# Topological degree for equivariant gradient perturbations of an unbounded self-adjoint operator in Hilbert space 

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#### Abstract

We present a version of the equivariant gradient degree defined for equivariant gradient perturbations of an equivariant unbounded self-adjoint operator with purely discrete spectrum in Hilbert space. Two possible applications are discussed.


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## 1. Introduction

To obtain new bifurcation results, N. Dancer [5] introduced in 1985 a new topological invariant for $\mathrm{S}^{1}$ equivariant gradient maps, which provides more information than the usual equivariant one. In 1994 S . Rybicki $[14,16]$ developed the complete degree theory for $S^{1}$-equivariant gradient maps and 3 years later K . Gęba extended this theory to an arbitrary compact Lie group. In 2001 S. Rybicki [15] defined the degree for $S^{1}$-equivariant strongly indefinite functionals in Hilbert space. 10 years later A. Gołębiewska and S. Rybicki [8] generalized this degree to compact Lie groups. The relation between equivariant and equivariant gradient degree theories were studied in $[1,2,7]$.

The main goal of this paper is to present a construction and properties of a new degree-type topological invariant $\operatorname{Deg}_{G}^{\nabla}$, which is defined for equivariant gradient perturbations of an equivariant unbounded selfadjoint Hilbert operator with a purely discrete spectrum (in the general case a compact Lie group). As far as we know, the idea of the construction of such an invariant should be attributed to K. Gęba.

It is worth pointing out that equivariant gradient perturbations of an equivariant unbounded self-adjoint operator with a purely discrete spectrum appear naturally in a variety of problems in nonlinear analysis, such as the search for periodic solutions of Hamiltonian systems or the study of Seiberg-Witten equations for three dimensional manifolds. The purpose of our work is to provide a topological tool that allows us to solve problems similar to the above mentioned ones.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we present the construction that leads to the definition of the degree $\operatorname{Deg}_{\mathrm{G}} \nabla_{\text {. The correctness of this definition is proved in }}$. Section 4. The properties of the degree $\operatorname{Deg}_{\mathrm{G}}^{\nabla}$ are examined in Section 5. Finally, in Section 6 we discuss two examples of possible applications.

## 2. Preliminaries

The preliminaries are divided into five brief subsections.

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### 2.1. Unbounded self-adjoint operators in Hilbert space

This subsection is based on [17]. Let $E$ be a real separable Hilbert space with inner product $\langle\cdot \mid \cdot\rangle$ and $A: D(A) \subset E \rightarrow E$ be a linear operator (not necessarily bounded) such that its domain $D(A)$ is dense in $E$. Set

$$
\mathrm{D}\left(A^{*}\right)=\{\mathrm{y} \in \mathrm{E} \mid \exists \mathfrak{u} \in \mathrm{E} \forall x \in \mathrm{D}(\mathrm{~A})\langle\mathrm{Ax} \mid \mathrm{y}\rangle=\langle x \mid \mathfrak{u}\rangle\} .
$$

Since $D(A)$ is dense in $E$, the vector $u \in E$ is uniquely determined by $y$. Therefore by setting $A^{*} y=u$ we obtain a well-defined linear operator from $D\left(A^{*}\right)$ to $E$. The operator $A^{*}$ is called the adjoint operator of $A$. We say that $A$ is self-adjoint if $A=A^{*}$. By the Hellinger-Toeplitz theorem, if $A$ is self-adjoint and $D(A)=E$ then $A$ is bounded.

It is easy to see that

$$
\langle x \mid y\rangle_{1}=\langle x \mid y\rangle+\langle A x \mid A y\rangle
$$

defines an inner product on the domain $D(A)$. Under this product $D(A)$ becomes a Hilbert space, which will be denoted by $E_{1}$. Thus $D(A)$ and $E_{1}$ are equal as sets but equipped with different inner products. Note that $A$ treated as an operator from $E_{1}$ to $E$ is bounded.

We say that a self-adjoint operator $A$ has a purely discrete spectrum if its spectrum consists only of isolated eigenvalues of finite multiplicity. If $E$ is an infinite dimensional Hilbert space then following conditions are equivalent:

1. A has a purely discrete spectrum.
2. There is a real sequence $\left\{\lambda_{n}\right\}$ and an orthonormal basis $\left\{e_{n}\right\}$ such that $\lim \left|\lambda_{n}\right|=\infty$ and $A e_{n}=\lambda_{n} e_{n}$ for $n \in \mathbb{N}$.
3. The embedding $\imath: \mathrm{E}_{1} \rightarrow \mathrm{E}$ is compact.

### 2.2. Local maps in Hilbert space

Let

- E be a real Hilbert orthogonal representation of a compact Lie group G,
- $A: D(A) \subset E \rightarrow E$ be an unbounded self-adjoint operator with a purely discrete spectrum,
- $D(A)$ be invariant and $A$ equivariant.

Definition 2.1. We write $f \in \mathcal{G}_{G}(E)$ if

- $f: D_{f} \subset E_{1} \rightarrow E$, where $D_{f}$ is an open invariant subset of $E_{1}$,
- $f(x)=A x-\nabla \varphi(x)$, where $\varphi: E \rightarrow \mathbb{R}$ is $C^{1}$ and invariant,
- $f^{-1}(0)$ is compact.

Elements of $\mathcal{G}_{G}(E)$ will be called local maps.

### 2.3. Otopies in Hilbert space

Let $\mathrm{I}=[0,1]$. Assume that G acts trivially on I. A map $h: \Lambda \subset \mathrm{I} \times \mathrm{E}_{1} \rightarrow \mathrm{E}$ is called an otopy if

- $\Lambda$ is an open invariant subset of $I \times E_{1}$,
- $h(t, \cdot) \in \mathcal{G}_{G}(E)$ for each $t \in I$,
- $h^{-1}(0)$ is compact.

Given an otopy $h: \Lambda \subset I \times E_{1} \rightarrow E$ we can define for each $t \in I$ :

- sets $\Lambda_{t}=\left\{x \in E_{1} \mid(t, x) \in \Lambda\right\}$,
- maps $h_{t}: \Lambda_{t} \rightarrow E$ with $h_{t}(x)=h(t, x)$.

If $h$ is an otopy, we say that $h_{0}$ and $h_{1}$ are otopic. The relation of being otopic is an equivalence relation in $\mathcal{G}_{G}(\mathrm{E})$.

Observe that if $f$ is a local map and $U$ is an open subset of $D_{f}$ such that $f^{-1}(0) \subset U$, then $f$ and $f \upharpoonright u$ are otopic. This property of local maps is called the restriction property. In particular, if $f^{-1}(0)=\emptyset$ then $f$ is otopic to the empty map.

### 2.4. Euler-tom Dieck ring

Recall the notion of the Euler-tom Dieck ring following [19]. For a compact Lie group G let $\mathfrak{U}(\mathrm{G})$ denote the set of equivalence classes of finite G-CW-complexes. Two complexes $X$ and $Y$ are identified if the quotients $X^{\mathrm{H}} / W \mathrm{H}$ and $\mathrm{Y}^{\mathrm{H}} / \mathrm{WH}$ have the same Euler characteristic for all closed subgroups H of G . Recall that $X^{\mathrm{H}}$ stands here for the $H$-fixed point set of $X$, i.e. $X^{H}:=\{x \in X \mid h x=x$ for all $h \in H\}$ and $W H$ for the Weyl group of $H$, i.e. $W H=N H / H$. Addition and multiplication in $\mathfrak{U}(G)$ are induced by disjoint union and cartesian product with diagonal G-action, i.e.

$$
[\mathrm{X}]+[\mathrm{Y}]=[\mathrm{X} \sqcup \mathrm{Y}], \quad[\mathrm{X}] \cdot[\mathrm{Y}]=[\mathrm{X} \times \mathrm{Y}],
$$

where the square brackets stand for an equivalence class of finite G-CW-complexes. In this way $\mathfrak{U}(\mathrm{G})$ becomes a commutative ring with unit and is called the Euler-tom Dieck ring of G.

Additively, $\mathfrak{l}(G)$ is a free abelian group with basis elements $[G / H]$, where $H$ is a closed subgroup of $G$. In consequence, each element of $\mathfrak{U}(G)$ can be uniquely written as a finite sum $\sum d_{(H)}[G / H]$, where $d_{(H)}$ is an integer, which depends only on the conjugacy class of $H$. The ring unit is [G/G].

### 2.5. Finite dimensional equivariant gradient degree $\operatorname{deg}_{\mathrm{G}}^{\nabla}$

Assume that V is a real finite dimensional orthogonal representation of a compact Lie group G . We write $f \in \mathcal{G}_{G}(V)$ if $f$ is an equivariant gradient map from an open invariant subset of $V$ to $V$ and $f^{-1}(0)$ is compact. In the papers $[1,2,6,16]$ the authors defined the equivariant gradient degree

$$
\operatorname{deg}_{G}^{\nabla}: \mathcal{G}_{G}(\mathrm{~V}) \rightarrow \mathfrak{U}(\mathrm{G})
$$

and proved that the degree has the following properties: additivity, otopy invariance, existence and normalization. The product property formulated below was proved in [6] and [9].
Theorem 2.2 (Product property). Let V and W be real finite dimensional orthogonal representations of a compact Lie group $G$. If $f \in \mathcal{G}_{G}(V)$ and $f^{\prime} \in \mathcal{G}_{G}(W)$, then $f \times f^{\prime} \in \mathcal{G}_{G}(V \oplus W)$ and

$$
\operatorname{deg}_{G}^{\nabla}\left(f \times f^{\prime}\right)=\operatorname{deg}_{G}^{\nabla}(f) \cdot \operatorname{deg}_{G}^{\nabla}\left(f^{\prime}\right) \text { in } \mathfrak{U}(G)
$$

In the next section we will make use of the following result, which can be found in [8, Cor. 2.1].
Theorem 2.3. Let V be a real finite dimensional orthogonal representation of a compact Lie group G . If B is an equivariant self-adjoint isomorphism of V then $\operatorname{deg}_{\mathrm{G}}^{\nabla}(\mathrm{B})$ is invertible in $\mathfrak{U}(\mathrm{G})$.
Remark 2.4. Note that Theorem 2.3 holds even if $V$ is trivial. In this case $\operatorname{deg}_{G}^{\nabla}(B)$ is equal to the unit of $\mathfrak{U}(G)$.

## 3. Definition of degree

In this section we present the construction of the degree $\operatorname{Deg}_{\mathrm{G}}^{\nabla}$ using finite dimensional approximations.

### 3.1. Finite dimensional approximations

Let us start with some notations:

- for $\lambda \in \sigma(A)$ denote by $V(\lambda)$ the corresponding eigenspace;
- for $n \in \mathbb{N}$ write $V_{n}=\oplus|\lambda| \leqslant n ~ V(\lambda), V^{n}=\oplus_{n-1<|\lambda| \leqslant n} V(\lambda)$ and $A_{n}=A \upharpoonright V^{n}$; hence $V_{n}=V_{n-1} \oplus V^{n}$;
- let $P_{n}: E \rightarrow V_{n}$ denote the orthogonal projection.

Assume that $U$ is an open bounded invariant subset of $D_{f}$ such that

$$
\mathrm{f}^{-1}(0) \subset \mathrm{U} \subset \mathrm{clu} \subset \mathrm{D}_{\mathrm{f}}
$$

Set $\mathrm{U}_{\mathrm{n}}=\mathrm{U} \cap \mathrm{V}_{\mathrm{n}}$. Finally, let $\mathrm{f}_{\mathrm{n}}: \mathrm{U}_{\mathrm{n}} \rightarrow \mathrm{V}_{\mathrm{n}}$ be given by

$$
f_{n}(x)=A x-P_{n} F(x)
$$

where $F(x)=\nabla \varphi(x)$.
The following two lemmas are needed to prove Lemma 3.3, which is crucial for the definition of $\operatorname{Deg}_{\mathrm{G}}^{\nabla}$.

Lemma 3.1. There is $\epsilon>0$ such that $|f(x)| \geqslant 2 \epsilon$ for all $x \in \partial u$.
Proof. The fact F is compact and $\partial \mathrm{U}$ is closed and bounded implies our claim.
Let us introduce an auxiliary map $\tilde{f}_{n}: D_{f} \rightarrow$ E given by $\tilde{f}_{n}(x)=A x-P_{n} F(x)$. By definition, $\widetilde{f}_{n} \upharpoonright u_{n}=f_{n}$.
Lemma 3.2. There is $N$ such that for $n \geqslant N$ we have

1. $\left|f(x)-\tilde{f}_{n}(x)\right|<\epsilon$ for $x \in \operatorname{clU}$,
2. $\left|\tilde{f}_{n}(x)\right|>\in$ for $x \in \partial u$.

Proof. Since F is compact, $F$ is close to $P_{n} F$, which gives (1). In turn (2) follows from (1) and Lemma 3.1.
Lemma 3.3. For $n \geqslant N$ we have $f_{n} \in \mathcal{G}_{G}\left(V_{n}\right)$ and, in consequence, $\operatorname{deg}_{G}^{\nabla}\left(f_{n}\right) \in \mathfrak{U}(G)$ is well-defined.
Proof. Since $f_{n}$ is obviously gradient, it is enough to check that $f_{n}^{-1}(0)$ is compact. Note that $\tilde{f}_{n}$ can be considered as an extension of $f_{n}$ on $\mathrm{cl}_{n}$. By (2) from Lemma 3.2, $\widetilde{f}_{n}$ does not have zeroes in $\partial U_{n} \subset \partial U$, which implies that $f_{n}^{-1}(0)=\tilde{f}_{n}^{-1}(0) \cap U_{n}$ is compact.

### 3.2. Degree definition

Observe that $A_{n}$ is an equivariant self-adjoint isomorphism for $n \geqslant 1$. By Theorem 2.3, elements $a_{n}:=$ $\operatorname{deg}_{G}^{\nabla}\left(A_{n}\right)$ are invertible in $\mathfrak{U}(G)$. Set $m_{n}:=a_{1}^{-1} \cdot a_{2}^{-1} \cdot \ldots \cdot a_{n}^{-1}$.
Definition 3.4. Let $\operatorname{Deg}_{G}^{\nabla}: \mathcal{G}_{G}(E) \rightarrow \mathfrak{U}(G)$ be defined by

$$
\operatorname{Deg}_{G}^{\nabla}(f):=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right)
$$

for $\mathrm{n} \geqslant \mathrm{N}$.
An alternative definition of $\operatorname{Deg}_{\mathrm{G}}^{\nabla}$ in terms of the direct limit is given in Appendix A.

## 4. Correctness of the definition

We have to prove that our definition does not depend on the choice of $n$ and the neighbourhood $U$.
Independence from the choice of $n$. To show this we will need the following lemma.
Lemma 4.1. For $n$ large enough $f_{n+1}$ is otopic to $f_{n} \times A_{n+1}$ in $\mathcal{G}_{G}\left(V_{n+1}\right)$ and hence

$$
\operatorname{deg}_{G}^{\nabla}\left(f_{n+1}\right)=\operatorname{deg}_{G}^{\nabla}\left(f_{n} \times A_{n+1}\right)
$$

Proof. First observe there is an open $\mathrm{W} \subset \mathrm{U}$ and natural number N such that

- $\mathrm{f}^{-1}(0) \subset W \subset \mathrm{U}$,
- $\mathrm{P}_{\mathrm{n}}(\mathrm{cl} W) \subset \mathrm{U}_{\mathrm{n}}$ for all $\mathrm{n} \geqslant \mathrm{N}$.

Define $h_{n+1}: I \times c l W_{n+1} \rightarrow V_{n+1}$ by

$$
h_{n+1}(t, x)=(1-t) f_{n+1}(x)+t\left(f_{n} \times A_{n+1}\right)(x) .
$$

We set $n$ sufficiently large. One can show that $h_{n+1}(t, x) \neq 0$ for $t \in I$ and $x \in \partial W_{n+1}$. In consequence, $h_{n+1} \upharpoonright I \times W_{n+1}$ is a finite dimensional equivariant gradient otopy between $f_{n+1} \upharpoonright W_{n+1}$ and $f_{n} \times A_{n+1} \upharpoonright W_{n+1}$ (otherwise there would be a point $x_{0} \in \partial W$ such that $f\left(x_{0}\right)=0$, a contradiction). On the other hand, by the restriction property, $f_{n+1}$ and $f_{n} \times A_{n+1}$ are otopic to their restrictions to $W_{n+1}$, which completes the proof.

From Lemma 4.1 and Theorem 2.2 we can easily conclude that

$$
\operatorname{deg}_{G}^{\nabla}\left(f_{n+1}\right) \stackrel{4.1}{=} \operatorname{deg}_{G}^{\nabla}\left(f_{n} \times A_{n+1}\right) \stackrel{2.2}{=} \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right) \cdot \operatorname{deg}_{G}^{\nabla}\left(A_{n+1}\right)=a_{n+1} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right)
$$

This gives

$$
m_{n+1} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n+1}\right)=m_{n+1} \cdot a_{n+1} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right)
$$

which shows that $\operatorname{Deg}_{G}{ }_{\mathrm{G}}(f)$ does not depend on the choice of $n$ large enough.
Independence from the choice of $U$. According to our definition $\operatorname{Deg}_{G}^{\nabla}(f)=\operatorname{Deg}_{G}^{\nabla}(f \upharpoonright u)$. Now we will prove that in fact $\operatorname{Deg}_{G}^{\nabla}(f)$ is independent from the choice of the neighbourhood $U$.
Lemma 4.2. Let $W$ and $U$ be open bounded sets such that

$$
\mathrm{f}^{-1}(0) \subset \mathrm{W} \subset \mathrm{u} \subset \mathrm{clu} \subset \mathrm{D}_{\mathrm{f}}
$$

Then $\operatorname{Deg}_{G}^{\nabla}(f \mid w)=\operatorname{Deg}_{G}^{\nabla}(f \mid u)$.
Proof. By the analogue of Lemma 3.1 (with $\partial \mathrm{U}$ replaced by $\mathrm{cl} U \backslash W$ ), $|f(x)| \geqslant 2 \epsilon$ for $x \in \operatorname{clU} \backslash W$ and by Lemma 3.2, $\left|f(x)-\widetilde{f}_{n}(x)\right|<\epsilon$ for $x \in \operatorname{clU}$. Hence $\widetilde{f}_{n}(x) \neq 0$ for $x \in \operatorname{clU} \backslash W$. In consequence, $f_{n}(x) \neq 0$ for $x \in \mathrm{cl} \mathrm{U}_{\mathrm{n}} \backslash \mathrm{W}_{\mathrm{n}}$. Therefore

$$
\operatorname{Deg}_{G}^{\nabla}(f \upharpoonright u)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n} \upharpoonright u_{n}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n} \upharpoonright w_{n}\right)=\operatorname{Deg}_{G}^{\nabla}(f \upharpoonright w) .
$$

Corollary 4.3. Let U and $\mathrm{U}^{\prime}$ be open bounded subsets of $\mathrm{D}_{\mathrm{f}}$ such that

$$
\mathrm{f}^{-1}(0) \subset \mathrm{U} \cap \mathrm{u}^{\prime} \subset \mathrm{cl}\left(\mathrm{U} \cup \mathrm{u}^{\prime}\right) \subset \mathrm{D}_{\mathrm{f}}
$$

Then $\operatorname{Deg}_{G}^{\nabla}(f \upharpoonright u)=\operatorname{Deg}_{G}^{\nabla}\left(f \upharpoonright u n u^{\prime}\right)=\operatorname{Deg}_{G}^{\nabla}\left(f \upharpoonright u^{\prime}\right)$.
In this way we have proved that $\operatorname{Deg}_{G}^{\nabla}(f)$ does not depend on the choice of admissible U.

## 5. Degree properties

In this section we prove that our degree $\operatorname{Deg}_{G}^{\nabla}: \mathcal{G}_{G}(E) \rightarrow \mathfrak{U}(G)$ has all properties analogous to the wellknown properties of the finite dimensional equivariant gradient degree $\operatorname{deg}_{\mathrm{G}}{ }^{\nabla}$.

Additivity property. If $\mathrm{f}, \mathrm{f}^{\prime} \in \mathcal{G}_{G}(\mathrm{E})$ and $\mathrm{D}_{\mathrm{f}} \cap \mathrm{D}_{f^{\prime}}=\emptyset$ then

$$
\operatorname{Deg}_{G}^{\nabla}\left(f \sqcup f^{\prime}\right)=\operatorname{Deg}_{G}^{\nabla}(f)+\operatorname{Deg}_{G}^{\nabla}\left(f^{\prime}\right) .
$$

Otopy invariance property. Let $\mathrm{f}, \mathrm{f}^{\prime} \in \mathcal{G}_{\mathrm{G}}(\mathrm{E})$. If f is otopic to $\mathrm{f}^{\prime}$ then

$$
\operatorname{Deg}_{G}^{\nabla}(f)=\operatorname{Deg}_{G}^{\nabla}\left(f^{\prime}\right)
$$

Existence property. If $\operatorname{Deg}_{G}^{\nabla}(f) \neq 0$ then $f(x)=0$ for some $x \in D_{f}$.
Normalization property.

$$
\operatorname{Deg}_{G}^{\nabla}\left(A+P_{0}\right)=[G / G]=1_{\mathfrak{U}(G)}
$$

where $P_{0}: E_{1} \rightarrow V_{0}=\operatorname{ker} A$ is the orthogonal projection.
Product property. Let E and $\mathrm{E}^{\prime}$ be real Hilbert orthogonal representations of a compact Lie group G . If $\mathrm{f} \in \mathcal{G}_{\mathrm{G}}(\mathrm{E})$ and $f^{\prime} \in \mathcal{G}_{G}\left(E^{\prime}\right)$, then $f \times f^{\prime} \in \mathcal{G}_{G}\left(E \oplus E^{\prime}\right)$ and

$$
\operatorname{Deg}_{G}^{\nabla}\left(f \times f^{\prime}\right)=\operatorname{Deg}_{G}^{\nabla}(f) \cdot \operatorname{Deg}_{G}^{\nabla}\left(f^{\prime}\right),
$$

where the dot here denotes the multiplication in $\mathfrak{U}(\mathrm{G})$.

Proof. Additivity. Immediately from the additivity of $\operatorname{deg}_{G}^{\nabla}$ we obtain

$$
\operatorname{Deg}_{G}^{\nabla}\left(f \sqcup f^{\prime}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n} \sqcup f_{n}^{\prime}\right)=m_{n} \cdot\left(\operatorname{deg}_{G}^{\nabla}\left(f_{n}\right)+\operatorname{deg}_{G}^{\nabla}\left(f_{n}^{\prime}\right)\right)=\operatorname{Deg}_{G}^{\nabla}(f)+\operatorname{Deg}_{G}^{\nabla}\left(f^{\prime}\right) .
$$

Otopy invariance. Let the map $h: \Lambda \subset I \times E_{1} \rightarrow E$ given by $h(t, x)=A x-F(t, x)$ be an otopy. We introduce the following notation:

$$
\begin{aligned}
& \Lambda^{\mathrm{t}}=\left\{x \in \mathrm{E}_{1} \mid(\mathrm{t}, \mathrm{x}) \in \Lambda\right\}, \\
& h^{\mathrm{t}}: \Lambda^{\mathrm{t}} \rightarrow \mathrm{E}, \\
& h^{t}(x)=h(t, x), \\
& \Lambda_{n}=\Lambda \cap\left(\mathrm{I} \times \mathrm{V}_{\mathrm{n}}\right) \text {, } \\
& \Lambda_{n}^{t}=\Lambda^{t} \cap V_{n}, \\
& h_{n}: \Lambda_{n} \rightarrow V_{n} \text {, } \\
& h_{n}(t, x)=A x-P_{n} F(t, x), \\
& h_{n}^{t}: \Lambda_{n}^{t} \rightarrow V_{n}, \\
& h_{n}^{t}(x)=h_{n}(t, x) .
\end{aligned}
$$

Note that for the needs of this subsection the time parameter $t$ of the otopy is a superscript, not a subscript. According to the above notation we have to show that $\operatorname{Deg}_{G}^{\nabla}\left(h^{0}\right)=\operatorname{Deg}_{G}^{\nabla}\left(h^{1}\right)$. Since $h^{-1}(0)$ is compact, there is an open bounded set $W \subset I \times E_{1}$ such that

$$
\mathrm{h}^{-1}(0) \subset W \subset \operatorname{cl} W \subset \Lambda
$$

Hence for $i=0,1$ we have

$$
\left(h^{i}\right)^{-1}(0) \subset W^{i} \subset \operatorname{cl} W^{i} \subset \Lambda^{i}
$$

where $W^{i}=\left\{x \in E_{1} \mid(i, x) \in W\right\}$. Similarly as in Lemma 3.1, there is $\epsilon>0$ such that $|h(z)| \geqslant 2 \epsilon$ for $z \in \partial W$. On the other hand, similarly as in Lemma 3.2, there is $N$ such that $\left|h(z)-\widetilde{h}_{n}(z)\right|<\epsilon$ for $z \in \operatorname{cl} W$ and $n \geqslant N$, where $\widetilde{h}_{n}: \Lambda \rightarrow E$ is given by $\widetilde{h}_{n}(t, x)=A x-P_{n} F(t, x)$. Therefore $\left|h_{n}(z)\right| \geqslant \epsilon$ for $z \in \partial W_{n} \subset \partial W$. From the above:

- $h_{n} \upharpoonright W_{n}$ is a finite dimensional equivariant gradient otopy,
- $\operatorname{Deg}_{G}^{\nabla}\left(h^{i}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(h_{n}^{i} \upharpoonright w_{n}^{i}\right)$,
which, by the otopy invariance of $\operatorname{deg}_{\mathrm{G}}^{\nabla}$, gives

$$
\operatorname{Deg}_{G}^{\nabla}\left(h^{0}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(h_{n}^{0} \upharpoonright w_{n}^{0}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(h_{n}^{1} \upharpoonright w_{n}^{1}\right)=\operatorname{Deg}_{G}^{\nabla}\left(h^{1}\right)
$$

Existence. If $f^{-1}(0)=\emptyset$ then $f$ is otopic with the empty map. Hence

$$
\operatorname{Deg}_{G}^{\nabla}(f)=\operatorname{Deg}_{G}^{\nabla}(\emptyset)=0
$$

Normalization. Observe that $A+P_{0}$ is an injection and

$$
\operatorname{deg}_{G}^{\nabla}\left(\left(A+P_{0}\right)_{n}\right)=\operatorname{deg}_{G}^{\nabla}\left(\operatorname{Id} \upharpoonright v_{0}\right) \cdot \operatorname{deg}_{G}^{\nabla}\left(A_{1}\right) \cdot \ldots \cdot \operatorname{deg}_{G}^{\nabla}\left(A_{n}\right)=m_{n}^{-1}
$$

for any $n \geqslant 1$. Hence

$$
\operatorname{Deg}_{G}^{\nabla}\left(A+P_{0}\right)=m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(\left(A+P_{0}\right)_{n}\right)=[G / G] .
$$

Product formula. Let $f(x)=A x-F(x)$ and $f^{\prime}(x)=A^{\prime} x-F^{\prime}(x)$. Observe that, by Theorem 2.2, if $f_{n} \in \mathcal{G}_{G}\left(V_{n}\right)$ and $f_{n}^{\prime} \in \mathcal{G}_{G}\left(V_{n}^{\prime}\right)$ then $f_{n} \times f_{n}^{\prime} \in \mathcal{G}_{G}\left(V_{n} \oplus V_{n}^{\prime}\right)$ and

$$
\operatorname{deg}_{G}^{\nabla}\left(f_{n} \times f_{n}^{\prime}\right)=\operatorname{deg}_{G}^{\nabla}\left(f_{n}\right) \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}^{\prime}\right)
$$

Moreover, for n large enough

$$
\begin{aligned}
\operatorname{Deg}_{G}^{\nabla}(f) & =m_{n} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right), \\
\operatorname{Deg}_{G}^{\nabla}\left(f^{\prime}\right) & =m_{n}^{\prime} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}^{\prime}\right) .
\end{aligned}
$$

Since for any $i \geqslant 1$

$$
\operatorname{deg}_{G}^{\nabla}\left(\left(A \times A^{\prime}\right)_{i}\right)=\operatorname{deg}_{G}^{\nabla}\left(A_{i} \times A_{i}^{\prime}\right)=\operatorname{deg}_{G}^{\nabla}\left(A_{i}\right) \cdot \operatorname{deg}_{G}^{\nabla}\left(A_{i}^{\prime}\right)
$$

we have

$$
\operatorname{Deg}_{G}^{\nabla}\left(f \times f^{\prime}\right)=m_{n} \cdot m_{n}^{\prime} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n} \times f_{n}^{\prime}\right)=m_{n} \cdot m_{n}^{\prime} \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}\right) \cdot \operatorname{deg}_{G}^{\nabla}\left(f_{n}^{\prime}\right)=\operatorname{Deg}_{G}^{\nabla}(f) \cdot \operatorname{Deg}_{G}^{\nabla}\left(f^{\prime}\right)
$$

Remark 5.1. The normalization property can be formulated more generally, but the proof of this fact will appear elsewhere. Namely, let $x_{0} \in V_{n}$ and, in consequence, $G x_{0} \subset V_{n}$. Define

$$
\mathrm{U}=\left\{\mathrm{x}+\mathrm{y}+z\left|\mathrm{x} \in \mathrm{G} x_{0}, \mathrm{y} \in\left(\mathrm{~T}_{\mathrm{x}_{0}}\left(\mathrm{G} x_{0}\right)\right)^{\perp} \subset \mathrm{V}_{\mathrm{n}},|\mathrm{y}|<\epsilon, z \in\left(\mathrm{~V}_{\mathrm{n}}\right)^{\perp} \subset \mathrm{E}_{1}\right\}\right.
$$

and $\mathrm{f}: \mathrm{U} \rightarrow \mathrm{E}$ by

$$
f(x+y+z)=\left(A+P_{0}\right)(y+z)
$$

Then $\operatorname{Deg}_{G}^{\nabla}(f)=\left[G / G_{x_{0}}\right]$.

## 6. Possible applications

We should emphasize that this section contains not real applications of the theory but only two exemplary situations illustrating potential applications.

### 6.1. Applications to Hamiltonian systems

The search for periodic solutions in Hamiltonian systems is one of the fundamental problems in nonlinear analysis (see for instance [3, 12, 13, 20]). Consider the Hamiltonian system of ODE

$$
\frac{d p}{d t}=-H_{q}, \quad \frac{d q}{d t}=H_{p}
$$

where $H \in C^{1}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)$ and $p, q \in \mathbb{R}^{n}$ or equivalently

$$
\frac{\mathrm{d} z}{\mathrm{dt}}=\mathcal{J H}_{z}
$$

where $z=(p, q)$ and

$$
\mathcal{J}=\left(\begin{array}{rr}
0 & -\mathrm{I} \\
\mathrm{I} & 0
\end{array}\right) .
$$

The function H is called the hamiltonian or energy.
Rewrite the Hamiltonian system as

$$
\begin{equation*}
\dot{z}=\mathcal{J} \nabla \mathrm{H}(z), \quad z \in \mathbb{R}^{2 n} \tag{*}
\end{equation*}
$$

or equivalently $-\mathfrak{J} \dot{z}-\nabla \mathrm{H}(z)=0$.
We are searching for solutions $z \in H_{T}^{1}$ of the equation $(*)$, where $H_{T}^{1}(T>0)$ denotes the completion of the set of smooth T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^{2 n}$ in the norm associated to the inner product $(u \mid v)_{\mathrm{H}_{\mathrm{T}}^{1}}=$ $\int_{0}^{T} u v d t+\int_{0}^{T} \dot{u} \dot{v} d t$. For this purpose we apply the method of the topological degree $\operatorname{Deg}_{S^{1}}^{\nabla}$. Namely, let $E=L^{2}\left(S^{1}, \mathbb{R}^{2 n}\right)$ and $E_{1}=H^{1}\left(S^{1}, \mathbb{R}^{2 n}\right)$. Moreover, denote by $D$ the set $E_{1}$ equipped with the inner product from $E$.

Observe that

- $E$ and $E_{1}$ are Hilbert spaces and orthogonal representations of the group $S O(2)=S^{1}$ with the $S^{1}$-action given by the shift in time,
- $A: D \rightarrow E$ given by $A z=-\partial \dot{z}$ is an equivariant unbounded self-adjoint operator with a purely discrete spectrum,
- $\nabla \mathrm{H}(z)$ is a gradient of the invariant functional $\varphi: E \rightarrow \mathbb{R}$ defined by $\varphi(z)=\int_{0}^{2 \pi} \mathrm{H}(z(\mathrm{t})) \mathrm{dt}$,
- $\nabla \mathrm{H} \circ \imath: \mathrm{E}_{1} \rightarrow \mathrm{E}$ is a compact map by the compactness of the inclusion $\imath: \mathrm{E}_{1} \rightarrow \mathrm{E}$.

We can now formulate the main result of this subsection.
Theorem 6.1. Assume that $\lambda>0$ and the set of zeros of the map $f_{\lambda}(z)=-\partial \dot{z}-\lambda \nabla \mathrm{H}(z)$ is compact. If $\operatorname{Deg}_{S^{1}}^{\nabla}\left(f_{\lambda}\right) \neq 0$ then the equation $(*)$ has a solution in $\mathrm{H}_{2 \pi \lambda}^{1}$.
Proof. First note that if $f_{\lambda}^{-1}(0)$ is compact then $f_{\lambda}$ is an element of $\mathcal{G}_{S^{1}}(E)$. By the existence property, $\operatorname{Deg}_{S^{1}}\left(f_{\lambda}\right) \neq 0$ implies that $f_{\lambda}(z)=0$ for some $z \in E_{1}$. Hence a lift $\widetilde{z} \in H_{2 \pi \lambda}^{1}$ of $z$ given by $\widetilde{z}(t)=z(\rho(t))$, where $\rho: \mathbb{R} \rightarrow S^{1}$ is the standard covering projection, is a solution of $(*)$, which is our claim.

### 6.2. Applications to the Seiberg-Witten equations

The description of the Seiberg-Witten equations presented here is necessarily sketchy (for more details we refer the reader to $[4,10,11,18]$ ). Let $M$ be a closed oriented Riemannian 3-manifold. A Spin ${ }^{\text {c }}$-structure on $M$ consists of rank two Hermitian vector bundle $S \rightarrow M$ called the spinor bundle. We write $\Omega^{1}(M, i \mathbb{R})$ for the space of smooth imaginary-valued 1-forms on $M$ and $\Gamma(S)$ for the space of smooth cross-sections of the spinor bundle $S \rightarrow M$. For each $a \in \Omega^{1}(M, i \mathbb{R})$ there is an associated Dirac operator $D_{a}: \Gamma(S) \rightarrow \Gamma(S)$.

Recall that, in what follows, $d$ stands for the exterior derivative and $*$ denotes the Hodge star. For a pair $(a, \varphi) \in \Omega^{1}(M, i \mathbb{R}) \oplus \Gamma(S)$ the Seiberg-Witten equations are

$$
\left\{\begin{array}{l}
D_{a} \varphi=0 \\
* \mathrm{da}=\mathrm{Q}(\varphi),
\end{array}\right.
$$

where $Q(\varphi) \in \Omega^{1}(M, i \mathbb{R})$ is a certain quadratic form (nonlinear part of the equations). The solutions of Seiberg-Witten equations are zeros of the Seiberg-Witten map

$$
\mathrm{SW}: \Omega^{1}(\mathrm{M}, \mathrm{i} \mathbb{R}) \oplus \Gamma(\mathrm{S}) \rightarrow \Omega^{1}(M, \mathfrak{i}) \oplus \Gamma(\mathrm{S})
$$

given by

$$
\operatorname{SW}(a, \varphi)=\left(* \operatorname{da}-Q(\varphi),-D_{a} \varphi\right)
$$

After suitable Sobolev completion the Seiberg-Witten map SW can be written in the form $A-F$, where $A=\left(* d a,-D_{a} \varphi\right)$ is an unbounded self-adjoint operator and $F$ is a gradient map. Moreover, the SeibergWitten map is equivariant for the action of the group $S^{1}$, which acts trivially on the component arising from the differential forms and as complex multiplication on the spinor component. It suggests that the SW map should fit to our abstract setting of the degree $\operatorname{Deg}_{\text {S }^{1}}^{\nabla}$. Unfortunately, the set of zeros of the SW map is not compact and its operator part does not have a purely discrete spectrum. However, we hope that it is possible to reduce our problem to some subspace of $\Omega^{1}(M, i \mathbb{R})$ in such a way that the reduced SW map will have a compact set of zeros and its operator part will have a purely discrete spectrum. Verifying this claim is, however, still in progress.

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## Appendix A.

Definition 3.4 may be seen as a simple particular case of a more general construction called the direct limit of a direct system of groups. Namely, for $i=0,1, \ldots$ let $G_{i}$ denote an abelian group and $\alpha_{i}: G_{i} \rightarrow G_{i+1}$ a group homomorphism. With this notation we get the sequence

$$
\mathrm{G}_{0} \xrightarrow{\alpha_{0}} \mathrm{G}_{1} \xrightarrow{\alpha_{1}} \mathrm{G}_{2} \xrightarrow{\alpha_{2}} \mathrm{G}_{3} \rightarrow \cdots
$$

Let $\widetilde{G}:=\coprod_{i=0}^{\infty} G_{i}$ denote a disjoint union, i.e.

$$
\widetilde{G}=\left\{(i, m) \mid i \in \mathbb{N}, m \in G_{i}\right\}
$$

We introduce in $\widetilde{G}$ an equivalence relation. For $\mathfrak{i}>\mathfrak{j}$ we write $(i, m) \sim(j, l)$ if

$$
\alpha_{i-1} \circ \cdots \circ \alpha_{j+1} \circ \alpha_{j}(l)=m .
$$

The direct limit of groups is the set of equivalence classes of the above relation, denoted by

$$
\lim _{\longrightarrow} G_{i}=\widetilde{G} / \sim .
$$

Let $\underset{\longrightarrow}{\lim } \mathfrak{U}(\mathrm{G})$ denote a direct limit of groups, where

- $G_{i}=\mathfrak{U}(G)$ for all $i$,
- $\alpha_{i}$ is multiplication by an element $a_{i}=\operatorname{deg}_{G}^{\nabla}\left(A_{i}, V^{i}\right) \in \mathfrak{l}(G)$.

With this notation we can alternatively define our degree as a function $\operatorname{Deg}_{G}^{\nabla}: \mathcal{G}_{G}(E) \rightarrow \underset{\longrightarrow}{\lim } \mathfrak{U}(G) \approx \mathfrak{U}(G)$ given by

$$
\operatorname{Deg}_{G}^{\nabla}(f):=\left[\left(n, \operatorname{deg}_{G}^{\nabla}\left(f_{n}, U_{n}\right)\right)\right]
$$

for $n$ large enough.

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