is a star, so $\gamma_r(G) = n(G) = \widetilde{\gamma}_c(G) + 1$.

(ii) The result follows immediately from (i).

(iii) Let $G = (k+1)K_2 + K_1$. It is an easy exercise to verify that $\widetilde{\gamma}_c(G) - \gamma_r(G) = k$. □

6 Complexity issues for $\widetilde{\gamma}_c$

In this section we consider the decision problem of the OUTER-CONNECTED DOMINATING SET as follows

OUTER-CONNECTED DOMINATING SET (OCDS)
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Does G have an outer-connected dominating set of cardinality at most k?

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The decision problem of OCDS stays NP–complete even when restricted to connected bipartite graphs.

To prove that the decision problem for arbitrary graphs is NP-complete, we need to use a well-known NP-completeness result, called Exact Three Cover (X3C), which is defined as follows.

EXACT COVER BY 3-SETS (X3C)
INSTANCE: A finite set X with $|X| = 3q$ and a collection $\mathcal{C}$ of 3-element subsets of X.
QUESTION: Does $\mathcal{C}$ contain an exact cover for X, that is, a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that every element of X occurs in exactly one member of $\mathcal{C}'$?

Garey and Johnson in [4] proved that X3C is NP–complete.

Theorem 14 *OCDS for bipartite graphs is NP–complete.*

Proof. We know that the OCDS problem for bipartite graphs is in the NP class of decision problems as it is easy to verify in polynomial time whether a given subset of vertices of G is an outer-connected dominating set of G. To show that OCDS is an NP-complete problem, we will establish a polynomial transformation from X3C. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of X3C.

We will construct a bipartite graph G and a positive integer k such that this instance of X3C will have an exact three cover if and only if G has an outer-connected dominating set of cardinality at most k.

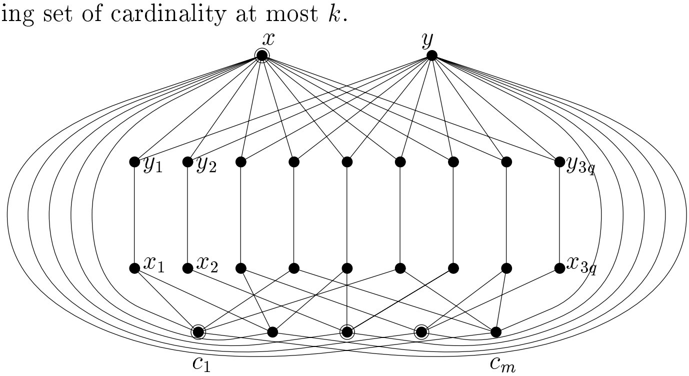

Figure 3: Reduction from X3C to OCDS

Now we describe the construction of G and k as follows:

$$
\begin{aligned}
V(G) &= \{x_1, x_2, \ldots, x_{3q}\} \cup \{y_1, y_2, \ldots, y_{3q}\} \cup \{c_1, c_2, \ldots, c_m\} \cup \{x, y\},\\
E(G) &= \{x_i y_i : i \in \{1, 2, \ldots, 3q\}\}\\
&\quad \cup \{x_i c_j : x_i \in C_j,\ i \in \{1, 2, \ldots, 3q\},\ j \in \{1, 2, \ldots, m\}\}\\
&\quad \cup \{x y_i, y y_i : i \in \{1, 2, \ldots, 3q\}\}\\
&\quad \cup \{x c_j, y c_j : j \in \{1, 2, \ldots, m\}\}\\
k &= q + 1.
\end{aligned}
$$

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The graph G so obtained is connected and bipartite.

Assume first that $\mathcal{C}$ has an exact 3-cover, say $\mathcal{C}'$. Then $\{c_j : C_j \in \mathcal{C}'\} \cup \{x\}$ is an outer-connected dominating set of cardinality $q+1$.

Now assume that D is an outer-connected dominating set of cardinality at most $q+1$. If x and y do not belong to D, then since D is dominating, at least $3q$ vertices of G belong to D to dominate y_i, $i = 1, 2, \ldots, 3q$, so $|D| \geq 3q$, a contradiction. Hence, at least one of x and y, say x, belongs to D. Notice that $N[x] \cap X = \emptyset$. Moreover, for each vertex u belonging to $\{x_1, \ldots, x_{3q}, y_1, \ldots, y_{3q}\}$, $|N_G(u) \cap X| = 1$ and for every vertex v belonging to $\{c_1, \ldots, c_m\}$, $|N_G(v) \cap X| = 3$. Hence $y \notin D$ and exactly q vertices of $\{c_1, \ldots, c_m\}$, say vertices $c_{j_1}, \ldots, c_{j_q}$, must belong to D in such a way that the corresponding set $\{C_{j_1}, \ldots, C_{j_q}\}$ is an exact cover of X. □

References

[1] G. Chartrand and L. Leśniak, *Graphs and Digraphs*, Wadsworth and Brooks/Cole, Monterey, CA, third edition, 1996.

[2] J. Cyman, M. Lemańska and J. Raczek, On doubly connected domination of a graph, *Central European J. Math.* 4(1) (2006), 34–45.

[3] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar and L.R. Markus, Restrained domination in graphs, *Discrete Math.* 203 (1999), 61–69.

[4] M.R. Garey and D.S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness*, Freeman, New York, 1979.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[6] O. Ore, Theory of Graphs, *Amer. Math. Soc. Colloq. Publ.* 38 (Amer. Math. Soc, Providence, RI), 1962.

[7] E. Sampathkumar and H.B. Walikar, The connected domination number of a graph, *J. Math. Phys. Sci.* 13 (1979), 607–613.

[8] H.B. Walikar, B.D. Acharya and E. Sampathkumar, Recent developments in the theory of domination in graphs, *MRI Lecture Notes in Math.* Mahta Research Instit., Allahabad, vol. 1, 1979.

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