Cent. Eur. J. Phys. • 12(5) • 2014 • 315-322
DOI: 10.2478/s11534-014-0449-8

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Central European Journal of **Physics**

Thermal self-action effects for acoustic beams containing fronts in a Maxwell relaxing fluid

Research Article

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Received 06 October 2013; accepted 24 February 2014

Abstract: This paper examines the thermal self-action of acoustic beams in a Maxwell relaxing fluid. This type of thermal self-action differs from that in a Newtonian fluid and behaves differently depending on a ratio of sound period and time of thermodynamic relaxation. The self-action which relates to sound beams containing shock fronts is also discussed. In addition, stationary and non-stationary types of self-action are considered.

PACS (2008): 43.25; 43.35.Fj

Keywords: thermal self-focusing of acoustic beam • shock acoustic wave • thermal lens • thermodynamic relaxation

1. Introduction

Self-focusing of powerful light waves has attracted considerable attention to self-action in wave theory [1, 2]. Self-action of optic waves arises from a dependence of the complex dielectric constant on the intensity of a wave. Hence, the local sound speed also depends on the wave's intensity. Theoretical studies on self-focusing of optic waves had considerable impact on nonlinear acoustics. Optic waves are strongly dispersive, this allows one to consider propagation of quasi-harmonic waves individually. On the contrary, the spectrum of sound waves is spread due to nonlinear generation of higher harmonics, and their profile becomes distorted because they typically propagate over weakly dispersive media [3, 4]. Nonlinear self-action is especially significant in the case of intense ultrasound waves in weakly attenuating media. Nonlinearity of sound may be weak but it increases with increasing distance from a transducer. In order to describe nonlinear dynamics of sound, the general wave theory has been enriched by analytical methods allowing one to describe acoustic pressure in the paraxial region of Gaussian beams [5] .

Reference [6] records that acoustic beams can arise from thermal self-action similar to laser beams. The nonlinear transfer of acoustic energy into that of a non-acoustic thermal mode, leads to variations in background temperature during propagation of sound over a medium. The typical attenuation specific to Newtonian fluids always causes the temperature to rise. This influences the sound speed and, as a consequence, yields refraction of the sound in a thermally inhomogeneous medium, altering the width of a sound beam. This kind of self-action is also associated with nonlinearity because the transfer of energy is a nonlinear process. However, the second specific neces-

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sary condition for this transfer is absorption of a medium, which along with nonlinearity, is a reason for interaction of acoustic and non-acoustic modes. In a Newtonian gas, where sound velocity increases with increasing temperature T, an acoustic beam is defocused, while in a liquid (except for water) with negative thermal coefficient $\delta = (\partial c/\partial T)_p/c < 0$, it is focused ($c$ denotes the infinitely small-signal sound speed in a fluid). The first theoretical results were reviewed in Ref. [7], and the first experiments confirming the theory were described in Ref. [8, 9]. Considerable attention has been paid to the thermal self-action of quasi-harmonic sound waves because results obtained in nonlinear optics are related to the field of acoustics [10]. The comprehensive review by Rudenko and Sapozhnikov [11] concentrates on the thermal self-action of beams containing shock fronts in media with quadratic and cubic nonlinearities. The scale of thermal inhomogeneities is much larger than the acoustic wavelength, and they form slowly, being characterized by a time of formation much longer than the wave period. This allows one to treat these inhomogeneities as almost stationary compared with quick acoustic perturbations. The approach of geometric acoustics implies a weak diffraction.

The issue of "sound self-action" consists of two parts: firstly, to describe the sound pressure, and secondly, to account for slow variations of background temperature due to sound propagation over a relaxing fluid and influence of these variations on a sound beam itself. A simplified system of equations includes the analogue of the Khokhlov-Zabolotskaya-Kuznetsov [KZK] [4, 5, 7] equation supplemented by the term responsible for relaxation, and an equation which describes slow dynamics of an excess temperature of the thermal mode. There are two different equations, the first describing the low-frequency sound propagation, and the second describing the high-frequency sound propagation. The mathematical content of description of sound thermal self-action is similar to that which has been developed by Rudenko et al. in studies of self-action of sound beams with discontinuities in a Newtonian fluid [11]. The thermodynamic model of a Maxwell fluid, which is refers to the form of the viscous stress tensor, which is different from a Newtonian fluid, is described in detail in Refs. [12, 13]. The viscous stress tensor takes the form:

$$P_{i,k} = 2m\rho_0 c_0^2 \int_{-\infty}^{t} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \exp(-(t-t')/t_R)\, dt', \tag{1}$$

where t denotes time, x_i are spatial co-ordinates ($i, k = 1, 2, 3$), m is the parameter responsible for relaxation, ρ_0 is the unperturbed density of a fluid, t_R is the characteristic time of relaxation, v_i denote components of a fluid velocity, and c_0 denotes the equilibrium speed of an infinitely small-signal sound in a gas. The equilibrium sound speed equals $\sqrt{\frac{C_P}{C_V k \rho_0}}$, where C_P and C_V are specific heats under constant pressure and volume respectively, and $k = \rho_0^{-1} \left(\frac{\partial \rho}{\partial p} \right)_T$ is the compressibility of a fluid.

2. The foundations and governing equations

The system of equations describing thermal self-action in an axially symmetric flow of a relaxing fluid take the form [4, 11, 14]:

$$\frac{\partial}{\partial \tau} \left(\frac{\partial p}{\partial x} - \frac{\delta T}{c_0} \frac{\partial p}{\partial \tau} - \frac{\varepsilon}{c_0^3 \rho_0} p \frac{\partial p}{\partial \tau} - \frac{m}{2c_0} \frac{\partial}{\partial \tau} \int_{-\infty}^{\tau} \frac{\partial p}{\partial \tau'} e^{-(\tau-\tau')/t_R} d\tau' \right) = \frac{c_0}{2} \Delta_\perp p. \tag{2}$$

$$\frac{\partial T}{\partial t} - \frac{\chi}{\rho_0 C_P} \Delta_\perp T = \frac{m}{\rho_0^2 c_0^2 C_P} \langle \frac{\partial p}{\partial \tau} \int_{-\infty}^{\tau} \frac{\partial p}{\partial \tau'} e^{-(\tau-\tau')/t_R} d\tau' \rangle, \tag{3}$$

where x and r are cylindrical coordinates, the x axis coincides with the axis of a beam, p is acoustic pressure, $\tau = t - x/c_0$ is the retarded time in the reference frame which moves with the sound speed c_0 in the positive direction of axis x, $\Delta_\perp$ is the Laplacian with respect to the radial coordinate, ε is the parameter of nonlinearity, and χ is the thermal conductivity. The angle brackets denote averaging over fast acoustic oscillations. The parameter responsible for the thermodynamic relaxation, m, may be expressed in terms of c_0 and the linear sound speed at infinitely large frequency, c_∞:

$$m = \frac{c_\infty^2 - c_0^2}{c_0^2} \tag{4}$$

Eq. (2) describes an acoustic pressure in a beam which propagates in the positive direction of the axis Ox. In contrast to the KZK equation, it accounts for variations in the wave speed due to changes in the temperature (the second term) [15]. Eqs. (2), (3) account for relaxation which is represented by integrals in the both equations. The term reflecting relaxation in Eq. (2), is well established [4]. With regard to the right-hand side of Eq. (3), its derivation is explained in detail in [14]. In Ref. [14], this is Eq. (23) with m denoting $\frac{c_\infty^2 - c_0^2}{2c_0^2}$. Also, the leading-order ratio for acoustic pressure and acoustic density, ρ_a, $p = c_0^2 \rho_a$, has been used to eliminate acoustic density in deriving the Eq. (3) above. Eq. (3) includes a linear term in its left-hand side originating from the thermal conduction. A discussion of the incorporation of the first and second viscosities and the thermal conduction of a fluid in the dynamic equation for acoustic heating, is found in [14]. In this study, we account for thermal conduction in the linear part of Eq. (3) which governs the entropy mode, but consider attenuation of sound itself in Eq. (2) and the acoustic source of heating only due to relaxation. In the case when the acoustic nonlinearity is important, and a beam is slightly divergent, the approximation of the geometrical acoustics is successful. For the validity of approximation of geometrical acoustics, diffraction should be insignificant over the characteristic length of self-focusing. An acoustic pressure may be found in the form which follows from the theory of geometrical acoustics [11],

$$p = p(x, r, \theta = \tau - \psi(x, r)/c_0). \tag{5}$$

This leads to equations for unknown eikonal ψ and p,

$$\frac{\partial p}{\partial x} - \frac{\varepsilon}{c_0^3\rho_0} p\frac{\partial p}{\partial \theta} - \frac{m}{2c_0}\frac{\partial}{\partial \theta}\int_{-\infty}^{\tau}\frac{\partial p}{\partial \theta'}e^{-(\theta-\theta')/t_R}d\theta' + \frac{\partial \psi}{\partial r}\frac{\partial p}{\partial r} + \frac{\Delta_\perp \psi}{2}p = 0, \tag{6}$$

$$\frac{\partial \psi}{\partial x} + \frac{1}{2}\left(\frac{\partial \psi}{\partial r}\right)^2 + \delta T = 0. \tag{7}$$

The form of solution of Eqs. (6), (7) depends on the product ωt_R, where ω is the sound frequency.

2.1. Low-frequency sound

Here, we consider $\omega t_R << 1$. In the low-frequency regime, $e^{-(\theta-\theta')/t_R}$ varies much more quickly than $\partial p/\partial\theta'$, and

$$\frac{\partial p}{\partial \theta'} \approx \frac{\partial p}{\partial \theta} + \frac{\partial^2 p}{\partial \theta^2}(\theta' - \theta), \tag{8}$$

so that Eq. (6), (3) may be rearranged as

$$\frac{\partial p}{\partial x} - \frac{\varepsilon}{c_0^3\rho_0} p\frac{\partial p}{\partial \theta} - \frac{m t_R}{2c_0}\frac{\partial^2 p}{\partial \theta^2} + \frac{m t_R^2}{2c_0}\frac{\partial^3 p}{\partial \theta^3} + \frac{\partial \psi}{\partial r}\frac{\partial p}{\partial r} + \frac{\Delta_\perp \psi}{2}p = 0, \tag{9}$$

$$\frac{\partial T}{\partial t} - \frac{\chi}{\rho_0 C_p}\Delta_\perp T = \frac{b\omega^2}{\pi^2\rho_0^3 c_0^4 C_P}A^2, \tag{10}$$

where A denotes the magnitude of acoustic pressure in a shock wave. One period of the shock wave profile may be considered as a sum of a jump and straight sawtooth portion, described by the formula [4]

$$p(x, r, \theta) = A(x, r)\left(-\frac{\omega\theta}{\pi} + f\left(\frac{\varepsilon\theta}{m t_R p_0}A(x, r)\right)\right), \tag{11}$$

where $p/p_0 = f(\theta/t_R)$ is the function which follows from the equality describing the stationary dynamics of an acoustic impulse, and

$$\frac{\theta}{t_R} = \ln\frac{(1 + p/p_0)^{m/(2\varepsilon M)-1}}{(1 - p/p_0)^{m/(2\varepsilon M)+1}}, \tag{12}$$

where M is the initial acoustic Mach number. In account of low-frequency relaxation, asymmetry in the wave profile decreases at increasing distances from a transducer. Note that the formula for acoustic pressure in a Newtonian fluid includes the term $\tanh\left(\frac{\varepsilon\theta}{b}A(x, r)\right)$ instead of the second term in the brackets in Eq. (11), where b is the total attenuation of a Newtonian fluid. The profile of the shock wave in a Newtonian fluid is always symmetric.
The term similar to the Newtonian attenuation in Eq. (9) (which is proportional to $\frac{\partial^2 p}{\partial\theta^2}$), is much larger than the term responsible for dispersion. In this limit, a relaxing medium behaves as a Newtonian fluid with

$$b = m t_R \rho_0 c_0^2. \tag{13}$$

We will consider the case of relatively strong nonlinearity , $m/(2\varepsilon M) < 1$. This allows one to consider the saw-tooth wave profile as a limit of viscous shock in a Newtonian fluid when its width tends to zero. The saw-tooth wave is periodic with the period $2\pi/\omega$. One period takes the form:

$$p(x, r, \theta) = A(x, r)\cdot\begin{cases} -\frac{\theta\omega}{\pi} - 1, -\pi < \theta\omega < 0, \\ -\frac{\theta\omega}{\pi} + 1, 0 < \theta\omega < \pi \end{cases}. \tag{14}$$

2.2. High-frequency sound

Here, we consider $\omega t_R >> 1$, where $e^{-(\theta-\theta')/t_R}$ varies insignificantly over one period of sound and may be expanded in a series as $1 - (\theta - \theta')/t_R + \ldots$. Hence Eq. (2) takes the form

$$\frac{\partial p}{\partial x} - \frac{\varepsilon}{c_0^3\rho_0} p\frac{\partial p}{\partial \Theta} + \frac{m}{2c_0 t_R}p + \frac{\partial \psi}{\partial r}\frac{\partial p}{\partial r} + \frac{\Delta_\perp \psi}{2}p = 0, \tag{15}$$

where $\Theta = \omega(t - x/c_\infty) - \psi(x, r)/c_\infty$. Eqs (15), (7) in the following new variables $P = \exp(Bx)p$, $\Psi = -\exp(Bx)\psi/B$, $X = \exp(-Bx) - 1$, where

$$B = \frac{m}{2c_0 t_R}, \tag{16}$$

may be readily rearranged into the set

$$\frac{\partial P}{\partial X}+\frac{\varepsilon}{Bc_0^3\rho_0}P\frac{\partial P}{\partial \Theta}+\frac{\partial \Psi}{\partial r}\frac{\partial P}{\partial r}+\frac{\Delta_\perp \Psi}{2}P=0, \quad (17)$$

$$(X+1)\frac{\partial}{\partial X}((X+1)\Psi)+\frac{1}{2}\left(\frac{\partial \Psi}{\partial r}\right)^2(X+1)^2+\delta T B^{-2}=0. \quad (18)$$

The dimensionless distance from a transducer where the shock wave forms (if the initial wave emitted by a transducer is sinusoidal), X_s, equals

$$X_s=-\frac{\rho_0 c_0^3 B\pi}{P_0\varepsilon\omega}=-\frac{c_0 B\pi}{M\varepsilon\omega}. \quad (19)$$

We highlight that X_s is negative, but the corresponding dimensional distance from a transducer, $-\frac{1}{B}\ln(1+X_s)$ (if it exists in real numbers) should always be positive for a beam propagating in the positive direction of axis Ox. For large negative B responsible for strong attenuation, a discontinuity does not form at all. This corresponds to $X_s\leq -1$. Otherwise, discontinuity always forms when

$$\omega t_R>\frac{m}{2\varepsilon M}. \quad (20)$$

Assuming that the saw-like wave is periodic with period $2\pi/\omega$, and that its amplitude varies in space, the waveform over one period is described by an equality

$$P(X,r,\Theta)=A(X,r)\cdot\begin{cases}-\frac{\Theta\omega}{\pi}-1, -\pi<\Theta\omega<0,\\ -\frac{\Theta\omega}{\pi}+1, 0<\Theta\omega<\pi\end{cases}. \quad (21)$$

Hence, Eqs. (17), (3) result in the following equations:

$$\frac{\partial A}{\partial X}-\frac{\varepsilon\omega}{B\pi c_0^3\rho_0}A^2+\frac{\partial \Psi}{\partial r}\frac{\partial A}{\partial r}+\frac{\Delta_\perp\Psi}{2}A=0, \quad (22)$$

$$\begin{aligned}\frac{\partial T}{\partial t} &- \frac{\chi}{\rho_0 C_P}\Delta_\perp T\\ &= \frac{2Bt_R}{\rho_0^2 c_0 C_P}(X+1)^2\langle\frac{\partial P}{\partial \Theta}\int_{-\infty}^{\Theta}\frac{\partial P}{\partial \Theta'}e^{-(\Theta-\Theta')/t_R}d\Theta'\rangle\\ &= \frac{2B(\omega t_R)^2}{\pi^2\rho_0^2 c_0 C_P}(X+1)^2A^2. \end{aligned} \quad (23)$$

3. Non-stationary thermal self-action of a sound beam

If the heat conduction is small then the self action is not stationary and the diffusion term in Eq. (3) may be neglected. This occurs at initial stage of evolution, when $t<t_0$, where

$$t_0=\frac{\rho_0 C_p a^2}{12\chi} \quad (24)$$

is the characteristic time of temperature establishment, and a is an initial beam's radius at a transducer ($x=0$).

3.1. Low-frequency sound

The self-focusing of saw-tooth waves in Newtonian fluids is well-studied in the stationary and non-stationary regimes of propagation of the beams which are Gaussian at a transducer. The details of the self-focusing may be found in the review [11]. In this review by Rudenko and Sapozhnikov, a Newtonian viscosity in the acoustic source of heating was eliminated by use of sound periodicity, which in the leading order yields

$$\varepsilon\langle p^2\frac{\partial p}{\partial\theta}\rangle=\frac{b}{2}\langle p\frac{\partial^2 p}{\partial\theta^2}\rangle=-\frac{b}{2}\langle\left(\frac{\partial p}{\partial\theta}\right)^2\rangle. \quad (25)$$

This allows one to eliminate viscosity, because an acoustic source may be considered to be proportional to $\varepsilon A^3(x,r)$ instead of $bA^2(x,r)$. In this study, we consider the acoustic source proportional to $bA^2(x,r)$ in order to compare self-focusing in the low and high-frequency regimes in dependence on dispersion.

In account of Eq. (14), Eq. (10) takes the form

$$\frac{\partial T}{\partial t}-=\frac{b\omega^2}{\pi^2\rho_0^3 c_0^4 C_P}A^2. \quad (26)$$

Eq. (9) can be solved by-considering the parabolic wave front in the eikonal described by Eq. (7)

$$\psi(x,r,t)=\psi_0(x,t)+\frac{r^2}{2}\frac{\partial}{\partial x}\ln F(x,t). \quad (27)$$

Eq. (27) reflects the sphericity of the wave front, only its curvature may vary during propagation of a beam. The unknown function of two variables $F(x,t)$ is responsible for these variations, and $\psi_0(x,t)$ is a phase shift of the wavefront at the axis of a beam. In accordance with Eqs. (7), (27), an evolution of eikonal ψ is described by equation

$$\frac{1}{F}\left(\frac{\partial^2 F}{\partial x^2}\right)=\delta T_2, \quad (28)$$

where $T_2(x,t)$ is the coefficient in the transverse-coordinate expansion of temperature,

$$T = T_0 - \frac{r^2}{2} T_2 + \dots \qquad (29)$$

With regard to Eq. (27), the exact solution of the nonlinear equation (with respect to Eqs. (9), (14))

$$\frac{\partial A}{\partial x} + \frac{\varepsilon\omega}{\pi c^3 \rho_0} A^2 + \frac{\partial A}{\partial r}\frac{\partial \psi}{\partial r} + \frac{A}{2}\Delta_\perp \psi = 0, \qquad (30)$$

is

$$A(x,r) = \frac{P_0}{F}\Phi\left(\frac{r}{aF}\right)\left[1 + \frac{1}{x_s}\Phi\left(\frac{r}{aF}\right)\int_0^x \frac{dx'}{F(x',t)}\right]^{-1}, \qquad (31)$$

where P_0 is the initial amplitude at the beam axis, and x_s denotes the distance at which a break of a wave which is sinusoidal at a transducer, occurs. x_s determines the scale of the nonlinear absorption,

$$x_s = \frac{\rho_0 c_0^3 \pi}{P_0 \varepsilon \omega} = -\frac{X_s}{B}. \qquad (32)$$

With regard to the function Φ which describes the initial transverse distribution, $A(x=0,r) = P_0 \Phi\left(\frac{r}{a}\right)$ we will initially consider Gaussian beams, where $\Phi(\xi) = \exp(-\xi^2)$. Using Eqs. (28), (29) and performing the expansion of A in the transverse coordinate in the vicinity of a beam axis, one arrives at the equation for $F(x,t)$:

$$\frac{\partial}{\partial t}\left(\frac{1}{F}\frac{\partial^2 F}{\partial x^2}\right) = \frac{4 m t_R \delta \omega^2 P_0^2}{a^2 \pi^2 \rho_0^2 c_0^2 C_P F^4 \left(1 + \frac{1}{x_s}\int_0^x F^{-1}(x')dx'\right)^2}, \qquad (33)$$

which in dimensionless variables

$$\eta = \frac{t}{t_0}, \quad z = \frac{x}{x_0}, \quad z_s = \frac{x_s}{x_0} \qquad (34)$$

takes the form

$$\frac{\partial}{\partial \eta}\left(\frac{1}{F}\frac{\partial^2 F}{\partial z^2}\right) = \frac{sgn(\delta)\Pi}{F^4\left(1 + \frac{1}{z_s}\int_0^z F^{-1}(z')dz'\right)^2}, \qquad (35)$$

where

$$\Pi = B x_0 = -\frac{3 m \pi^2 \chi}{4 \rho_0 |\delta| t_R (\omega t_R)^2 M^2 c_0^4}. \qquad (36)$$

Eq. (35) may be solved numerically under conditions

$$F(z=0,\eta) = F(z,\eta=0) = 0, \; \frac{\partial F}{\partial z}(z=0,\eta) = \frac{x_0}{R}, \qquad (37)$$

where R^{-1} is the initial curvature of a beam which equals zero for planar beams.

3.2. High-frequency sound

As outlined and similar to the previous subsection, Eq. (22) can be solved by assuming the parabolic wave front

$$\Psi(X,r,t) = \Psi_0(X,t) + \frac{r^2}{2}\frac{\partial}{\partial X}\ln F(X,t). \qquad (38)$$

Eq. (38) reflects the sphericity of the wave front, only its curvature may vary during propagation of a beam. In accordance with Eqs. (18),(38), an evolution of eikonal Ψ is described by the equation

$$\frac{1}{F}\left(\frac{\partial^2 F}{\partial X^2} + \frac{1}{X+1}\frac{\partial F}{\partial X}\right) = \frac{\delta T_2}{B^2 (X+1)^2}, \qquad (39)$$

where $T_2(X,t)$ is the coefficient in the transverse-coordinate expansion of the temperature, Eq. (29). With regard to Eq. (38), the exact solution of nonlinear Eq. (22) is

$$A(X,r) = \frac{P_0}{F}\Phi\left(\frac{r}{aF}\right)\left[1 + \frac{1}{X_s}\Phi\left(\frac{r}{aF}\right)\int_0^X \frac{dX'}{F(X',t)}\right]^{-1}, \qquad (40)$$

where notations P_0 and $\Phi(\xi)$ are the same as defined in the previous subsection. Using Eqs (39), (29) and performing expansion of A in the transverse coordinate in the vicinity of a beam axis, one arrives at the equation for $F(X,t)$:

$$\frac{\partial}{\partial t}\left(F^{-1}\left(\frac{\partial^2 F}{\partial X^2} + \frac{1}{X+1}\frac{\partial F}{\partial X}\right)\right) = \frac{8\delta(\omega t_R)^2 P_0^2}{\pi^2 B \rho_0^2 c_0 C_P a^2 F^4 \left(1 + \frac{1}{X_s}\int_0^X F^{-1}(X')dX'\right)^2}, \qquad (41)$$

which in dimensionless variables, Eq. (34) takes the form

$$\frac{\partial}{\partial \eta}\left(F^{-1}\frac{\partial^2 F}{\partial z^2}\right) = \frac{sgn(\delta)\Pi\exp(-2\Pi z)}{F^4\left(1 + \frac{1}{z_s}\int_0^z F^{-1}(z')\exp(-\Pi z')dz'\right)^2}. \qquad (42)$$

Eq. (42) may be solved numerically under initial conditions as expressed in Eq. (37). Hence, in the high-frequency case, where dispersion is strong, the solution can not be expressed in terms of one parameter.

4. Stationary thermal self-action of a sound beam

At the later stages of evolution, $t > t_0$, the temporal derivative of an excess temperature in Eq. (3) may be defined as zero.

4.1. Low-frequency sound

With regard to Eq. (14), Eq. (10) is rearranged as

$$-\frac{\chi}{\rho_0 C_p}\Delta_\perp T = \frac{b\omega^2}{\pi^2\rho_0^3 c_0^4 C_P}A^2. \tag{43}$$

Expanding T and A in the series in the vicinity of the axis of a beam propagation, one arrives at

$$\begin{aligned}\frac{1}{F}\frac{\partial^2 F}{\partial x^2} &= \delta T_2 = \frac{\delta b\omega^2}{2\chi\pi^2\rho_0^2 c_0^4}A^2 \\ &= \frac{6\delta M^2 m t_R t_0 \omega^2 c_0^2}{a^2\pi^2 C_P F^2}\left(1+\frac{1}{x_s}\int_0^x F^{-1}(x')dx'\right)^{-2},\end{aligned} \tag{44}$$

which in dimensionless variables takes the form

$$F\frac{d^2F}{dz^2} = \frac{3sgn(\delta)\Pi}{2\left(1+\frac{1}{z_s}\int_0^z F^{-1}(z')dz'\right)^2}, \tag{45}$$

which may be readily solved numerically with initial conditions as expressed in Eq. (37).

4.2. High-frequency sound

Using Eqs. (39), (29) and performing expansion of A in the transverse coordinate in the vicinity of a beam axis, one arrives at the equation for $F(X,t)$:

$$F\left(\frac{\partial^2 F}{\partial X^2}+\frac{1}{X+1}\frac{\partial F}{\partial X}\right)\left(1 + \frac{1}{X_s}\int_0^X \frac{dX'}{F(X')}\right)^2 = \frac{\delta(\omega t_R)^2}{B\chi\pi^2\rho_0 c_0}P_0^2, \tag{46}$$

which in dimensionless variables as stated in Eqs. (34) takes the form

$$F\frac{d^2F}{dz^2} = \frac{3sgn(\delta)\Pi\exp(-2\Pi z)}{2\left(1+\frac{1}{z_s}\int_0^z F^{-1}(z')\exp(-\Pi z')dz'\right)^2}. \tag{47}$$

5. Discussion

Liquids are mostly weakly viscous and relaxation times are very small. For example, the relaxation time of benzene equals $2,7\cdot 10^{-10}$ s. In compounds such as carbon tetrachloride, benzene, and chloroform, the relaxation lies in the frequency range of the order of 10^9-10^{10} Hz, where ordinary ultrasonic methods of measurement are not applicable and dispersion of sound can be measured only by use of optical methods. Acoustic methods are the only way to measure the second viscosity of a fluid, which depends on sound frequency, while the first viscosity does not as a rule (exceptions are very weakly damping liquids at very high frequencies), and may be measured by means of other methods [16]. Thermal conductivity of liquids is relatively weak. With respect to the low-frequency domain of sound frequencies, $\omega t_R << 1$, where liquids behave as Newtonian, the second viscosity may be dominant compared with the first viscosity and thermal conduction. For benzene, the first viscosity μ equals $6\cdot 10^{-4}$Pa · s, and $b = mt_R\rho_0 c_0^2 = 8\cdot 10^{-2}$Pa · s, so that only the relaxing second viscosity may be accounted for. For the ten-centimeter transducer, the characteristic time in benzene, t_0, equals 145 minutes under normal conditions. Practically, the non-stationary self-focusing is of importance, which is described by Eq. (35). The stationary and non-stationary thermal self-action of shock waves in a Newtonian fluid has been considered in detail by Rudenko and Sapozhnikov [11].

Polyatomic gases, to a greater extent, are relaxing. Their thermal conductivity is much greater than that of standard liquids. Since the characteristic relaxation time of polyatomic gases are typically much smaller than that of the majority of liquids, the high-frequency regime takes place at megahertz frequencies. As an example, we consider carbon dioxide which is probably the oldest object of investigation of dispersion since first reports by Pierce and Abello [17, 18]. The physical properties of carbon dioxide are determined at temperature 18°C and atmospheric pressure [19]. The relaxation time of carbon dioxide is $3,6\cdot 10^{-5}$s. The characteristic time of temperature establishment, t_0, equals 80s. For smaller times, the non-stationary regime takes place. If $\omega = 10^4$Hz (this corresponds to the low-frequency regime) and $M = 0.1$, $\Pi = 0.07$, $z_s = 198$, and for the characteristic radius of a transducer $a = 0.1$m, then $z_d = 56$. The Mach number $M = 0.1$ is associated with strong nonlinearity as compared with relaxation, which presumes propagation of the shock waves, and $m/2\varepsilon M = 0.16$. Figure 1 illustrates a width of a beam and its amplitude at the axis of propagation in the low-frequency and high-frequency regimes, non-stationary and stationary, as a function of distance from a transducer. The second plot in Fig. 1 relates to the high-frequency regime with $\omega = 10^6$Hz, $M = 5 \cdot 10^{-3}$, $\Pi = 0.0003$, $z_s = 991$, and $z_d = 1.4 \cdot 10^5$ corresponding to the same size of a transducer. In this case, $X_s = -0.3$, so that discontinuity forms at a distance of 0.16m. In both plots, the approach of geometrical acoustics is valid because the characteristic length of a broadening beam is much shorter than the diffraction length ($z = 0$ corresponds to the distance of discontinuity forming). The initial wavefront is supposed to be planar with $R^{-1} = 0$.

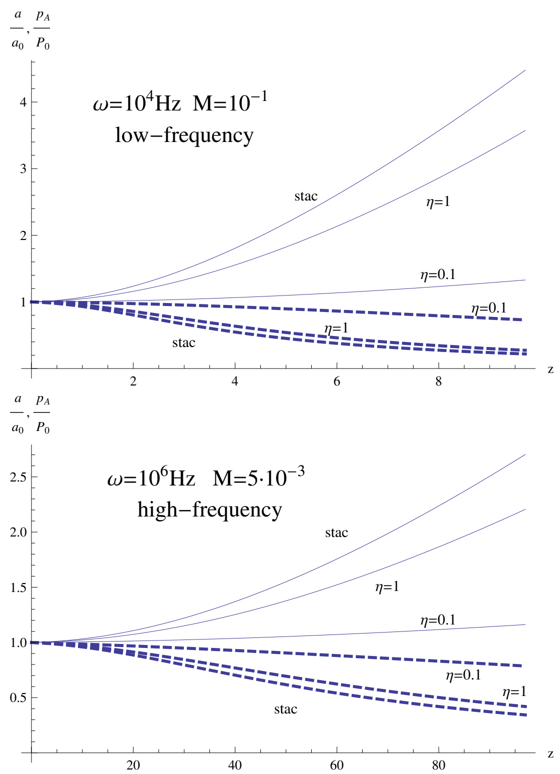

Figure 1. The dimensionless width of a beam (solid lines) and amplitude of sound pressure (dashed lines) at the axis of a beam with an initial planar front in the non-stationary and stationary regimes.

The width of a beam increases with time, and its amplitude decreases. This is a common property of a beam propagating over attenuating gases, where the thermal coefficient is positive. At some distances from a transducer, the cross section of a beam which is Gaussian at a transducer, becomes almost flat, as it was reported in the paper by Rudenko and co-authors, devoted to Newtonian fluids [20]. The shear viscosity of carbon dioxide under atmospheric pressure and temperature $18^o C$ equals $1,4 \cdot 10^{-5}$ Pa · s, while $b = m t_R \rho_0 c_0^2 = 0,17$ Pa · s, which is responsible for the bulk viscosity, is much larger.

6. Concluding remarks

With respect to the estimation of the relaxation time, the issue is to separate different relaxation processes in a fluid. The rates of individual relaxation processes may differ strongly. Generally speaking, molecular collisions in a gas are accompanied by variations in the translational, rotational, vibrational and electronic energy of the collision partners. The physical difference in relaxation times follows from the variance of probabilities (or cross sections) of the different elementary events. At temperatures up to the order of 10^3K, the characteristic times of the individual relaxation processes in a molecular gas form the following hierarchy:

$$\tau_{TT} < \tau_{RT} \ll \tau_{VV} \ll \tau_{VT}, \qquad (48)$$

where τ_{TT},τ_{RT},τ_{VT} are characteristic times of establishment of equilibrium among translational, rotational and vibrational degrees of freedom, τ_{VV} is the characteristic time of exchange of vibrations among molecules. The main difficulty is to separate different relaxation processes especially if their relaxation times are close. Often, chemical reactions are accompanied by a non-equilibrium excitation of the internal degrees of freedom of molecules [21]. The dispersive properties of chemically reacting gases where additional relaxation of molecules' vibrational degrees of freedom takes place, are considered in [22].

In fact, thermodynamic relaxation imposes dissipation, and vice versa. The Kramers-Kronig relations in optics may be recalled, which connect relaxation and attenuation of light waves [23]. As for attenuation of sound over its wavelength in a Maxwell fluid, it depends on the sound frequency and achieves a maximum for the frequencies $\omega = 1/t_R$ [4]. The low-frequency sound propagates over a Maxwell fluid similar to a Newtonian fluid. The high-frequency sound almost does not attenuate, but its speed increases, $c_\infty = (1 + m/2)c_0$. The thermal self-action of sound also depends strongly on the sound frequency.

Similarly to the thermal self-action, another inertial self-action process can occur by means of formation of hydrodynamic streams in a medium ("acoustic streaming") due to the loss of momentum of an intense sound wave. The stream velocity in the paraxial area coincides with the direction of beam propagation. This mechanism always leads to additional divergence because the drift caused by streaming causes the wave velocity to increase in the central part of a beam. The sound beam in a gas is divergent due to nonlinear generation of both non-acoustic motions: the entropy mode (acoustic heating), which forms a thermal lens, and the vortex mode (acoustic streaming), which is responsible for a bulk motion of a gas. Since shock positive pulses travel in unperturbed media with supersonic velocities, this leads to instantaneous self-refraction of shock pulse beams. This kind of self-action is known to be the reason for limiting of the maximal intensity that can be achieved in strong focused signals [4]. This kind of self-action is not considered in this study.

In this study, we assume that the thermal self-action occurs in a static medium. The effects associated with the occurrence of flows in sawtooth wave fields in Newtonian fluids were discussed in Ref.[24]. The relaxing Maxwell gases with dispersive second viscosity in the low-frequency regime behave as Newtonian fluids. In the high-frequency regime, the term responsible for attenuation is different, and is proportional to acoustic pressure but not to its second derivative with respect to the retarded time (Eq. (15)). Figure 1 reveals some important features of thermal self-action of the shock sound beams propagating over a relaxing gas. The width of a beam always increases, and amplitude of acoustic pressure decreases along the axis of a beam. The nonlinear broadening of a beam can be explained by flattening of the transverse beam profile due to stronger absorption near the axis (the so-called isotropization of the directional distribution). In the non-stationary self-action, the thermal lens becomes stronger with time and the focal point moves towards the transducer, which is more significant in the low-frequency nonlinear regime.

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