

## TOTAL OUTER-CONNECTED DOMINATION IN TREES

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### Abstract

Let  $G = (V, E)$  be a graph. Set  $D \subseteq V(G)$  is a total outer-connected dominating set of  $G$  if  $D$  is a total dominating set in  $G$  and  $G[V(G) - D]$  is connected. The total outer-connected domination number of  $G$ , denoted by  $\gamma_{tc}(G)$ , is the smallest cardinality of a total outer-connected dominating set of  $G$ . We show that if  $T$  is a tree of order  $n$ , then  $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$ . Moreover, we constructively characterize the family of extremal trees  $T$  of order  $n$  achieving this lower bound.

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### 1. INTRODUCTION

Graph theory terminology not presented here can be found in [1, 5].

Let  $G = (V, E)$  be a simple graph. The *neighbourhood* of a vertex  $v$ , denoted by  $N_G(v)$ , is the set of all vertices adjacent to  $v$  in  $G$  and the integer  $d_G(v) = |N_G(v)|$  is the *degree* of  $v$  in  $G$ . A vertex of degree one is called an *end-vertex*. A *support* is the unique neighbour of an end-vertex.

Let  $P_n$  denotes the path of order  $n$ . For a vertex  $v$  of  $G$ , we shall use the expression, *attach a  $P_n$  at  $v$* , to refer to the operation of taking the union of  $G$  and a path  $P_n$  and joining one of the end-vertices of this path to  $v$  with an edge.

Set  $D \subseteq V(G)$  is a *dominating set* in  $G$  if  $N_G(v) \cap D \neq \emptyset$  for every vertex  $v \in V(G) - D$ . The *domination number* of  $G$ , denoted  $\gamma(G)$ , is the cardinality of a minimum dominating set of  $G$ .

Set  $D \subseteq V(G)$  is a *total dominating set* of  $G$  if each vertex of  $V(G)$  has a neighbour in  $D$ . The cardinality of a minimum total dominating set in  $G$  is the *total domination number* of  $G$  and is denoted by  $\gamma_t(G)$ . Total domination in graphs is currently well studied in graph theory (for examples, see [2, 6]).

Set  $D \subseteq V(G)$  is said to be a *total outer-connected dominating set* of  $G$  if  $D$  is a total dominating set and  $G[V(G) - D]$  is connected. The cardinality of a minimum total outer-connected dominating set in  $G$  is called the *total outer-connected domination number* of  $G$  and is denoted by  $\gamma_{tc}(G)$ . Observe that every graph  $G$  without isolates has a total outer-connected dominating set, since the set of all vertices of  $G$  is a total outer-connected dominating set in  $G$ .

We will show that if  $T$  is a tree of order  $n$ , then  $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$ . Moreover, we will constructively characterize the extremal trees  $T$  of order  $n \geq 3$  achieving this lower bound.

Similar bounds for various domination numbers in trees are given in [2, 6].

## 2. THE LOWER BOUND

**Theorem 1.** *If  $T$  is a tree of order  $n \geq 2$ , then*

$$\gamma_{tc}(T) \geq \left\lceil \frac{2n}{3} \right\rceil.$$

**Proof.** The result is obvious for  $n = 2$ . Assume that  $n \geq 3$  and let  $D$  be a minimum total outer-connected dominating set of  $T$ . Let us denote by  $S$  any component of  $T[D]$ . Since  $T$  is a tree, no two vertices of  $V(T) - D$  have a common neighbour in  $S$ . Hence  $|N_T(S) \cap (V(T) - D)| \leq 1$ . Moreover,  $D$  is dominating in  $T$  and isolate free, and thus

$$\begin{aligned} n(T) &= |V(T) - D| + |D| \\ &\geq |V(T) - D| + 2|V(T) - D| \\ &\geq n - \gamma_{tc}(T) + 2n - 2\gamma_{tc}(T). \end{aligned}$$

Finally, we have  $\gamma_{tc}(T) \geq \frac{2}{3}n$ , and so  $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$ . ■



## 3. THE CHARACTERIZATION OF THE EXTREMAL TREES

For  $n \geq 2$ , let  $\mathcal{T}_n = \{T \mid T \text{ is a tree of order } n \text{ such that } \gamma_{tc}(T) = \lceil \frac{2n}{3} \rceil\}$ ,  $\mathcal{T} = \bigcup_{n \geq 2} \mathcal{T}_n$ . We will present a constructive characterization of the family  $\mathcal{T}$ . For this purpose, we define a type (1) operation on a tree  $T$  as attaching  $P_3$  at  $v$  where  $v$  is a vertex of  $T$  not belonging to some minimum total outer-connected dominating set of  $T$ , and a type (2) operation as attaching  $P_1$  at  $v$  where  $v$  belongs to some minimum total outer-connected dominating set of  $T$ .

We now define families of trees as follows. Let  $\mathcal{C}_n = \{T \mid T \text{ is a tree of order } n \text{ which can be obtained from the path } P_3 \text{ by a finite sequence of operations of type (1) and (2), where the operation of type (2) appears in the sequence exactly } n \pmod{3} \text{ times}\}$ ,  $n \geq 3$ , and  $\mathcal{C}_2 = \{P_2\}$ .

We shall establish:

**Theorem 2.** For  $n \geq 2$ ,  $\mathcal{T}_n = \mathcal{C}_n$ .

We prove Theorem 2 by establishing eight lemmas.

**Lemma 3.** If  $D$  is a minimum total outer-connected dominating set of a tree  $T$  of order at least 6 and  $T \in \mathcal{T}$ , then every end-vertex of  $T$  and every support of  $T$  belongs to  $D$ .

**Lemma 4.** If  $T \in \mathcal{T}$ , then  $|\Omega(T)| \leq |S(T)| + 2$ , where  $\Omega(T)$  is the set of all end-vertices of  $T$  and  $S(T)$  is the set of all supports of  $T$ .

**Proof.** Let  $D$  be a minimum total outer-connected dominating set of a tree  $T$  belonging to  $\mathcal{T}$ . Then for some positive integer  $n$  we have  $T \in \mathcal{T}_n$  and  $|D| = \lceil \frac{2n}{3} \rceil$ . Suppose  $|\Omega(T)| = |S(T)| + t$ ,  $t > 2$ . Denote by  $s_1, \dots, s_m$  the supports of  $T$  and by  $l_1, \dots, l_m, l_{m+1}, \dots, l_{m+t}$  the end-vertices of  $T$ , where  $l_i \in N_T(s_i)$ ,  $1 \leq i \leq m$ . Notice that  $D - \{l_{m+1}, l_{m+2}, l_{m+3}\}$  is a total outer-connected dominating set of a tree  $T' = T - \{l_{m+1}, l_{m+2}, l_{m+3}\}$ . Hence  $\gamma_{tc}(T') \leq |D| - 3 = \lceil \frac{2n-9}{3} \rceil$ . On the other hand, by Theorem 1, we have  $\gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil$  and consequently  $\lceil \frac{2(n-3)}{3} \rceil \leq \gamma_{tc}(T') \leq \lceil \frac{2n-9}{3} \rceil$ , which is impossible. ■

Thus we have what follows.

**Corollary 1.** If  $T \in \mathcal{T}$ , then exactly one of the following conditions holds:

- (i) every support of  $T$  is a neighbour of exactly one end-vertex;
- (ii) exactly one support of  $T$  is a neighbour of exactly two end-vertices, while every other support is a neighbour of exactly one end-vertex;
- (iii) exactly one support of  $T$  is a neighbour of three end-vertices, while every other support is a neighbour of exactly one end-vertex or exactly two supports of  $T$  are the neighbours of exactly two end-vertices, while every other support is a neighbour of exactly one end-vertex.

**Lemma 5.** *If  $T \in \mathcal{T}_n$ ,  $n \geq 3$ , and  $T'$  is obtained from  $T$  by a type (1) operation, then  $T' \in \mathcal{T}_{n+3}$ .*

**Proof.** By definition of a type (1) operation on a tree  $T$ , there exists a minimum total outer-connected dominating set of  $T$  such that adding a new end-vertex of  $T'$  and a new support of  $T'$  to it produces a total outer-connected dominating set of  $T'$ . Hence, since  $T \in \mathcal{T}_n$ ,  $\gamma_{tc}(T') \leq \gamma_{tc}(T) + 2 = \lceil \frac{2n+6}{3} \rceil$ . However,  $T'$  is a tree of order  $n+3$ , and so, by Theorem 1,  $\gamma_{tc}(T') \geq \lceil \frac{2(n+3)}{3} \rceil$ . Consequently,  $\gamma_{tc}(T') = \lceil \frac{2(n+3)}{3} \rceil$ , and hence  $T' \in \mathcal{T}_{n+3}$ . ■

Notice that  $\mathcal{C}_3 = \{P_3\} = \mathcal{T}_3$ . Hence an immediate consequence of Lemma 5 now follows.

**Lemma 6.** *If  $n \geq 3$  and  $n \equiv 0 \pmod{3}$ , then  $\mathcal{C}_n \subseteq \mathcal{T}_n$ .*

We will now prove the inverse inclusion.

**Lemma 7.** *If  $n \geq 3$  and  $n \equiv 0 \pmod{3}$ , then  $\mathcal{T}_n \subseteq \mathcal{C}_n$ .*

**Proof.** We proceed by induction on  $n \geq 3$ . Since  $\mathcal{T}_3 = \{P_3\} = \mathcal{C}_3$ , the result is true for  $n = 3$ . Let  $n \geq 6$  satisfy  $n \equiv 0 \pmod{3}$  and assume that  $\mathcal{T}_k \subseteq \mathcal{C}_k$  for all integers  $k \equiv 0 \pmod{3}$ , where  $3 \leq k < n$ . Let  $T \in \mathcal{T}_n$ . We show that  $T \in \mathcal{C}_n$ . Let  $D$  be a minimum total outer-connected dominating set of  $T$ . Let  $P = (v_1, v_2, \dots, v_m)$  be a longest path in  $T$ . By Lemma 3,  $\{v_1, v_2, v_{m-1}, v_m\} \subseteq D$ .

We will show that  $d_T(v_2) \equiv 2$  and  $\{v_3, v_4\} \cap D = \emptyset$ . Suppose that  $v_2$  is adjacent to two end-vertices, say  $v_1$  and  $l_1$ . Then  $D' = D - \{l_1\}$  is a total outer-connected dominating set of  $T' = T - l_1$ . Hence, since  $T \in \mathcal{T}_n$ ,  $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 1 = \frac{2n}{3} - 1$ . However,  $T'$  is a tree of order  $n-1 \equiv 2 \pmod{3}$ , and so, by Theorem 1,  $\gamma_{tc}(T') \geq \lceil \frac{2(n-1)}{3} \rceil = \frac{2n}{3}$ , a contradiction.

Suppose now  $v_3 \in D$ . Then the set  $D' = D - \{v_1\}$  is a total outer-connected dominating set of  $T' = T - v_1$  and  $\frac{2n}{3} \leq \gamma_{tc}(T') \leq \frac{2n}{3} - 1$  — a contradiction. Hence  $d_T(v_2) = 2$  and  $v_3 \notin D$ . From Lemma 3 and from the fact that  $V(T) - D$  is a tree we conclude that  $m \geq 6$  and  $v_4 \notin D$ .

We will now prove that  $d_T(v_3) = 2$ . Since  $v_3 \notin D$ ,  $v_3$  is not a support. Suppose there exists a path  $P' = (u_1, u_2, v_3)$  in  $T$  such that  $u_2 \notin \{v_2, v_4\}$ . By Lemma 4,  $\{u_1, u_2\} \subseteq D$ . Moreover  $D' = D - \{u_1, u_2\}$  is a total outer-connected dominating set of  $T' = T - \{u_1, u_2\}$ . Hence  $\gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 = \frac{2n}{3} - 2$ , which contradicts the fact that (by Theorem 1)  $\gamma_{tc}(T') \geq \lceil \frac{2(n-2)}{3} \rceil$ .

Let us consider tree  $T' = T - \{v_1, v_2, v_3\}$ . The set  $D' = D - \{v_1, v_2\}$  is a total outer-connected dominating set of  $T'$ . Hence  $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 2 = \lceil \frac{2n-6}{3} \rceil$ . Moreover by Theorem 1,  $\gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil$  and so  $T' \in \mathcal{T}_{n-3}$ . Thus, by the inductive hypothesis,  $T' \in \mathcal{C}_{n-3}$ . Since  $v_4$  does not belong to some minimum total outer-connected dominating set of  $T'$ , namely  $D'$ ,  $T$  is constructed from  $T'$  by a type (1) operation. Hence  $T \in \mathcal{C}_n$ . ■

**Lemma 8.** *If  $T \in \mathcal{T}_n$ ,  $n \geq 3$ , and  $n \not\equiv 2 \pmod{3}$ , then a tree  $T'$  obtained from  $T$  by a type (2) operation belongs to  $\mathcal{T}_{n+1}$ .*

**Proof.** By definition of a type (2) operation on a tree  $T$ , there exists a minimum total outer-connected dominating set of  $T$  such that adding to it the new end-vertex of  $T'$  produces a total outer-connected dominating set of  $T'$ . Hence, since  $T \in \mathcal{T}_n$  and  $n \not\equiv 2 \pmod{3}$ ,  $\gamma_{tc}(T') \leq \gamma_{tc}(T) + 1 = \lceil \frac{2n+3}{3} \rceil = \lceil \frac{2n+2}{3} \rceil$ . However,  $T'$  is a tree of order  $n+1$ , and so, by Theorem 1,  $\gamma_{tc}(T') \geq \lceil \frac{2(n+1)}{3} \rceil$ . Consequently,  $\gamma_{tc}(T') = \lceil \frac{2n+2}{3} \rceil$  and  $T' \in \mathcal{T}_{n+1}$ . ■

**Lemma 9.** *If  $n \geq 4$  and  $n \not\equiv 0 \pmod{3}$ , then  $\mathcal{C}_n \subseteq \mathcal{T}_n$ .*

**Proof.** We proceed by induction on  $n \geq 4$ . The base case is true since  $\mathcal{C}_4 = \{K_{1,3}, P_4\} \subseteq \mathcal{T}_4$  and  $\mathcal{C}_5 = \{K_{1,4}, P_5, T_1\} \subseteq \mathcal{T}_5$ , where  $T_1$  is a tree obtained from a star  $K_{1,3}$  by subdivision of exactly one of its edges.

Assume now that the result is true for  $k \not\equiv 0 \pmod{3}$ ,  $4 \leq k < n$ . Let  $T$  be a tree belonging to the family  $\mathcal{C}_n$ . Thus  $T$  can be obtained from a tree  $T'$  by either one operation of type (1) or one operation of type (2). If  $T$  is obtained from  $T'$  as a result of operation of type (1), then  $T'$  is a tree of order  $n-3$  and by our induction hypothesis  $T' \in \mathcal{T}_{n-3}$ . Therefore, by Lemma 5,  $T \in \mathcal{T}_n$ .

If  $T$  is obtained from  $T'$  by one operation of type (2), then  $T'$  is a tree of order  $n-1$ . We consider two cases:



*Case 1.* If  $n \equiv 1 \pmod{3}$ , then the construction of  $T'$  is accomplished by using only type (1) operations starting with the path  $P_3$  and thus  $T' \in \mathcal{C}_{n-1}$ . From Lemma 6 we conclude that  $T' \in \mathcal{T}_{n-1}$ . Hence, by Lemma 8,  $T \in \mathcal{T}_n$ .

*Case 2.* If  $n \equiv 2 \pmod{3}$ , then  $T' \in \mathcal{C}_{n-1}$  and by our induction hypothesis  $T' \in \mathcal{T}_{n-1}$ . Finally, by Lemma 8,  $T \in \mathcal{T}_n$ . ■

**Lemma 10.** *If  $n \geq 4$  and  $n \not\equiv 0 \pmod{3}$ , then  $\mathcal{T}_n \subseteq \mathcal{C}_n$ .*

**Proof.** We proceed by induction on  $n \geq 4$ . Since  $\mathcal{P}_4 = \{P_4, K_{1,3}\} = \mathcal{C}_4$  and  $\mathcal{P}_5 = \{K_{1,4}, P_5, T_1\} = \mathcal{C}_5$ , where  $T_1$  is a tree obtained from a star  $K_{1,3}$  by subdivision of exactly one of its edges, the result is true for  $n = 4$  and  $n = 5$ . Let  $n \geq 7$  satisfy  $n \not\equiv 0 \pmod{3}$ , and assume that  $\mathcal{T}_k \subseteq \mathcal{C}_k$  for all integers  $k \not\equiv 0 \pmod{3}$ , where  $4 \leq k < n$ . Let  $T \in \mathcal{T}_n$  and let  $D$  be a minimum total outer-connected dominating set of  $T$ . Let  $P = (v_1, v_2, \dots, v_m)$  be the longest path in  $T$ . By Lemma 3,  $\{v_1, v_2, v_{m-1}, v_m\} \subseteq D$ . We consider two cases:

*Case 1.* One of the vertices  $v_2$  or  $v_{m-1}$  is adjacent to at least two end-vertices. Without loss of generality, we can assume that  $|N_T(v_2) \cap \Omega(T)| \geq 2$ . Let  $l_1 \in N_T(v_2) \cap \Omega(T)$ ,  $l_1 \neq v_1$ . In this case  $D' = D - \{l_1\}$  is a total outer-connected dominating set of  $T' = T - l_1$  and hence  $\gamma_{tc}(T') \leq \gamma_{tc}(T) - 1 = \lceil \frac{2n-3}{3} \rceil = \lceil \frac{2n-2}{3} \rceil$ . Thus, Theorem 1 implies  $\gamma_{tc}(T') = \lceil \frac{2n-2}{3} \rceil$ . Depending on whether  $n \equiv 1 \pmod{3}$  or  $n \equiv 2 \pmod{3}$  we have  $T' \in \mathcal{C}_{n-1}$  from Lemma 7 or by our induction hypothesis, respectively. Hence we obtain  $T \in \mathcal{C}_n$ .

*Case 2.* The vertices  $v_2$  and  $v_{m-1}$  have degree 2. Suppose that  $v_3$  or  $v_{m-2}$ , say  $v_3$ , belongs to  $D$ . Then for tree  $T' = T - v_1$  and for  $D' = D - \{v_1\}$ , similarly to Case 1, we have that  $T \in \mathcal{C}_n$ . Hence we can assume that  $\{v_3, v_{m-2}\} \cap D = \emptyset$ . Thus from connectivity of  $V(T) - D$  we have  $\{v_4, v_{m-3}\} \cap D = \emptyset$ .

We will now show that  $v_3$  or  $v_{m-2}$  is of degree two. Suppose to the contrary, that neither  $v_3$  nor  $v_{m-2}$  is of degree 2. Let  $y$  be the neighbour of  $v_3$ ,  $y \neq v_2$  and  $y \neq v_4$ , and let  $z$  be the neighbour of  $v_{m-2}$ ,  $z \neq v_{m-1}$  and  $z \neq v_{m-3}$ . Then neither  $y$  nor  $z$  is not an end-vertex – otherwise we would have  $v_3 \in D$  or  $v_{m-2} \in D$ . From that and from our choice of path  $(v_1, v_2, \dots, v_m)$  it is straightforward that  $y$  and  $z$  are supports and  $A = N_T(y) - \{v_3\} \subseteq \Omega(T)$ ,  $B = N_T(z) - \{v_{m-2}\} \subseteq \Omega(T)$ . We also have that  $D - (A \cup B \cup \{y, z\})$  is a total outer-connected dominating set of  $T' = T - (A \cup B \cup \{y, z\})$ , and so  $\lceil \frac{2(n-2-|A|-|B|)}{3} \rceil \leq \gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 - |A| - |B| \leq \lceil \frac{2n}{3} \rceil - 2 - |A| - |B|$ , which

is impossible. Therefore, without the loss of generality, we may assume that  $\deg_T(v_3) = 2$ .

Let us consider  $T' = T - \{v_1, v_2, v_3\}$ . The set  $D' = D - \{v_1, v_2\}$  is a total outer-connected dominating set of  $T'$ , and hence  $\gamma_{tc}(T') \leq \lceil \frac{2n}{3} \rceil - 2 = \lceil \frac{2n-6}{3} \rceil$ . Moreover, by Theorem 1,  $\gamma_{tc}(T') \geq \lceil \frac{2(n-3)}{3} \rceil$  and so  $T' \in \mathcal{T}_{n-3}$ . Therefore, by the inductive hypothesis,  $T' \in \mathcal{C}_{n-3}$ . However,  $T$  is constructed from  $T'$  by a type (1) operation. Hence  $T \in \mathcal{C}_n$ . ■

Theorem 2 now follows immediately from Lemmas 6, 7, 9 and 10.

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